Question 1. **Marks**

(a) Sketch $y = 2 - e^{-x}$, showing all essential detail. **2**

(b) Evaluate $\lim_{x \to 0}\left(\dfrac{3x}{\sin 5x}\right)$. **2**

(c) Find the acute angle formed between the two lines: $y = 2$ and $y = x\sqrt{3} - 2$. **2**

(d) Solve $2\cos\theta + \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$. **2**

(e) Express $2\sin x + \cos x$ in the form **2**

$$R\sin(x+\alpha), \text{ where } R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}.$$

(f) Find the number of ways in which the letters of the word **EPSILON** can be

(i) arranged in a circle. **1**

(ii) arranged in a row so that the three vowels are all next to each other. **2**

(g) (i) Show that $x - 2$ is a factor of the polynomial expression: $x^3 - 3x^2 + 4$. **1**

(ii) Hence express $x^3 - 3x^2 + 4$ as a product of three linear factors. **1**

Question 2. **[START A NEW PAGE]**

(a) Show that $\tan 2x + \tan x \equiv \dfrac{\sin 3x}{\cos 2x \cos x}$. **2**

(b) Find $\dfrac{dy}{dx}$ in the following:

(i) $y = \sec(x^3)$. **1**

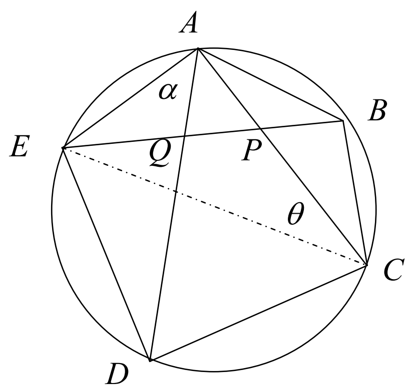
(ii) $y = 10x - x\ln x$. **2**

(c) Given that $\cos 75^0 = \dfrac{\sqrt{6} - \sqrt{2}}{4}$ and using the double angle results, **2**

or otherwise, find the exact value of $\cos\left(37\frac{1}{2}^0\right)$.

Question **2** continued over the page

(c) $ABCDE$ is a pentagon inscribed in a circle, where $AB = AE$, BE meets AC and AD at P and Q respectively.



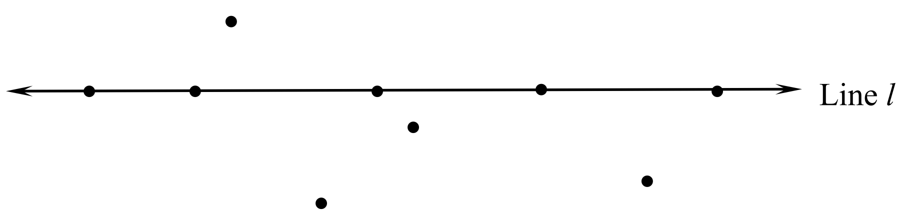
Not to scale

Given $\angle ACE = \theta$ and $\angle DAE = \alpha$,

(i) Copy the diagram onto your writing paper and show that $\angle AEB = \theta$, give reasons. **2**

(ii) Hence, or otherwise show that $CPQD$ is a cyclic quadrilateral. **3**

(d) The diagram shows 9 points lying in the plane, 5 of which lie on the line l. The remaining 4 points do not lie on line l and no other set of 3 points is collinear.

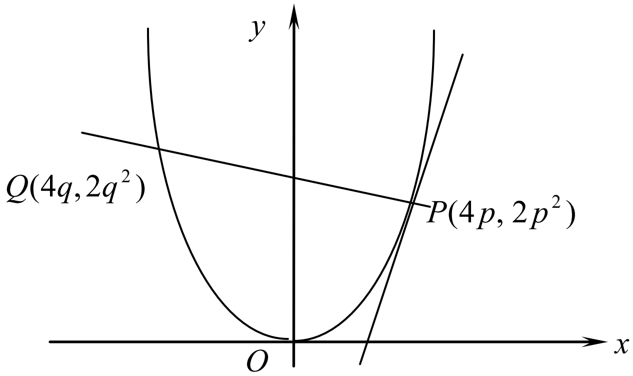


(i) How many sets of 3 points can be chosen from the 5 points lying on l? **1**

(ii) How many distinct triangles can be formed using any three of the 9 points as vertices? **2**

Question 3. **[START A NEW PAGE]** **Marks**

(a) Point $P(4p, 2p^2)$ lies on the parabola shown. The normal at P intersects the parabola at the point $Q(4q, 2q^2)$.



Not to scale

(i) Show that the Cartesian equation for this parabola is $x^2 = 8y$. **1**

(ii) Show that the equation of the normal at P is given by: **3**

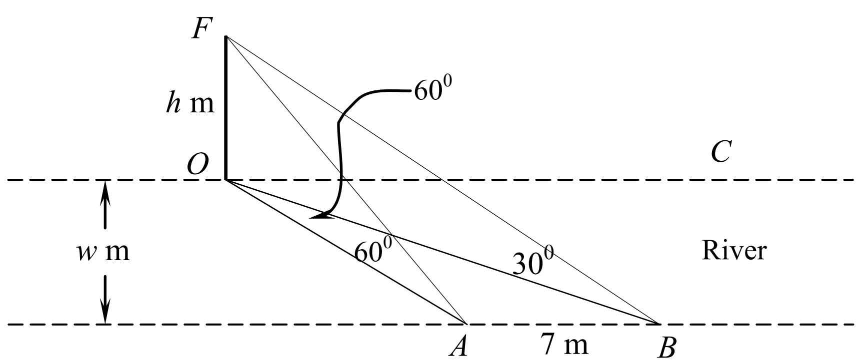
$$x + py = 2p^3 + 4p.$$

(iii) Hence, or otherwise, show that $p + q = -\frac{2}{p}$, where $p \neq 0$. **2**

(iv) Find the coordinates of the midpoint M of chord PQ. **1**

(v) Hence find the locus of M as points P and Q move on the parabola. **3**

(b) A river has level parallel riverbanks OC and AB of width w metres. OF is a vertical flagpole of height h metres which stands with its base O on the edge of riverbank OC. Positions A and B are two points on the other riverbank such that $AB = 7$ metres and $\angle AOB = 60^0$.
The angle of elevation to the top of the flagpole from A and B are 60^0 and 30^0 respectively, as shown below.



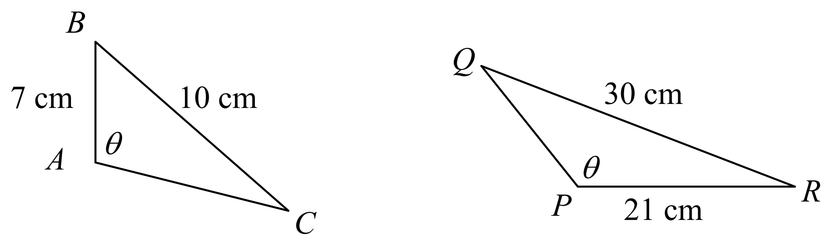
(i) Show that the height of the flagpole is $\sqrt{21}$ metres. **3**

(ii) By finding the area of $\Delta\, AOB$, or otherwise, find the width of the river. **2**

Question 4. **[START A NEW PAGE]** **Marks**

(a) For what value(s) of k is the line $y = 12x + k$ a tangent to $y = x^3$. **3**

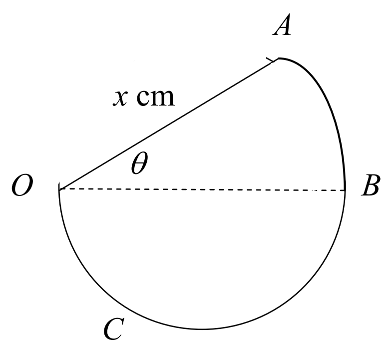
(b) Given the triangles ABC and PQR where $\angle BAC = \angle RPQ = \theta$ and $\theta \geq 90^0$.



Not to scale

Copy the diagrams onto your writing paper and using the Sine rule, prove that $\Delta ABC \,|||\, \Delta PRQ$. **3**

(c) A cam for a motor is made with a cross-section as shown below.



Not to scale

The cam's cross-section consists of a semi-circle OCB and the sector of a circle OAB of centre O with a radius of x cm and with an included angle $BOA = \theta$.

(i) Determine the formula for the perimeter P cm of the cam $OABC$, in terms of x and θ. **2**

(ii) Given that the area of the cross-section of the cam is 1 cm^2, show that **2**

$$x^2 = \frac{8}{\pi + 4\theta}.$$

(iii) Hence, or otherwise, show that the perimeter P is given by: **1**

$$P = \frac{(\pi + 2 + 2\theta)\sqrt{2}}{\sqrt{\pi + 4\theta}}.$$

(iv) Determine the value of θ for least perimeter of the cam, (answer correct to the nearest minute). **4**

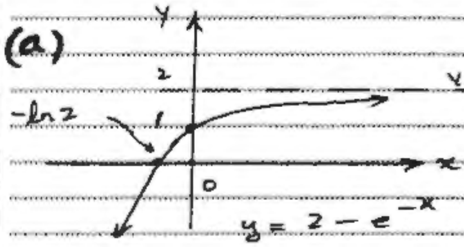
THE END ☺ 😐 ☹

YEAR 11 MATH EXT 1, T3, 2007
PRELIMINARY SOLUTIONS

MATHEMATICS Extension 1 : Question 1

Suggested Solutions	Marks	Marker's Comments

(a)



$\boxed{2}$

Marks: $\frac{1}{2}$ H.A.; $\frac{1}{2}$ y-int; $\frac{1}{2}$ x-int; $\frac{1}{2}$ shape

(b) $\lim \dfrac{3x}{\sin 5x} = \lim \dfrac{3x(5x)}{5x\sin(5x)}$ $\boxed{2}$

$= \dfrac{3}{5}\lim \dfrac{(5x)}{\sin(5x)}$

$= \dfrac{3}{5} \times 1 = \dfrac{3}{5}$

Marker's Comments: Awl For recognising; Awl For showing 1 and then $\frac{3}{5}$

(c) METHOD 1 $\quad y = 2 \quad m_1 = 0$

$y = x\sqrt{3} - 2 \quad m_2 = \sqrt{3}$ $\boxed{2}$

$\tan\theta = \left|\dfrac{0-\sqrt{3}}{1+0\times\sqrt{3}}\right| = \dfrac{\sqrt{3}}{1}$

$\therefore \angle\theta = 60^\circ$

Marks: $\frac{1}{2}$, $\frac{1}{2}$, 1

Marker's Comments: Awl For getting to $\tan\theta = \sqrt{3}$; Awl For 60°

(d) $2\cos\theta = -\sqrt{3}$ $\boxed{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\angle\theta = \pi - \frac{\pi}{6}$ or $\pi + \frac{\pi}{6}$ ✓

$\therefore \angle\theta = \underline{\frac{5\pi}{6}}$ or $\underline{\frac{7\pi}{6}}$

Marks: 1, 1

(e) $R\sin(x+\alpha) \equiv 2\sin x + \cos x$

$R\sin x\cos\alpha + R\cos x\sin\alpha = 2\sin x + \cos x$

$\therefore R\cos\alpha = 2$ --- (1) $\qquad \cos\alpha = \frac{2}{\sqrt{5}}$

$R\sin\alpha = 1$ --- (2) $\qquad \sin\alpha = \frac{1}{\sqrt{5}}$

$\therefore R = \sqrt{2^2+1^2} = \underline{\sqrt{5}}$ $\qquad \therefore \alpha = \cos^{-1}\frac{2}{\sqrt{5}} / \sin^{-1}\frac{1}{\sqrt{5}} / \tan^{-1}\frac{1}{2} = 0.4636\ldots$

$\therefore \sqrt{5}\sin(x+\alpha)$, where $\alpha = \ldots\ldots$

Marker's Comments: Awl For $R = \sqrt{5}$; Awl For $\alpha = -$

(f) E I O P S L N $\quad n = 7$ $\boxed{1}$

(i) Nº of ways $= \dfrac{7!}{7} = 6! = \underline{720}$

(ii) | E | I | O | · | · | · | · | $\boxed{2}$ Nº of ways $= (4\times 3!)\times 4! = \underline{576}$ (Vowels)

Marker's Comments: Awl For 6! / 720; Awl For $4\times 3!$; Awl For 4! and ...

(g)(i) $P(x) = x^3 - 3x^2 + 4$ $\boxed{1}$

$P(2) = 8 - 3\times 4 + 4 = 0$ $\therefore$ by Factor thm $(x-2)$ is a factor ✓

(ii) $x^3 - 3x^2 + 4 = (x-2)(x^2-x-2)$ $\boxed{1}$

$= (x-2)(x-2)(x+1)$

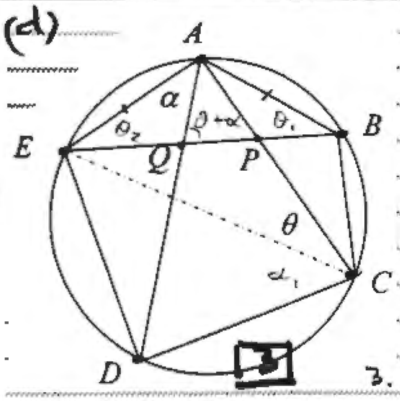
Marker's Comments: Awl For getting the answer

Using, eg LONG DIVISION,

$$
\begin{array}{r}
x^2 - x - 2 \\
x-2\;\big)\; x^3 - 3x^2 \qquad + 4 \\
x^3 - 2x^2 \qquad\quad \\
\hline
0 \;\; - x^2 \qquad\quad \\
-x^2 + 2x \quad \\
\hline
0 \;\; - 2x + 4 \\
-2x + 4 \\
\hline
0 \quad 0
\end{array}
$$

YEAR 11 MATH EXT 1, T3 2007
PRELIMINARY SOLUTIONS

MATHEMATICS Extension 1 : Question 2

Suggested Solutions	Marks	Marker's Comments
(a) LHS $\tan 2x + \tan x = \dfrac{\sin 2x}{\cos 2x} + \dfrac{\sin x}{\cos x}$ $= \dfrac{\sin 2x \cos x + \cos 2x \sin x}{\cos 2x \cos x}$ $\boxed{2}$ $= \dfrac{\sin(2x + x)}{\cos 2x \cos x} = \dfrac{\sin 3x}{\cos 2x \cos x}$	$\frac{1}{2} + \frac{1}{2}$ 1	
(b) using $\cos 2\theta = 2\cos^2\theta - 1$ or $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ $\cos^2 37\frac{1}{2}^\circ = \frac{1}{2}\left(1 + \dfrac{\sqrt6 - \sqrt2}{4}\right)$ $\boxed{2}$ $= \frac{1}{2}\left(\dfrac{4 + \sqrt6 - \sqrt2}{4}\right)$ $\cos 37\frac{1}{2}^\circ = \pm\dfrac{\sqrt{4 + \sqrt6 - \sqrt2}}{2\sqrt2}$ as $\cos 37\frac{1}{2}^\circ > 0$ $\cos 37\frac{1}{2}^\circ = \dfrac{\sqrt{4 + \sqrt6 - \sqrt2}}{2\sqrt2}$	1 1	
(c) (i) $\dfrac{dy}{dx} = \sec(x^3)\tan(x^3) \cdot 3x^2$ ✓ $\boxed{1}$ (ii) $\dfrac{dy}{dx} = 10 - \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$ ✓ $\boxed{2}$ $= 9 - \ln x$ ✓	1 1 1	
(d)  (i) 1. $\angle ABE = \theta$ (Angles in the same segment are equal when standing on same arc AE) ✓ 2. $\therefore \angle AEB = \theta$ (Equal angles opposite equal sides AE, AB) $\boxed{2}$ (ii) 1. $\angle ECD = \alpha$ (Angles in same segment are equal when standing on same arc DE) ✓ 2. $\angle AQB = \theta + \alpha$ (Exterior angle of ΔAQE equals sum of the two opposite interior angles) ✓ $\boxed{3}$ 3. As $\angle PCD = \angle AQB = \theta + \alpha$ $\therefore PCDQ$ is a cyclic quad. as the exterior angle at Q equals the interior opposite angle C.	1 1 1 1	
(e) (i) Nº of ways $= {}^5C_3 = \underline{10}$ ✓ $\boxed{1}$ (ii) Nº of Δs $= {}^9C_3 - {}^5C_3 = {}^9C_3 - 10$ $= 84 - 10$ $\boxed{2}$ $= \underline{74}$	1	Aw! Fn ${}^9C_3$ Awl Fn − (i)

MATHEMATICS Extension 1 : Question 3

Suggested Solutions	Marks	Marker's Comments

(a) (i) $x = 4p \Rightarrow p = \frac{x}{4}$ $\therefore y = 2 \times \frac{x^2}{16}$

$y = 2p^2$

i.e $8y = x^2$

$\boxed{1}$

Marker's comments: $(4p)^2 = 8\times(2p^2)$ Awl For showing

(ii) $y = \frac{x^2}{8}$ $\boxed{3}$

$y' = \frac{x}{4}$

Gradient of Tangent at P $m_T = \frac{4p}{4} = p$ ✓

" " Normal at P $m_N = -\frac{1}{p}$ ✓

$\therefore$ Equation of Normal at P:

$y - 2p^2 = -\frac{1}{p}(x - 4p)$

$py - 2p^3 = -x + 4p$ ✓

$\therefore x + py = 2p^3 + 4p$ qed.

Marker's comments: Awl For getting to m_T at; Awl For $m_N = -\frac{1}{p}$; Awl For getting to result.

(iii) $m_{PQ} = -\frac{1}{p}, p \neq 0$ $\boxed{2}$

$\frac{2p^2 - 2q^2}{4p - 4q} = -\frac{1}{p}$

$\frac{2(p-q)(p+q)}{4(p-q)} = -\frac{1}{p}$

$\frac{p+q}{2} = -\frac{1}{p}, q \neq p$

$\therefore p + q = -\frac{2}{p}$; $p \neq 0$

P + Q satisfies (ii)

$4q + p \cdot 2q^2 = 2p^3 + 4p$

$2q + pq^2 = p^3 + 2p$

$pq^2 - p^3 = 2p - 2q$

$p(q^2 - p^2) = 2(p - q)$

$p(q-p)(q+p) = 2(p-q)$

$\therefore q + p = -\frac{2}{p}$, $q \neq p \neq 0$

Marker's comments: Awl For using pt Q in m_{PQ} or (ii); Awl For getting to result via factorisation

(iv) $M = \left(\frac{4p+4q}{2}, \frac{2p^2+2q^2}{2}\right) = \left(2(p+q), p^2+q^2\right)$ 1

$\boxed{1}$

(v) Let $M = (x, y)$ be the general point on the required locus $\boxed{3}$

so $x \equiv 2(p+q)$ --- (1)

$y \equiv p^2 + q^2$ --- (2)

$p + q = -\frac{2}{p}$ --- (3)

$p + q = \frac{x}{2} = -\frac{2}{p}$ so $p = -\frac{4}{x}$ ✓ 1

Now $p^2 + q^2 \equiv (p+q)^2 - 2pq$ ✓ 1

$y = \left(\frac{x}{2}\right)^2 - 2pq$

but (3) $p^2 + pq = -2$

$pq = -2 - p^2$

$= -2 - \left(-\frac{4}{x}\right)^2$

$= -2 - \frac{16}{x^2}$

$\therefore y = \frac{x^2}{4} - 2\left(-2 - \frac{16}{x^2}\right)$

$y = \frac{x^2}{4} + \frac{32}{x^2} + 4$, $x \neq 0$. ✓ 1

this is the locus of M

Marker's comments: OR

$y = p^2 + q^2$

but $q = -\frac{2}{p} - p$

and $p = -\frac{4}{x}$

$q = \frac{x}{2} + \frac{4}{x}$

$\therefore y = \left(-\frac{4}{x}\right)^2 + \left(\frac{x}{2} + \frac{4}{x}\right)^2$

$= \frac{x^2}{4} + \frac{32}{x^2} + 8$

Awl For getting $p = -\frac{4}{x}$

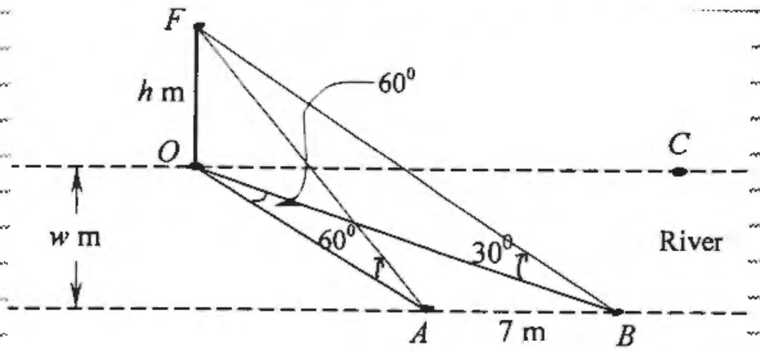
Awl For using $p^2 + q^2$ ...

Awl For getting locus.

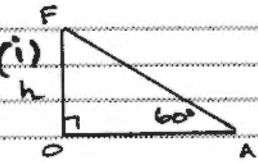
MATHEMATICS Extension 1 : Question 3

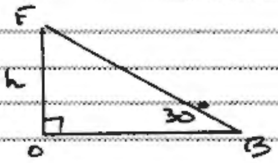
Suggested Solutions	Marks	Marker's Comments

(b)



(i) $\boxed{3}$



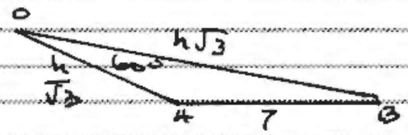


$\tan 60^\circ = \frac{h}{OA}$ $\qquad$ $\tan 30^\circ = \frac{h}{OB}$

$OA = \frac{h}{\tan 60^\circ} = h\tan 30^\circ$ $\qquad$ $OB = \frac{h}{\tan 30^\circ} = h\tan 60^\circ$

$OA = \frac{h}{\sqrt{3}}$ $\qquad$ $OB = h\sqrt{3}$ — **1** — Awl For getting $OA = \frac{h}{\sqrt{3}}$ and $OB = h\sqrt{3}$

In ΔAOB:



$7^2 = 3h^2 + \frac{h^2}{3} - 2\times h\sqrt{3}\times\frac{h}{\sqrt{3}}\cos 60^\circ$ — **1** — Awl For using cosine rule correctly in ΔAOB

$7^2 = h^2\left[3 + \frac{1}{3} - 2\times 1\times\frac{1}{2}\right]$

$= \frac{7}{3}h^2$

$h^2 = \frac{3\times 7^2}{7} = 21 \Rightarrow h = \sqrt{21}$ — Awl For getting to $\sqrt{21}$ correctly

$\therefore$ height is $\sqrt{21}$ m

(ii) Area $\Delta AOB = \frac{1}{2}\times h\sqrt{3}\times\frac{h}{\sqrt{3}}\times\sin 60^\circ = \frac{1}{2}\times 7\times w$

ie $\frac{1}{2}\times h^2\times\frac{\sqrt{3}}{2} = \frac{1}{2}\times 7\times w$ — **1** — Awl For using areas correctly

$21\times\frac{\sqrt{3}}{2} = 7w$

$\boxed{2}$

$\therefore w = \frac{3\sqrt{3}}{2}$

$\therefore$ width is $\frac{3\sqrt{3}}{2}$ m. — **1** — Awl For getting $w = \frac{3\sqrt{3}}{2}$

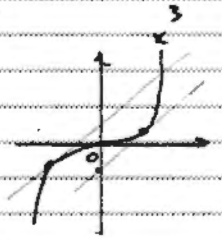
OTHERWISE:

MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments

(a) $y = x^3$, $y = 12x + k$ $\boxed{3}$

$\frac{dy}{dx} = 3x^2$ $\therefore m = 12$



so $3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

$\therefore y = (\pm 2)^3 = \pm 8$

$\Rightarrow \pm 8 = 12(\pm 2) + k = \pm 24 + k$

$\therefore \underline{k = \mp 16}$

Marker's Comments:

or: $x^3 - 12x - k = 0$

α, α and β

$2\alpha + \beta = 0$ ---- (1)

$\alpha^2 + 2\alpha\beta = -12$ ---- (2)

$\alpha^2\beta = k$ ---- (3)

(1) $\Rightarrow$ $4\alpha^2 + 2\alpha\beta = 0$

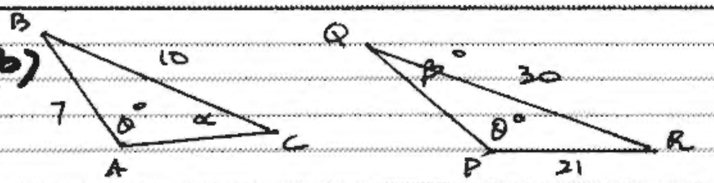
(2) $\alpha^2 + 2\alpha\beta = -12$

$-3\alpha^2 = 12$

$\alpha = \pm 2 \rightarrow \beta = \mp 4$

$k = (\pm 2)^2(\mp 4) = \underline{\mp 16}$

(b) $\boxed{3}$



In Δs ABC and PRQ

$\frac{\sin\alpha}{7} = \frac{\sin\theta}{10} = \frac{\sin B}{AC}$

and $\frac{\sin\beta}{21} = \frac{\sin\theta}{30} = \frac{\sin R}{PQ}$

$\therefore \sin\alpha = \frac{7}{10}\sin\theta \equiv \frac{21}{30}\sin\theta = \sin\beta$ — 1

so $\alpha = \beta$ or $\alpha = 180 - \beta$

but as $\theta \geq 90$ $\therefore \alpha = \beta$ only — 1

1. $\frac{AB}{PR} = \frac{7}{21} = \frac{1}{3}$ and $\frac{BC}{QR} = \frac{10}{30} = \frac{1}{3}$

2. $\angle C = \angle Q = \alpha$ — 1

$\therefore \Delta ABC \parallel\!\mid \Delta PRQ$ (sides about equal angles are in same ratio)

Marker's Comments:

NB: once shown $\alpha = \beta$ only, then

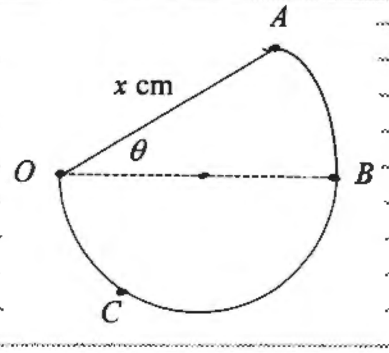
1. $\angle A = \angle P = \theta°$
2. $\angle C = \angle Q = \alpha$

$\therefore \Delta ABC \parallel\!\mid \Delta PRQ$ (equiangular)

Aw2 for showing $\alpha = \beta$ only

Aw1 for proving Δs similar.

(c)



$r = \frac{x}{2}$ for circle

(i) $\boxed{2}$

$P = OA + \text{arc}AB + BCO(\text{arc})$

$= x + x\theta + \frac{1}{2} \times 2 \times \pi \times \frac{x}{2}$

$P = \underline{x + x\theta + \frac{\pi}{2}x}$

(ii) $\boxed{2}$

Area $= \frac{1}{2} \times x^2\theta + \frac{1}{2} \times \pi \times \left(\frac{x}{2}\right)^2 = 1$

i.e. $\frac{1}{2}x^2\theta + \frac{1}{8}\pi x^2 = 1$

$x^2(4\theta + \pi) = 8$

$\therefore x^2 = \frac{8}{\pi + 4\theta}$ and

MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments

(iii) $x = \dfrac{\sqrt{8}}{\sqrt{\pi + 4\theta}} = \dfrac{2\sqrt{2}}{\sqrt{\pi+4\theta}}$ from (ii) — $\frac{1}{2}$

$\therefore P = x + x\theta + \frac{\pi}{2}x$ $\boxed{1}$

$= \frac{x}{2}[2 + 2\theta + \pi]$

$= \frac{1}{2} \times \dfrac{2\sqrt{2}}{\sqrt{\pi+4\theta}}[\pi + 2 + 2\theta]$ — $\frac{1}{2}$

$\therefore P = \dfrac{[\pi+2+2\theta]\sqrt{2}}{\sqrt{\pi+4\theta}}$ q.e.d.

(iv) $\dfrac{dP}{d\theta} = \sqrt{2}\left\{\dfrac{2\times\sqrt{\pi+4\theta} - (\pi+2+2\theta)\frac{1}{2}\times(\pi+4\theta)^{-\frac{1}{2}}\times 4}{(\pi+4\theta)}\right\}$

$= \sqrt{2}\left\{\dfrac{2\sqrt{\pi+4\theta} - \dfrac{2(\pi+2+2\theta)}{\sqrt{\pi+4\theta}}}{(\pi+4\theta)}\right\}$

$\boxed{4}$

$= \dfrac{\sqrt{2}}{(\pi+4\theta)^{3/2}}\{2(\pi+4\theta) - 2(\pi+2+2\theta)\}$ — Aw 2 for using Quotient rule correctly

$= \sqrt{2}\left\{\dfrac{2\pi+8\theta-2\pi-4-4\theta}{(\pi+4\theta)^{3/2}}\right\}$

$\dfrac{dP}{d\theta} = \dfrac{[4\theta-4]\sqrt{2}}{(\pi+4\theta)^{3/2}} = \dfrac{4(\theta-1)\sqrt{2}}{(\pi+4\theta)^{3/2}}$ — 2

For possible max/min values of P to occur $\frac{dP}{d\theta} = 0$

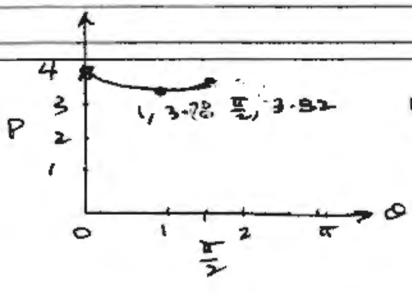
$\therefore \theta = 1$ $P = \sqrt{2}\times\sqrt{\pi+4} = \ldots$ — $\frac{1}{2}$ — Aw $\frac{1}{2}$ for θ

TEST

θ	$\frac{1}{2}$	1	$1\frac{1}{2}$
$\frac{dP}{d\theta}$	$\frac{4(-)\sqrt{2}}{(+)}$ -1.25 \	0 —	$\frac{4(+)\sqrt{2}}{(+)}$ 0.94 /

— 1 — Aw 1 for testing θ properly

$\therefore$ a relative min T.P at $\theta = 1$
and since P continuous for $0 \le \theta \le \frac{\pi}{2}$
$\therefore$ Least perimeter when $\theta = 1$
i.e $\angle\theta = 57°18'$ — $\frac{1}{2}$ — Aw 1 for answer



basic sketch.

$\theta = 0$ $P = \dfrac{(\pi+2)\sqrt{2}}{\sqrt{\pi}} \doteqdot 4.10$

$\theta = 1$ $P = \sqrt{2\pi+8} \doteqdot 3.78$

$\theta = \frac{\pi}{2}$ $P = \dfrac{(2+2\pi)\sqrt{2}}{\sqrt{3\pi}} \doteqdot 3.82$