Year 11 Extension 1 - Preliminary Examination 2005

QUESTION 1: START A NEW PAGE

Marks

(a) Find the acute angle to the nearest degree between the lines with the equations $\sqrt{3}x + y = 2$ and $x - y = 0$. **3**

(b) How many ways can the ten letters of the word REGULATION be arranged in a line if the vowels A, E, I, O, U occur:

(i) All together? **1**
(ii) In the order A, E, I, O, U, throughout the arrangement? **1**
(not necessarily next to each other)
Answers may be left unsimplified.

(c) If $\sin A = \dfrac{2}{3}$ and $90° < A < 180°$, find the exact value of $\tan 2A$. **3**

(d) A parabola has the equation $8y = x^2 + 6x + 1$.

(i) Write the equation of the parabola in the form: **2**
$(x-h)^2 = 4a(y-k)$.
(ii) Draw a neat sketch of the parabola on a number plane showing the:
(α) Vertex **1**
(β) Focus **1**
(γ) Directrix **1**

(e) Write $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$, **2**
where $0° < \alpha < 360°$ and R>0.

QUESTION 2: START A NEW PAGE

Marks

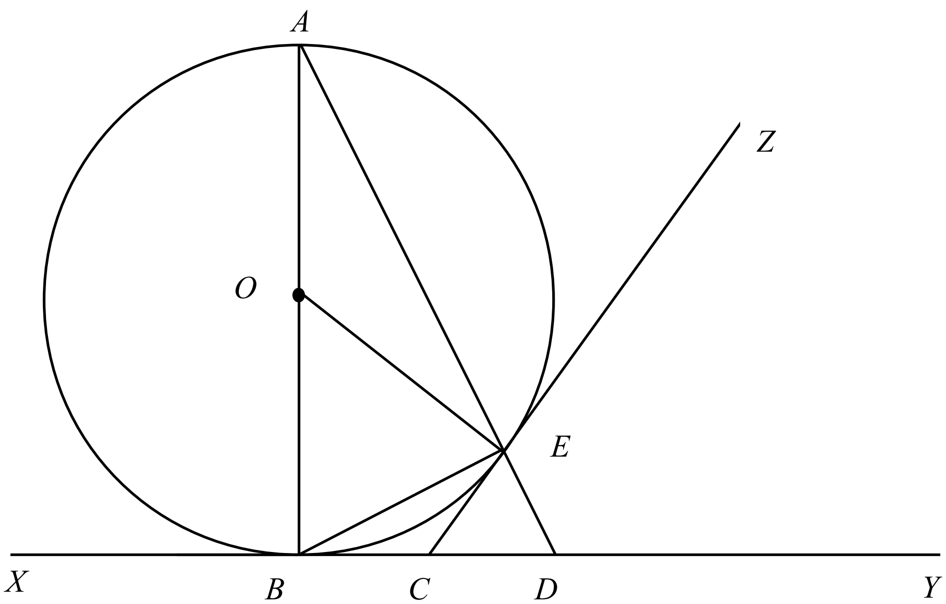
(a) Use the substitution $t = \tan\left[\dfrac{\theta}{2}\right]$ to solve to the nearest minute: **4**

$$2\sin\theta - \cos\theta = 1 \qquad \text{for } 0° \le \theta \le 360°.$$

(b) (i) Sketch the graph of $y = \dfrac{x-2}{x-4}$. **2**

(ii) Hence or otherwise, solve $\dfrac{x-2}{x-4} \le 2$. **2**

(c) *XY* and *CZ* are tangents at *B* and *E* respectively to circle centre *O*. Chord *AE* is produced to meet *XY* at *D*. E is joined to centre *O* and B.



(i) Copy the diagram and prove that $\angle EOB = \angle ECD$. **3**

(ii) Show that *C* is the centre of the circle through points *B, E* and *D*. **4**

QUESTION 3: START A NEW PAGE

Marks

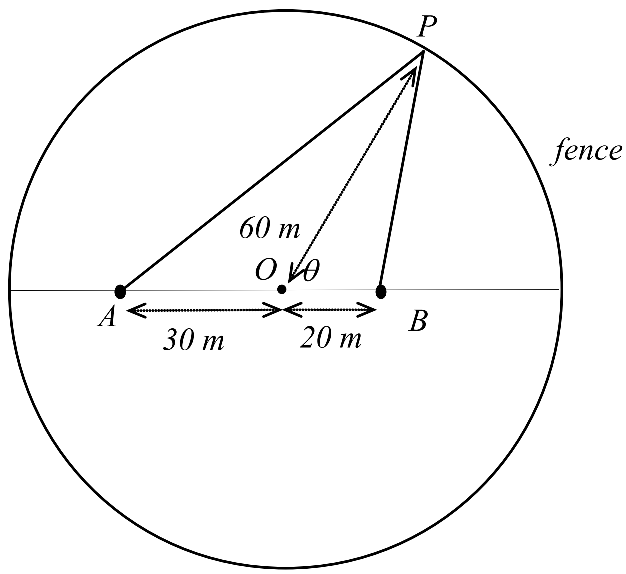
(a) If α, β and γ are the roots of the equation $2x^3 - 3x^2 - 4x + 1 = 0$ find the value of:

(i) $\alpha+\beta+\gamma$ **1**

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

(b) Find the general solution in degrees to $\sin 2\theta - \cos\theta = 0$. **3**

(c) In an activity during a sport class, students are to run from position A to the fence surrounding a circular field of radius 60 m and return to position B. Jason, who is a top Maths student, decides to calculate which direction will give him the shortest path.



(i) If he runs from A to P then to B show that the total path length D m is given by: **2**

$$D = 30\sqrt{5+4\cos\theta} + 20\sqrt{10-6\cos\theta}\ ,$$

where $\angle POB = \theta$ and $0 \le \theta \le \pi$.

(ii) Find the value of θ which will give Jason the shortest path. Justify your answer. **5**

(d) 16 people are invited to a party. They are to sit around 2 circular tables, with 8 people at each table.

(i) How many different ways can 2 groups of 8 be chosen from the 24 people? **1**

(ii) Find the number of different seating arrangements that are possible at the party? **1**

Answers may be left unsimplified.

QUESTION 4: START A NEW PAGE

Marks

(a) When the polynomial $P(x)$ is divided by $x-1$, the remainder is 2, and when divided by $x-2$ the remainder is 1. What is the remainder when $P(x)$ is divided by $(x-1)(x-2)$? **3**

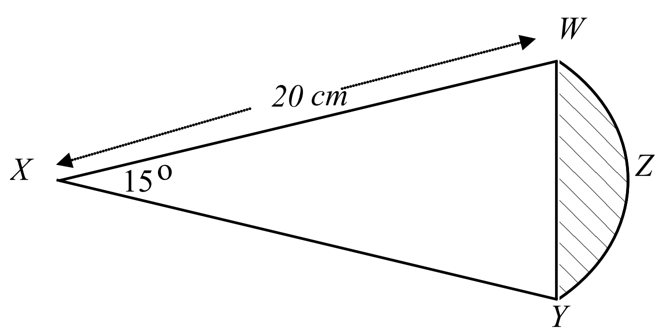
(b) Find the Cartesian equation of the curve whose parametric equations are: **3**

$$x = \cos 2t \text{ and } y = \cos t,\ 0 \le t \le 2\pi .$$

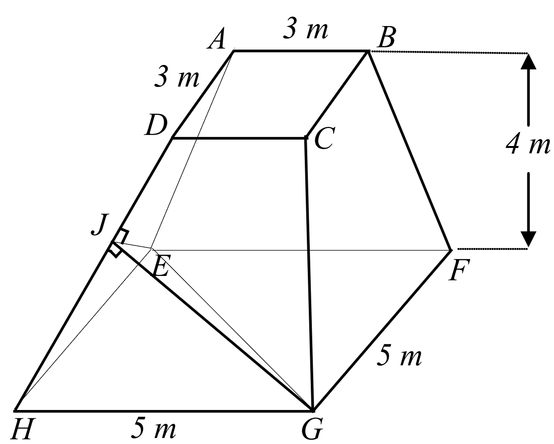
(c)

(i) Show that the exact value of $\sin 15^\circ$ is $\frac{\sqrt{3}-1}{2\sqrt{2}}$. **2**

(ii) If *WXYZ* is a sector of a circle of radius 20 cm, find the exact area of the segment shaded in the diagram below. **2**



(d) A frustrum of a right square pyramid has a base of side length 5m, and a square top of side length 3 m. Its vertical height is 4 m.



(i) Show that the length of the slant edge *DH* is $3\sqrt{2}$ m. **2**

(ii) Calculate the size of $\angle GJE$, the angle between the sloping faces, to the nearest minute. **3**

END OF EXAMINATION

Extension I 2005

1. (a) $\sqrt{3}x + y = 2 \quad m = -\sqrt{3}$ ①

$x - y = 0 \quad m = 1$

$\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \dfrac{-\sqrt{3} - 1}{1 - (\sqrt{3})(1)} \right|$ ①

$\theta = 75^\circ$ ①

(b)(i) $\boxed{AEIOU}$ $\boxed{R}$ $\boxed{G}$ $\boxed{L}$ $\boxed{T}$ $\boxed{N}$

6 elements arranged 6! ways

A, E, I, O, U can be arranged 5! ways

$\therefore$ total arrangements $= 6! \times 5!$ (86400) ways ①

(ii) 10 elements arranged 10! ways

For each arrangement of vowels only one has them in the correct order ie 1 out of 5!

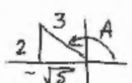
$\therefore$ no of ways with vowels in AEIOU order $= \dfrac{10!}{5!}$ ①

$= 30240$

(c) $\sin A = 2/3 \quad 90^\circ < A < 180^\circ$

$\therefore$ 2nd quadrant angle

$\therefore \tan A < 0 \quad \tan A = \frac{-2}{\sqrt{5}}$ ①



$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

① $= \dfrac{2(-2/\sqrt{5})}{1 - \left(\frac{-2}{\sqrt{5}}\right)^2} = \dfrac{-4/\sqrt{5}}{1 - \frac{4}{5}}$

① $= -4\sqrt{5}$

(d) $8y = x^2 + 6x + 1$

$\therefore x^2 + 6x = 8y - 1$

$x^2 + 6x + 9 = 8y - 1 + 9$ ①

$(x+3)^2 = 8(y+1)$

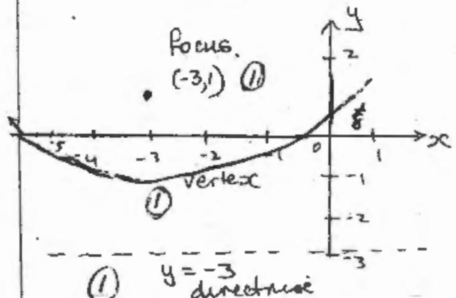
$(x+3)^2 = 4 \times 2(y+1)$ ①

vertex $= (-3, -1)$

$\therefore$ focal length = 2 units

$\therefore$ focus $= (-3, 1)$

directrix $y = -3$



(e) $\cos\theta + \sqrt{3}\sin\theta$

$= 2\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$= R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$

$R^2 = 1^2 + (\sqrt{3})^2 = 4$

$\therefore R = 2 \quad (\text{as } R > 0)$

$\cos\alpha = \frac{1}{2} \qquad \sin\alpha = -\frac{\sqrt{3}}{2}$

$\therefore \alpha$ is a 4th quad angle

$\therefore \alpha = 360^\circ - 60^\circ$

$= 300^\circ$

$\therefore$ Answer is

$2\cos(\theta + 300^\circ)$

① ①

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2 (a) $t = \tan(\frac{1}{2}\theta)$

$\therefore \sin\theta = \dfrac{2t}{1+t^2} \quad \cos\theta = \dfrac{1-t^2}{1+t^2}$

$\therefore 2\sin\theta - \cos\theta = 1$ ①

$\therefore 2\left(\frac{2t}{1+t^2}\right) - \frac{(1-t^2)}{(1+t^2)} = 1$

$\therefore 4t - 1 + t^2 = 1 + t^2$

$4t = 2$

$t = \frac{1}{2}$

$\tan\frac{1}{2}\theta = \frac{1}{2}$ ①

$\therefore \frac{1}{2}\theta = 26.56505118^\circ$

$\therefore \theta = 53.13010236^\circ$

$\theta = 53^\circ 8'$ (nearest minute)

Check $180^\circ = \theta$

LHS $= 2 \times \sin 180^\circ - \cos 180^\circ$

$= -(-1) = 1$

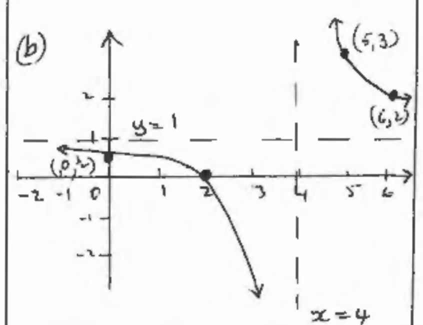
$=$ RHS

$\therefore 180^\circ$ is a solution

$\therefore \theta = 53^\circ 8'$ or 180°

① ①

(b)



asymptotes ①

intercepts ①

to solve $\dfrac{x-2}{x-4} \le 2$

consider $\dfrac{x-2}{x-4} = 2$

$x - 2 = 2x - 8$

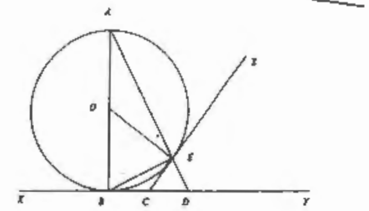
$6 = x$

From graph

$x < 4$ or $x \ge 6$ ①

①

(c)



(i) To prove $\angle EOB = \angle ECD$

① $\angle OBC = 90^\circ$ (angle between tangent and radius at B)

$\angle OEC = 90^\circ$ (similarly at E)

① $\therefore$ OECD is a cyclic quadrilateral (opposite angles of OECD sum to 180°)

① $\therefore \angle EOB = \angle ECD$ (exterior angle of OECD equals opposite interior angle)

(ii) To show C is centre of circle through B, E and D.

① CE = CB (intercepts of tangents from external point C are equal)

$\angle AEB = 90^\circ$ (angle subtended by diameter AB at circumference is 90°)

$\angle CED + \angle BEC + \angle AEB = 180^\circ$ (angle sum of straight angle at E is 180°)

$\therefore \angle CED + \angle BEC = 90^\circ$

$\angle BEC = \angle DAB$ (angle in alternate segment is equal to angle between tangent and chord at E)

① $\therefore \angle CED + \angle DAB = 90^\circ$

$\angle ABD = 90^\circ$ (angle between radius and tangent $= 90^\circ$ at B)

$\angle ABD + \angle DAB + \angle ADB = 180^\circ$ (angle sum of $\triangle ABD$ is 180°)

① $\angle DAB + \angle ADB = 90^\circ$

$\therefore \angle CED = \angle ADB$

① $\therefore$ CE = CD (equal sides opposite equal angles in $\triangle$ ECD)

$\therefore$ CE = CD = CB

$\therefore$ C is centre of circle through B, E, D

3 (c) α, β, γ roots of
$2x^3 - 3x^2 - 4x + 1 = 0$

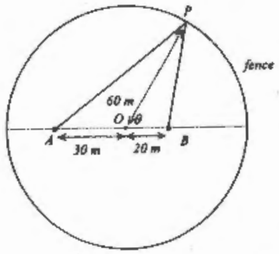
(i) $\alpha + \beta + \gamma = -\frac{b}{a}$ (1)
$= \frac{3}{2}$

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$
$= \frac{-4}{2} = -2$
$\alpha\beta\gamma = \frac{d}{a} = -\frac{1}{2}$ (1)

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
$= \frac{\beta\gamma + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$
$= \frac{-2}{-\frac{1}{2}} = 4$ (1)

(b) (b) $\sin 2\theta - \cos\theta = 0$
$2\sin\theta\cos\theta - \cos\theta = 0$
$\cos\theta(2\sin\theta - 1) = 0$ (1)
$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$
$\theta = 360n^\circ \pm 90^\circ$ or (1)
$\theta = 180n^\circ + (-1)^n 30^\circ$
n is an integer (1)

(c)



(i) $AP^2 = 30^2 + 60^2 - 2 \times 30 \times 60 \times \cos(180 - \theta)$
$= 900 + 3600 + 3600\cos\theta$
$= 4500 + 3600\cos\theta$
$AP = \sqrt{900(5 + 4\cos\theta)}$
$= 30\sqrt{5 + 4\cos\theta}$ (1)
$PB^2 = 20^2 + 60^2 - 2 \times 20 \times 60\cos\theta$
$= 400 + 3600 - 2400\cos\theta$
$PB = \sqrt{400(10 - 6\cos\theta)}$
$= 20\sqrt{10 - 6\cos\theta}$ (1)

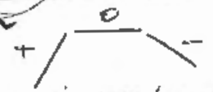
$\therefore D = AP + PB$
$= 30\sqrt{5 + 4\cos\theta} + 20\sqrt{10 - 6\cos\theta}$
$\therefore D = 30(5 + 4\cos\theta)^{\frac{1}{2}} + 20(10 - 6\cos\theta)^{\frac{1}{2}}$
$\frac{dD}{d\theta} = 30 \times \frac{1}{2}(5 + 4\cos\theta)^{-\frac{1}{2}}(-4\sin\theta)$
(1) $+ 20 \times \frac{1}{2}(10 - 6\cos\theta)^{-\frac{1}{2}}(6\sin\theta)$
$= \frac{-60\sin\theta}{\sqrt{5 + 4\cos\theta}} + \frac{60\sin\theta}{\sqrt{10 - 6\cos\theta}}$
$= -60\sin\theta\left[\frac{1}{\sqrt{5 + 4\cos\theta}} - \frac{1}{\sqrt{10 - 6\cos\theta}}\right]$

for stationary point
$\frac{dD}{d\theta} = 0$
$\therefore \sin\theta = 0$ or $\frac{1}{\sqrt{5 + 4\cos\theta}} = \frac{1}{\sqrt{10 - 6\cos\theta}}$
$\sin\theta = 0 \quad \theta = 0$ or π (end points) — (1)
$\sqrt{5 + 4\cos\theta} = \sqrt{10 - 6\cos\theta}$
$5 + 4\cos\theta = 10 - 6\cos\theta$
$10\cos\theta = 5$
$\cos\theta = \frac{1}{2}$
(1) $\therefore \theta = \frac{\pi}{3}$

Test stationary point at $\frac{\pi}{3}$
(1) (or calculate distance for $\theta = \pi/3$)

θ	$\frac{\pi}{4}$	$\pi/3$	$\pi/2$
$\frac{dD}{d\theta}$	3.56	0	−7.9

2.5~



function is continuous for $0 < \theta < \pi$, gradient changes from positive to negative and is stationary point at $\theta = \pi/3$
$\therefore$ local MAX at $\theta = \pi/3$
$\therefore$ As there is only one
(1) stationary point $0 < \theta < \pi$ 1
absolute min must be at an end point ie $\theta = \pi$ or $\theta = 0$
$\theta = 0 \quad D = 30\sqrt{5+4} + 20\sqrt{10-6}$
$= 130$ m
(1) $\theta = \pi \quad D = 30\sqrt{5-4} + 20\sqrt{10+6}$
$= 110$ m
$\therefore$ Min distance is when
$\theta = \pi$ (Alternative to testing $\frac{dD}{d\theta}$ at $\theta = \pi/3$ - calculate dist $= 132.3$ m) $\therefore$ longer not min

3 (d) (i) 2 groups of 8 from 16 people
(groups both equal)
$\frac{{}^{16}C_8 \times {}^8C_8}{2!} = \frac{12870}{2}$
$= 6435$ (1)

(ii)
Number of seating arrangements at each table is $7!$
$\therefore$ Total no. at 2 tables is
$\frac{{}^{16}C_8 \times {}^8C_8}{2!} \times (7!)^2$ (1)

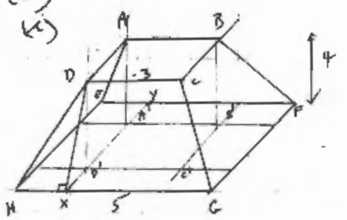
4 (a) $P(x) = (x-1)(x-2)Q(x) + ax + b$
$P(1) = 2$
$P(2) = 1$ (1)
$2 = a + b$
$1 = 2a + b$ (1)
$\therefore 1 = -a$
$\therefore a = -1$
$b = 3$
$\therefore$ Remainder is (1)
$-x + 3$

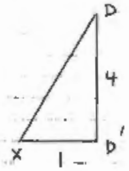
(b) $x = \cos 2t$
$-1 \le x \le 1$
$y = \cos t$
$-1 \le y \le 1$
$x = 2\cos^2 t - 1$ (1)
$x = 2y^2 - 1$
or $2y^2 = x + 1$ (1)
$-1 \le x \le 1$
or $(-1 \le y \le 1)$ (1)

4 (c) $\sin(45^\circ - 30^\circ) = \sin 15^\circ$
$= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$ (1)
$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$
$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

Area of segment
$A = \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 20^2\left(\frac{15 \times \pi}{180} - \sin 15^\circ\right)$
$= 200\left(\frac{\pi}{12} - \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)\right)$ (1)
cm^2

(d)
(i)



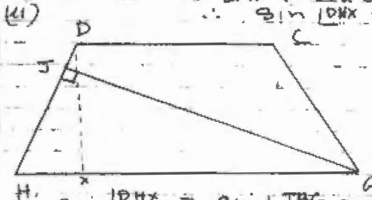


$XD' = A'Y = HX$
$= \frac{5-3}{2}$
$= 1$ cm

$\therefore XD = \sqrt{4^2 + 1^2}$
$= \sqrt{17}$ cm (1)

$HX = 1$ cm
$DH = \sqrt{17 + 1}$
$= \sqrt{18}$ cm
$\therefore DH = 3\sqrt{2}$ cm (1)
$\therefore \sin \angle DHX = \frac{\sqrt{17}}{\sqrt{18}}$

(ii)



$\sin \angle DHX = \sin \angle JHG = \frac{JG}{HG}$

Extension I 2005

1. (a) $\sqrt{3}x + y = 2 \quad m = -\sqrt{3}$ (1)

$x - y = 0 \quad m = 1$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{-\sqrt{3} - 1}{1 - (\sqrt{3})(1)}\right|$ (1)

$\theta = 75^\circ$ (1)

(b)(i) $\boxed{AEIOU}$ $\boxed{B}$ $\boxed{G}$ $\boxed{L}$ $\boxed{T}$ $\boxed{N}$

6 elements arranged 6! ways

A, E, I, O, U can be arranged 5! ways

∴ total arrangements $= 6! \times 5!$ (86400) ways (1)

(ii) 10 elements arranged 10! ways

For each arrangement of vowels only one has them in the correct order

ie 1 out of 5!

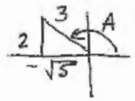
∴ no of ways with vowels in AEIOU order $= \dfrac{10!}{5!}$ (1)

$= 30240$

(c) $\sin A = \frac{2}{3} \quad 90^\circ < A < 180^\circ$

∴ 2nd quadrant angle.

∴ $\tan A < 0 \quad \tan A = \frac{-2}{\sqrt{5}}$ (1)



$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

(1) $= \dfrac{2(-\frac{2}{\sqrt{5}})}{1 - (\frac{-2}{\sqrt{5}})^2} = \dfrac{-\frac{4}{\sqrt{5}}}{1 - \frac{4}{5}}$

(1) $= -4\sqrt{5}$

(d) $8y = x^2 + 6x + 1$

∴ $x^2 + 6x = 8y - 1$

$x^2 + 6x + 9 = 8y - 1 + 9$ (1)

$(x+3)^2 = 8(y+1)$

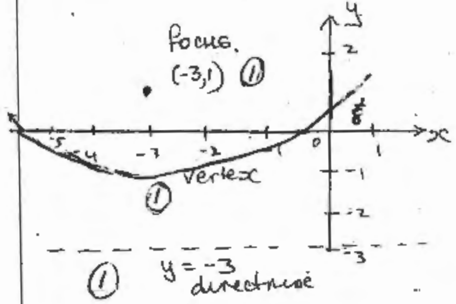
$(x+3)^2 = 4 \times 2(y+1)$ (1)

vertex $= (-3, -1)$

∴ focal length = 2 units

∴ focus $= (-3, 1)$

directrix $y = -3$



(e) $\cos\theta + \sqrt{3}\sin\theta$

$= 2\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$= R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$

$R^2 = 1^2 + (\sqrt{3})^2 = 4$

∴ $R = 2$ (as $R > 0$)

$\cos\alpha = \frac{1}{2} \quad \sin\alpha = -\frac{\sqrt{3}}{2}$.

∴ α is a 4th quad angle

∴ $\alpha = 360^\circ - 60^\circ$

$= 300^\circ$

∴ Answer is

$2\cos(\theta + 300^\circ)$

(1) (1)

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2 (a) $t = \tan(\frac{1}{2}\theta)$

∴ $\sin\theta = \dfrac{2t}{1+t^2} \quad \cos\theta = \dfrac{1-t^2}{1+t^2}$

∴ $2\sin\theta - \cos\theta = 1$ (1)

∴ $2\left(\frac{2t}{1+t^2}\right) - \frac{(1-t^2)}{(1+t^2)} = 1$

∴ $4t - 1 + t^2 = 1 + t^2$

$4t = 2$

$t = \frac{1}{2}$

$\tan\frac{1}{2}\theta = \frac{1}{2}$ (1)

∴ $\frac{1}{2}\theta = 26.565051\ldots^\circ$

∴ $\theta = 53.13010236^\circ$

$\theta = 53^\circ 8'$ (nearest minute)

Check $180^\circ = \theta$

$LHS = 2 \times \sin 180^\circ - \cos 180^\circ$

$= -(-1) = 1$

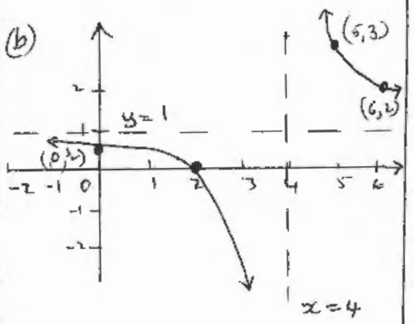
$= RHS$

∴ 180° is a solution

∴ $\theta = 53^\circ 8'$ or 180°

(1) (1)

(b)



asymptotes (1)

intercepts (1)

to solve $\dfrac{x-2}{x-4} \le 2$

consider $\dfrac{x-2}{x-4} = 2$.

$x - 2 = 2x - 8$

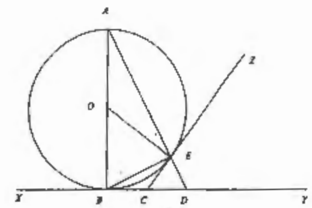
$6 = x$

From graph

$x < 4$ or $x \ge 6$ (1)

(1)

(c)



(i) To prove $\angle EOB = \angle ECD$

(1) $\angle OBC = 90^\circ$ (angle between tangent and radius at B)

$\angle OEC = 90^\circ$ (similarly at E)

(1) ∴ OECD is a cyclic quadrilateral (opposite angles of OECD sum to 180°)

(1) ∴ $\angle EOB = \angle ECD$ (exterior angle of OECD equals opposite interior angle)

(ii) To show C is centre of circle through B, E and D.

(1) $CE = CB$ (intercepts of tangents from external point C are equal)

$\angle AEB = 90^\circ$ (angle subtended by diameter AB at circumference is 90°)

$\angle CED + \angle BEC + \angle AEB = 180^\circ$ (angle sum of straight angle at E is 180°)

∴ $\angle CED + \angle BEC = 90^\circ$

$\angle BEC = \angle DAB$ (angle in alternate segment is equal to angle between tangent and chord at E)

(1) ∴ $\angle CED + \angle DAB = 90^\circ$

$\angle ABD = 90^\circ$ (angle between radius and tangent = 90° at B)

$\angle ABD + \angle DAB + \angle ADB = 180^\circ$ (angle sum of △ABD is 180°)

(1) ∴ $\angle DAB + \angle ADB = 90^\circ$

∴ $\angle CED = \angle ADB$

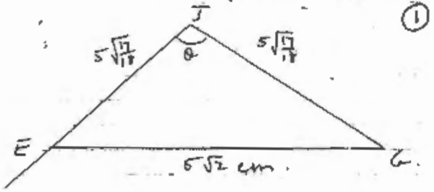
(1) ∴ $CE = CD$ (equal sides opposite equal angles in △ECD)

∴ $CE = CD = CB$

∴ C is centre of circle through B, E, D.

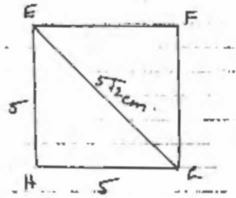
$\therefore \; JG = HG \times \sin \angle JHG$

$= 5 \times \sqrt{\frac{17}{18}}$ (can also do with similar triangle) ①



By symmetry $JE = JG$.

$EG = \sqrt{5^2 + 5^2}$
$EG = 5\sqrt{2}$ cm



$$\cos \angle GJE = \frac{\left(5\sqrt{\frac{17}{18}}\right)^2 + \left(5\sqrt{\frac{17}{18}}\right)^2 - (5\sqrt{2})^2}{2 \times 5\sqrt{\frac{17}{18}} \times 5\sqrt{\frac{17}{18}}} \quad ①$$

$$= \frac{25 \times \frac{17}{18} + 25 \times \frac{17}{18} - 25 \times 2}{2 \times 25 \times \frac{17}{18}}$$

$$= \frac{\cancel{25}\left(\frac{17}{18} + \frac{17}{18} - 2\right)}{\cancel{2 \times 25} \times \frac{17}{18}}$$

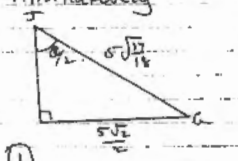
$$= \frac{\frac{17}{9} - 2}{\frac{17}{9}}$$

$$= \frac{17 - 18}{17}$$

$$= \frac{-1}{17}$$

$\therefore \angle GJE = 93°22.$ ①

Alternatively



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