KINCOPPAL ROSE-BAY
SCHOOL OF THE SACRED HEART

Year 11
Mathematics (Extension 1)
Preliminary Examination
Thursday 6 September 2001

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Answer questions 1 and 2 in book 1.
- Answer questions 3 and 4 in book 2.
- Answer questions 5, 6 and 7 in book 3.
- Apporoved calculators may be used.
- Write using blue or black pen.

- All questions are of equal value.

Question 1

(a) A is the point (1, 4) and C is the point (7, -2). Draw a reasonably accurate number plane that reflects this information. Join AC. **1**

(i) The point B divides AC internally in the ratio 1:2. Find the coordinates of B. **3**

(ii) Suppose D is a point in the first quadrant such its perpendicular distance from AC is 6 units. Without calculating any areas explain why the area of $\Delta ABD = \frac{1}{2} \times$ Area ΔBCD. You may justify your answer using your diagram. **2**

(b) Solve $\frac{2x-1}{x+3} \leq 3$. Graph your solution on a number line **3**

(c) Draw $y = |x|$ and $y = x + 1$ on the same number plane. Hence or otherwise solve $|x| \leq x + 1$ **3**

Question 2

(a) Suppose that there are 30 students studying Extension 1 Mathematics. In how many ways can an Extension 2 class of 10 students be formed? **1**

(b) In the women's marathon at the Sydney Olympics there were 80 competitors. In how many ways can the gold, silver and bronze medals be awarded? (You may assume medals are not shared.) **1**

(c) In how many ways can the letters of the word EQUATIONS be arranged if no two vowels are to be next to each other. (The vowels are A, E, I, O and U.) **2**

(d) A committee of 6 girls is to be chosen from 10 girls. Two of the girls are fighting and refuse to be on the same committee together. In how many ways can the committee be chosen? **2**

(e) A tangent is drawn to $x^2 = 4ay$ at the point $P(2ap, ap^2)$.

(i) Prove that the equation of the tangent is given by $y = px - ap^2$. **2**

(ii) Find the equation of the normal at P. **2**

(iii) The normal at P intersects the $y-$axis at Q. The tangent at P intersects the $y-$axis at R. Find the area of the triangle PQR. **2**

Question 3 start a new booklet

(a) Find the derivative of the following:

(i) $y=\left(4x^3-1\right)^5$ **2**

(ii) $f(x)=\dfrac{x}{2x+1}$ **2**

(b) (i) Show that if $y=(2x+1)^3\left(x^3-2\right)^4$ then $\dfrac{dy}{dx}=6(2x+1)^2\left(x^3-2\right)^3\left(5x^3+2x^2-2\right)$ **3**

(ii) Hence find the equation of the normal to $y=(2x+1)^3\left(x^3-2\right)^4$ at the point where $x=0$ **2**

(c) Use the method of first principles to find $f'(x)$ if $f(x)=x-3x^2$ **3**

Question 4

(a) If you were looking at a function drawn on a number plane where would you find the 'roots'? **1**

(b) (i) Explain in words what it means to say a quadratic is 'positive definite'. **1**

(ii) Prove that $y=x^2-4x+10$ is positive definite. **3**

(iii) If the roots of $x^2-4x+10=0$ are α and β, evaluate $\alpha+\beta$ **1**

(iv) Draw $y=x^2-4x+10$ on a number plane showing its axis of symmetry and vertex. **2**

(v) Angel notices that the parabola you just drew has no roots. Buffy wonders how you can have an answer to $\alpha+\beta$ if this is so. How would you explain it to her? **1**

(c) If $x^2\equiv A(x-1)^2+B(x-1)+C$, find A, B and C. **3**

Question 5

start a new booklet

(a) Draw an interval AB that is 5cm long. As accurately as possible draw the locus of all points which are 1cm from AB. **2**

(b) A point P moves so that it is equidistant from $A(1,-1)$ and $B(3,5)$. Find the equation of the locus and give a geometric description of it. (A diagram will prove to be useful). **4**

(c) (i) C is the point (1,0) and D is the point (-1,0). The locus of all points P, such that the gradients have a product of -1, ($m(CP)\times m(DP)=-1$) is given by $x^2+y^2=1$.

Without doing any working we should have expected this locus would be a circle with CD as the diameter. Why is this? (Use a diagram as part of your explanation.) **1**

(ii) When the locus is drawn there are two points which should be excluded from the circle. What are they and why? **2**

(d) Solve $4^x-65\times 2^x+64=0$ **3**

Question 6

(a) If $\tan\alpha=\dfrac{-3}{4}$ and $\cos\alpha>0$, find the exact value of $\sin 2\alpha$. **3**

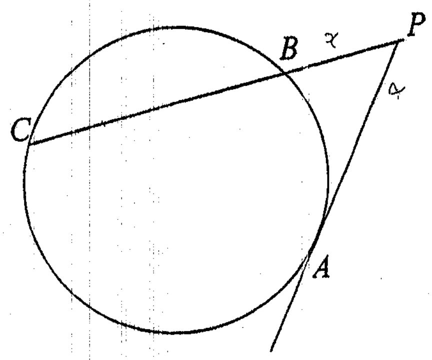
(b) Solve $2\cos\dfrac{x}{2}-1=0$ for $0\le x\le 2\pi$. **2**

(c) (i) Expand $A\sin(x-\alpha)$ **1**

(ii) By putting $2\sin x-\cos x$ in the form $A\sin(x-\alpha)$, solve $2\sin x-\cos x=1$ $(0\le x\le 360°)$. **3**

(d) Samantha stands due east of Centrepoint Tower and observes the angle of elevation of the top of the tower to be 70°. Miranda stands due south of the tower and observes the angle of elevation to be 60°. If Miranda and Samantha are 400m apart, find the height of the tower to the nearest metre. **3**

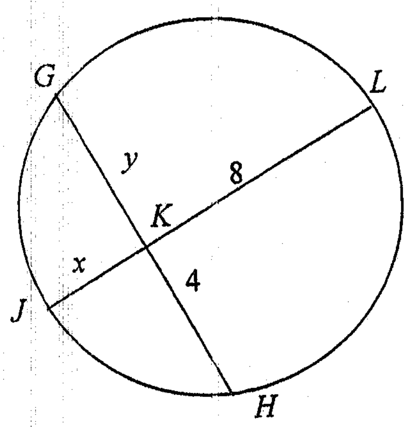
Question 7



(a) PA is a tangent. Use similar triangles to prove that $(PA)^2 = PB \times PC$. **4**

(b) If in the diagram above $PA = y$, $PB = x$ and $BC = 6$ write an equation that relates y and x **2**

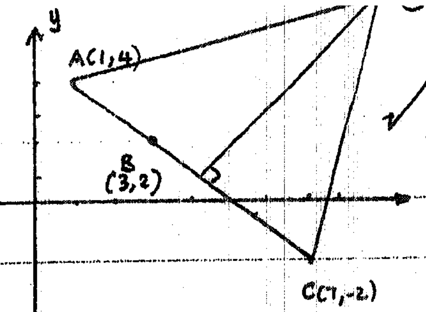
(c)



In the diagram above GH and JL are chords that intersect at K. Write a relationship between x and y. **2**

(d) Use your answers in (b) and (c) to solve the equations simultaneously and find possible values of the lengths of x and y. **4**

QUESTION ONE



(i) $\overset{(x_1,y_1)}{A(1,4)}$ $\overset{(x_2,y_2)}{B(7,-2)}$ $\overset{k:l}{1:2}$

$\therefore B = \left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$

$B = \left(\frac{7+2}{1+2}, \frac{-2+8}{1+2}\right)$

$B = (3, 2)$

(ii) Since B divides AC in the ratio 1:2

then $AB = \frac{1}{2}BC$

Area $\triangle ABD = \frac{1}{2} AB \times h$

$= \frac{1}{2}\left(\frac{1}{2}CB\right)h$

$= \frac{1}{2}\left[\frac{1}{2} \times CB \times h\right]$

$= \frac{1}{2}$ area $\triangle BDC$.

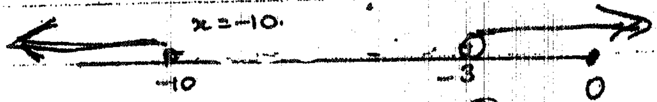
(b) $\frac{2x-1}{x+3} \leq 3$ $\quad x \neq 3$

let $\frac{2x-1}{x+3} = 3$

$2x-1 = 3x+9$

$-x = 10$

$x = -10$



test: $x = -11$ $\quad \frac{-23}{-8} \leq 3$

$x = -4$ $\quad \frac{-9}{-1} \leq 3$ ×

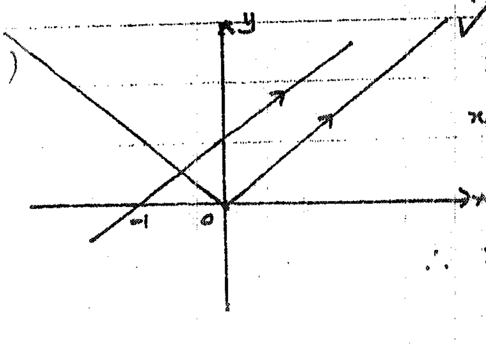
$x = 0$ $\quad -\frac{1}{3} \leq 3$ $\quad x > -3$

$x \leq -10$

$\therefore$

$x \leq -10, x > 3$.

(c) (diagram)



$y = -x \quad y = x+1$

$x + 1 = -x$

$2x = -1$

$x = -\frac{1}{2}$

$\therefore x \geq -\frac{1}{2}$

(a) No. of ways $= {}^{30}C_{10} = 30045015$

(b) No of ways $= \underline{80}\ \underline{79}\ \underline{78}$ or ${}^{80}P_3 = 492960$

(c) Consonants and vowels must alternate

$\therefore$ No of arrangements $= \underline{5! \times 4!}$

$= 2880$.

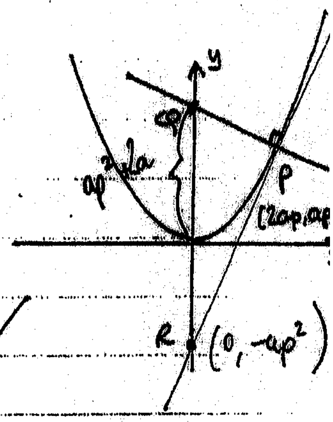
(d) Total no. of committees (no restriction) $= {}^{10}C_6 = 210$

Total no of committees (both girls in) $= {}^8C_4 = 70$

$\therefore$ No. of committees (without both) $= 210 - 70$

$= 140$

(e) (i) $x^2 = 4ay$



$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$ $\quad (2ap, ap^2)$

$m = \frac{2ap}{2a}$

$m = p$ $\quad (2ap, ap^2)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$.

$a + ap^2 + ap^2$

$= a + 2ap^2$

(ii) m (normal) $= -\frac{1}{p}$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$.

(iii) at Q normal has x-value of 0

$\therefore py = ap^3 + ap$

$y = ap^2 + a$.

at R tangent has x value of 0 — for either

$\therefore y = -ap^2$

Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times \cancel{(ap^3 + ap + ap^2)} \times 2ap$

$= \cancel{ap(ap^3 + ap + ap^2)}$

$= \cancel{a^2p^2(p^2 + 1 + p)}$.

$\frac{1}{\cancel{2}}(2a + 2ap^2)\, \cancel{2}ap$

$2a^2p + 2a^2p^3$

$2a^2p(1 + p^2)$

QUESTION 3

a) (i) $y = (4x^3 - 1)^5$

$\frac{dy}{dx} = 5(4x^3-1)^4 \times 12x^2$ ✓

$= 60x^2(4x^3-1)^4$ ✓ 2

(ii) $f(x) = \frac{x}{2x+1}$ ✓ $\left\{ \begin{array}{ll} u = x & v = 2x+1 \\ u' = 1 & v' = 2 \end{array} \right.$

$f'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{(2x+1)1 - 2x}{(2x+1)^2}$ 2

$= \frac{1}{(2x+1)^2}$ ✓

) (ii) $y = (2x+1)^3(x^3-2)^4$ $\quad u = (2x+1)^3 \quad v = (x^3-2)^4$ ✓ $\quad u' = 3(2x+1)^2 \cdot 2x \quad v' = 4(x^3-2)^3 \, 3x^2$

$y' = (x^3-2)^4 \, 6x(2x+1)^2 + (2x+1)^3 \, 12x^2(x^3-2)^3$ ✓

$= 6x(2x+1)^2(x^3-2)^3[x^3-2+2x^2(2x+1)]$ ✓

$= 6(2x+1)^2(x^3-2)^3(5x^3+2x^2-2)$

) at $x=0$ $\quad y = 1^3 \cdot (-2)^4 = 16$

at $x=0$ $\quad y' = 6(1)^2(-2)^3(-2)$

$m(\text{tangent}) = 96$

$\therefore m(\text{normal}) = \frac{-1}{96}$ ✓

$y - 16 = -\frac{1}{96}(x-0)$

$96y - 1536 = -x$

$\therefore x + 96y - 1536 = 0$ ✓

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ ✓

$= \lim_{h \to 0} \frac{x+h-3(x+h)^2 - (x-3x^2)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x}+h-3x^2-6xh-3h^2-\cancel{x}+3x^2}{h}$ ✓

$= \lim_{h \to 0} \frac{\cancel{h}(1-6x-3h)}{\cancel{h}}$ ✓

$= 1 - 6x$.

(no marks for just writing the answer, but mark generously)

QUESTION 4

(a) The roots are the values of x at which the curve crosses the x-axis. ✓

(b) (i) A quadratic is positive definite if all its y-values are positive. ✓

(ii) if $y = x^2 - 4x + 10$

$a = 1 > 0 \quad \therefore$ concave up ✓

$b^2 - 4ac = 16 - 4(1)(10) = -24$

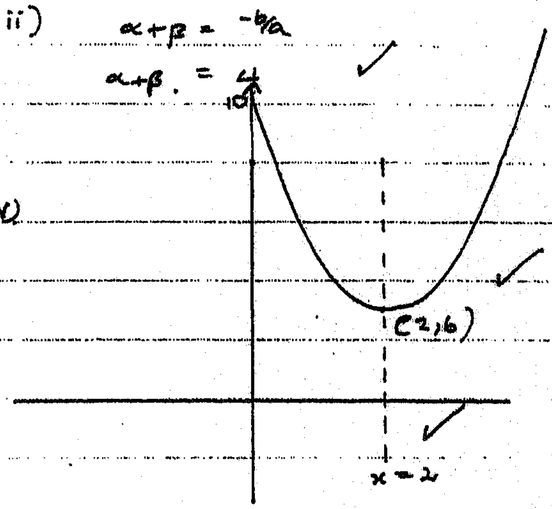
< 0 ✓

since $a > 0$ and $\Delta < 0$ it is pos. def ✓

(iii) $\alpha + \beta = -b/a$

$\alpha + \beta = 4$ ✓

(iv)



axis of symm: $x = -b/2a = 2$.

if $x = 2 \quad y = 2^2 - 4(2) + 10 = 6 \quad \therefore V = (2,6)$

(v) The "roots" are $x = \frac{4 \pm \sqrt{-24}}{2}$

which are not real but can still be added up.

$\frac{4+\sqrt{-24}}{2} + \frac{4-\sqrt{-24}}{2} = 4.$ ✓

(vi) $x^2 = A(x^2-2x+1) + Bx - B + C$

$x^2 = Ax^2 + x(-2A+B) + A - B + C$

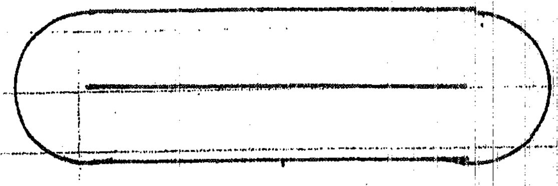
$\therefore A = 1$ ✓ $\quad -2A+B=0 \quad A-B+C=0$

$-2+B=0 \quad 1-2+C=0$

$B = 2$ ✓ $\quad C = 1$ ✓

QUESTION 5

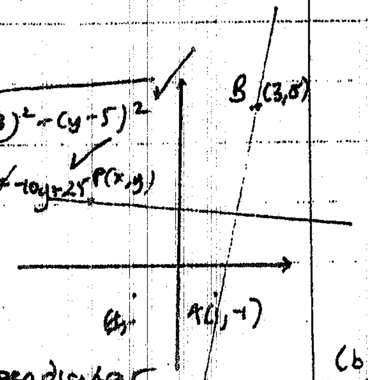
)



) $PA = PB$

$\sqrt{(x-1)^2+(y+1)^2} = \sqrt{(x-3)^2+(y-5)^2}$

$x^2-2x+1+y^2+2y+1 = x^2-6x+9+y^2-10y+25$



$4x + 12y - 32 = 0$

$x + 3y - 8 = 0$

The locus is the perpendicular bisector of AB.

(c)(i) Since $m(CP) \times m(DP) = -1$

then $CP \perp DP$

$\therefore \angle CPD = 90°$.

Since this is the angle in the semi-circle it is a circle with CD as diameter.

(name both points)

(ii) C and D themselves cannot be in the locus since at those points P coincides with C or D and $\angle CPD \neq 90°$.

(d) $4^x - 65 \times 2^x + 64 = 0$

let $u = 2^x$

$u^2 - 65u + 64 = 0$

$(u-64)(u-1) = 0$

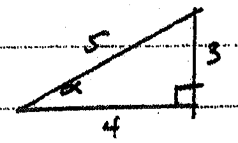
$u=1 \quad u=64$

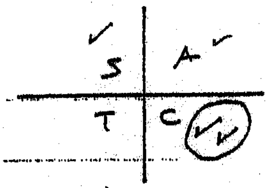
$2^x = 1 \quad 2^x = 64$

$x = 1, \quad x = 6.$

QUESTION 6

(a) 4th quadrant





(for both)

~~tan~~ $\sin\alpha = -\frac{3}{5} \quad \cos\alpha = \frac{4}{5}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \times -\frac{3}{5} \times \frac{4}{5}$

$= \frac{-24}{25}$.

(b) $2\cos\frac{x}{2} = 1 \qquad 0 \le \frac{x}{2} \le \pi$

$\cos\frac{x}{2} = \frac{1}{2}$

$\frac{x}{2} = 60° = \frac{\pi}{3}$

$x = 120° = \frac{2\pi}{3}$

(−1 if two answers are given).

(c)(i) $A\sin(x-\alpha) = A\sin x\cos\alpha - A\cos x\sin\alpha$

(ii) $A\sin x\cos\alpha - A\cos x\sin\alpha = 2\sin x - \cos x$

$\therefore A\cos\alpha = 2 \qquad A\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{2} \qquad A = \sqrt{2^2+1^2} = \sqrt{5}$

$\alpha = \tan^{-1}(\frac{1}{2})$

$\therefore 2\sin x - \cos x = \sqrt{5}\sin(x - 26°34')$

$\sqrt{5}\sin(x-26°34') = 1$

$\sin(x-26°34') = \frac{1}{\sqrt{5}}$

$x - 26°34' = \cancel{26°34'}, 180 - 26°34'$

$x = 53°8', 180°$

(d) $\tan 30 = \frac{x}{h}$

$x = h\tan 30$

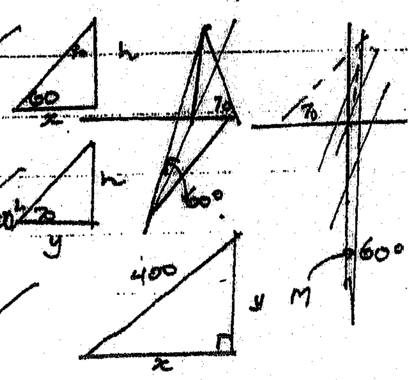
$\tan 20 = \frac{y}{h}$

$y = h\tan 20$

$\therefore h^2\tan^2 30 + h^2\tan^2 20 = 400^2$

$h^2 = \frac{400^2}{\tan^2 30 + \tan^2 20}$

$h = 586$m tall.



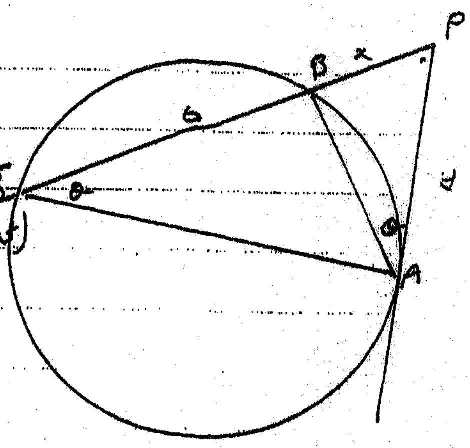
QUESTION 7

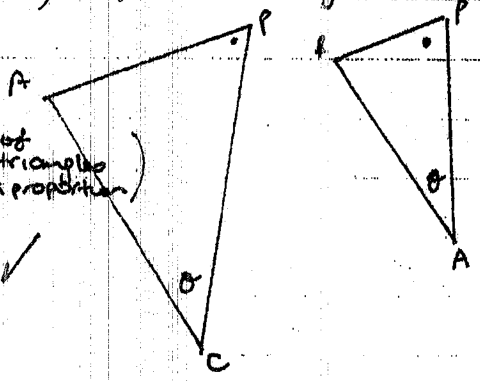
(a) In $\triangle$ PAB, $\triangle$ PAC

$\angle P$ is common

$\angle PAB = \angle PCA$ (angle between a tangent and a chord is equal to the angle in the alt segment)

$\therefore \triangle PAB \parallel\!\!\mid \triangle PAC$ (A·A·A)





$$\frac{PA}{PB} = \frac{PC}{PA}$$ (sides of similar triangles are in proportion)

$\therefore PA^2 = PB \cdot PC.$

(b) $y^2 = x(x+6)$

(c). $4y = 8x$ (the product of the intercepts of intersecting chords)

$y = 2x$

(d) $y = 2x$

$y^2 = x(x+6)$

$x(x+6) = 4x^2$

$x^2 + 6x = 4x^2$

$-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0 \quad x = 2$

but x is a length $\therefore x = 2$ cm. $x \neq 0$