Moriah College

Mathematics Extension 1

2016 Preliminary Examination

General Instructions

- Reading time - 5 minutes
- Working time - 2 hour
- Write using black or blue pen
- Board-approved calculators may be used

Total marks - 70

Section I
10 marks

- Attempt Questions 1-10
- Allow about 20 minutes for this section

Section II
60 marks

- Attempt Questions 11-14

Section I

10 marks
Attempt Questions 1 - 10
Allow about 20 minutes for this section

Answer each question on the multiple choice answer sheet provided.

1)	If α and β are the roots of $2x^2+3x-6=0$, what is the value of $\alpha\beta$? (A) $\frac{-3}{2}$ (B) 3 (C) $\frac{3}{2}$ (D) -3
2)	The quadratic equation $x^2+kx+k=0$ has its discriminant $\Delta = 0$. The value of k is: (A) 4 (B) 0 or 4 (C) 0 or -4 (D) 0
3)	A parabola has the equation $x^2 = 12(y-2)$. The coordinates of the focus of this parabola are: (A) $(0,5)$ (B) $(0,3)$ (C) $(0,2)$ (D) $(0,1)$
4)	Consider the function $y=\frac{4x^2-2x}{2x^2+2x}$ The value of $\lim_{x\to\infty}\left(\frac{4x^2-2x}{2x^2+2x}\right)$ is: (A) ∞ (B) 2 (C) 0 (D) 4

5) Again, consider the function $y = \frac{4x^2-2x}{2x^2+2x}$

The value of $\lim_{x \to 0} \left(\frac{4x^2-2x}{2x^2+2x}\right)$ is:

(A) 0 (B) ∞

(C) -1 (D) 1

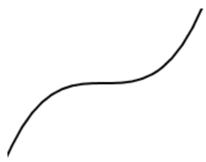
6) If $y = (x^3+1)^5$, then $\frac{dy}{dx} =$

(A) $15x^2(x^3+1)^4$ (B) $5(x^3+1)^4$

(C) $5(3x^2+1)^4$ (D) $15x^2(3x^2+1)^4$

7) For a certain function, it is known that $\frac{dy}{dx} = (x-1)^2(x-2)$.
The function has a minimum turning point at $x = 2$.

At $x = 1$, the function has which of the following shapes:

(A)  (B) 

(C)  (D) 

8) The function $y = 2x^3 - 24x + 1$ is concave down for

(A) $x < 0$ (B) $x > 0$

(C) $-2 < x < 2$ (D) $x < -2$

9)	The quadratic equation $x^2 + 6x + k = 0$ has roots $x = \alpha$ and $x = 2\alpha$ What is the value of k? (A) $k = -16$ (B) $k = 8$ (C) $k = -2$ (D) $k = -8$
10)	It can be proven by mathematical induction that: $2^{n+2} + 3^{2n+1}$ is always divisible by a certain number x for $n \geq 1$ Which of the following numbers could be the value of x? (A) $x = 2$ (B) $x = 3$ (C) $x = 5$ (D) $x = 7$

Section I Questions 11-14 60 marks

Question 11 (15 marks)

(a) Solve the equation $\boldsymbol{x^3 + \frac{8}{x^3} = 9}$ using the substitution $\boldsymbol{M = x^3}$ **3**

(b) For what values of k is the quadratic expression $\boldsymbol{kx^2 + 4x + k}$ positive for all values of x (ie positive definite). **3**

(c) Consider the function

$$\boldsymbol{y = \frac{2x+1}{4x-3}}$$

Find the equation of the tangent to this curve at the point where $\boldsymbol{x = 1}$ **3**

(d) α and β are the roots of the quadratic equation $\boldsymbol{x^2 - x - 1 = 0}$

(i) Find the value of $\alpha^2\beta + \beta^2\alpha$ **2**

(ii) Show that $\alpha^2 + \beta^2 = 3$ **2**

(iii) Find the value of $\alpha^3 + \beta^3$ **2**

Question 12 (16 marks)

(a) Consider the function $\boldsymbol{y = x^4 - 18x^2}$

(i) Find the x-coordinates of all stationary points of this function. **2**

(ii) ~~Find the x-coordinates of all points of inflexion of this function.~~ **2**

(iii) Sketch the curve of this function, showing the coordinates of the stationary and ~~inflexion points~~. **3**

(iv) Hence, find the values of k such that the function $\boldsymbol{y = x^4 - 18x^2 + k}$ has no x-intercepts. **1**

(b) Consider the function $\boldsymbol{y = x\sqrt{6 - x}}$

(i) Show that the derivative is given by: **2**

$$\frac{dy}{dx} = \frac{12 - 3x}{2\sqrt{6 - x}}$$

(ii) By finding any turning points, determine the maximum and minimum values of the function for the restricted domain $\boldsymbol{2 \le x \le 6}$ **3**

(c) A parabola has the equation $\boldsymbol{y = \frac{1}{8}x(x - 8)}$

(i) Find the vertex of this parabola. **2**

(ii) Find the equation of the directrix of this parabola. **1**

Question 13 (16 marks)

(a) The point *A* is $(-2, 0)$ and the point *B* is $(6, 0)$.

The variable point *P* has coordinates (x, y)

(i) Find an expression for the distance *PB* in terms of *x* and *y*. **1**

(ii) The point *P* moves so that the distance between *P* and *B* is three times the distance between *P* and *A* (*ie* $PB = 3PA$).

Show that the equation of the locus of *P* is given by the circle: **2**

$$x^2 + 6x + y^2 = 0$$

(iii) Find the centre of the circle. **1**

(b)

(i) Find the discriminant of the quadratic equation $x^2 - 2x + k = 0$. **1**

(ii) The line $y = 2x$ is a tangent to the parabola $y = x^2 + k$. **1**
Draw a diagram showing the tangent and the parabola.

(iii) Using part (i), find the value of *k*. **2**

(iv) Hence, find the coordinates of the point where the line touches the parabola. **2**

(c) Consider the following mathematical identity

$$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \dots\dots\dots \times \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

(i) Show that the identity holds true for n=2. **1**

(ii) Show that the identity holds true for n=3. **1**

(iii) Prove, by mathematical induction that the identity holds true for all positive integers $n \geq 2$. **3**

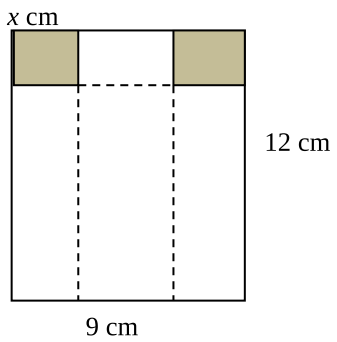
(iv) Using the identity with an appropriate value of *n*, find the value of **1**

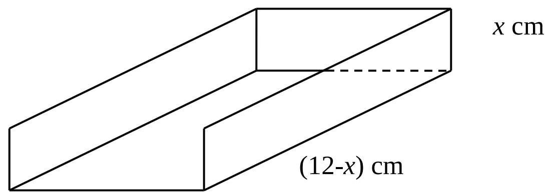
$$\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{24}{25} \times \dots\dots\dots \times \frac{9999}{10000}$$

Question 14 (13 marks)

(a) A rectangular sheet of cardboard measures 12cm by 9cm.

From two corners, squares of side x cm are removed as shown.
The remaining cardboard is folded along the dotted lines to form a tray as shown.
The height of the tray is x and the length of the tray is $(12\text{-}x)$.





(i) Show that the volume, V cm^3, of the tray is given by. **3**

$$\boldsymbol{V = 2x^3 - 33x^2 + 108x}$$

(ii) Find the maximum volume of the tray **3**

(iii) Find the range of values that the height x can take, in order for the tray to be able to be constructed. **1**

(b) Let $\boldsymbol{P(x\ ,y)}$ be a variable point on the parabola $\boldsymbol{x^2 = 4ay}$**, where $\boldsymbol{a}$ is the focal length**.
Let $\boldsymbol{A(0\ ,4a)}$ be a point on the y-axis.

(i) Let D be the distance PA.

Show that $\boldsymbol{D^2 = y^2 - 4ay + 16a^2}$ **3**

(ii) Show that the minimum value of D occurs when P is the point $(\boldsymbol{\pm 2\sqrt{2}a, 2a})$ **3**

Ext 1
Solutions

MULTIPLE CHOICE

1. D
2. B
3. A
4. B
5. C
6. A
7. D
8. A
9. B
10. D

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11 (a) $M = x^3 \rightarrow M + \frac{8}{M} = 9$

$$M^2 - 9M + 8 = 0$$

$$(M-8)(M-1) = 0$$

$$M = 8 \text{ OR } 1$$

$$x^3 = 8 \text{ OR } 1$$

$$\boxed{x = 2 \text{ OR } 1}$$

(b) $\Delta = 16 - 4k^2$ and $a = k$

We need $16 - 4k^2 < 0$ and $k > 0$

$$k^2 > 4$$

$k < -2$ OR $k > 2$ and $k > 0$

$$\boxed{\text{Hence } k > 2}$$

(c) $u = 2x + 1$; $v = 4x - 3$

$u' = 2$; $v' = 4$

$$\frac{dy}{dx} = \frac{2(4x-3) - 4(2x+1)}{(4x-3)^2}$$

$$= \frac{-10}{(4x-3)^2}$$

when $x = 1$, $y = 3$ and $\frac{dy}{dx} = -10$

Tangent $\longrightarrow$ $y - 3 = -10(x - 1)$

$$y = -10x + 13$$

(d) (i) $\alpha\beta(\alpha + \beta) = \frac{c}{a} \times \frac{-b}{a}$

$= -1 \times 1$

$= -1$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 1 - 2(-1)$

$= 3$

(iii) $(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = 1(3 + 1)$

$= 4$

(12) (a) $\frac{dy}{dx} = 4x^3 - 36x$ and $\frac{d^2y}{dx^2} = 12x^2 - 36$

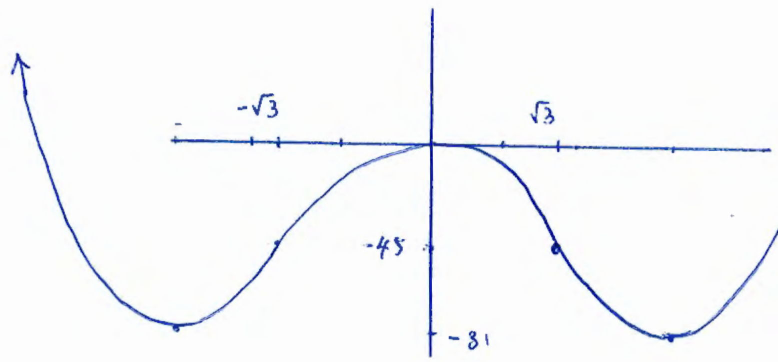
(i) $4x(x^2 - 9) = 0 \longrightarrow$ stationary points when

$x = 0, \pm 3$

(ii) $12(x^2 - 3) = 0 \longrightarrow$ inflexion points when

$x = \pm\sqrt{3}$

(iii)



y-coordinates of inflexion and stationary required

(iv) $k > 81$ to raise the curve above x-axis.

(b) (i) $y = x(6-x)^{1/2}$

$u = x \quad v = (6-x)^{1/2}$

$u' = 1 \quad v' = -\frac{1}{2}(6-x)^{-1/2}$

$= \frac{-1}{2\sqrt{6-x}}$

$\frac{dy}{dx} = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$ → enough for full marks

$= \frac{12-3x}{2\sqrt{6-x}}$

(ii) $\frac{dy}{dx} = 0 \longrightarrow x = 4$

$(4, 4\sqrt{2})$ is a max. turn. pt

$x = 2, \quad y = 4$

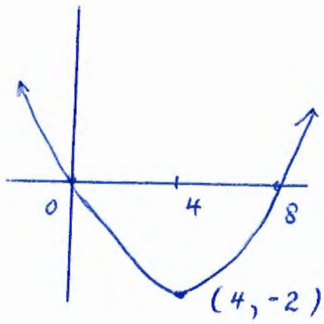
$x = 6, \quad y = 0$

$\therefore$ Max. value is $4\sqrt{2}$

Min. value is 0

(c) $y = \frac{1}{8}x^2 - x$

(i) Vertex is $(4, -2)$



$4a = 8$

$a = 2$

(ii) Directrix is $y = -4$

13 (a)

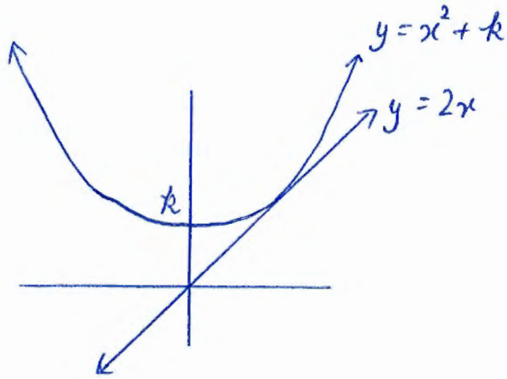
(i) distance $PB = \sqrt{(x-6)^2 + y^2}$

(ii)
$$\sqrt{(x-6)^2 + y^2} = 3\sqrt{(x+2)^2 + y^2}$$
$$x^2 - 12x + 36 + y^2 = 9x^2 + 36x + 36 + 9y^2$$
$$8x^2 + 48x + 8y^2 = 0 \longrightarrow$$ pay 2 marks to this point
$$x^2 + 6x + y^2 = 0$$

(iii)
$$(x+3)^2 + y^2 = 9$$
centre is $(-3, 0)$

(b) (i) $\Delta = 4 - 4k$

(ii)



(iii) $x^2 + k = 2x$ gives point(s) of intersection

$x^2 - 2x + k = 0$

$y = 2x$ is a tangent so $x^2 - 2x + k = 0$ has 1 root

$\therefore \quad 4 - 4k = 0 \longrightarrow k = 1$

(iv) $x^2 - 2x + 1 = 0 \longrightarrow (x-1)^2 = 0 \longrightarrow (1, 2)$

(c) (i) $n=2 \longrightarrow$ $\quad LHS = 1 - \dfrac{1}{2^2} = \dfrac{3}{4}$

$RHS = \dfrac{2+1}{2(2)} = \dfrac{3}{4}$

(ii) $n=3 \longrightarrow$ $\quad LHS = \left(1-\dfrac{1}{2^2}\right)\times\left(1-\dfrac{1}{3^2}\right) = \dfrac{3}{4}\times\dfrac{8}{9}$

$= \dfrac{2}{3}$

$RHS = \dfrac{3+1}{2(3)} = \dfrac{2}{3}$

(iii) Assume $\left(1-\dfrac{1}{2^2}\right)\times\left(1-\dfrac{1}{3^2}\right)\times\ldots\times\left(1-\dfrac{1}{k^2}\right) = \dfrac{k+1}{2k}$

RTP $\left(1-\dfrac{1}{2^2}\right)\times \ldots\ldots \times\left(1-\dfrac{1}{k^2}\right)\times\left(1-\dfrac{1}{(k+1)^2}\right) = \dfrac{k+2}{2(k+1)}$

$LHS = \dfrac{k+1}{2k}\times\left(1-\dfrac{1}{(k+1)^2}\right)$ by assumption

$= \dfrac{k+1}{2k}\times\dfrac{(k+1)^2-1}{(k+1)^2}$

$= \dfrac{k^2+2k}{2k(k+1)}$

$= \dfrac{k(k+2)}{2k(k+1)}$

$= \dfrac{k+2}{2(k+1)}$

By principles of M.I.
true for $n \geq 2$
$n=2$ case proven in part (i)

(iv) $1 - \dfrac{1}{n^2} = \dfrac{9999}{10000} \longrightarrow n = 100$

$\therefore$ Value of product is $\dfrac{101}{200}$

(14) (a)

(i) width of tray $= 9 - 2x$

Hence, $V = x(12-x)(9-2x)$

$= x(108 - 33x + 2x^2)$

$V = 2x^3 - 33x^2 + 108x$

(ii) $\frac{dV}{dx} = 6x^2 - 66x + 108$

solving $6x^2 - 66x + 108 = 0$

gives $x^2 - 11x + 18 = 0$

$(x-9)(x-2) = 0$

$x = 2$ OR 9 (only $x = 2$ is valid)

x	1	2	3
V'	+	0	−

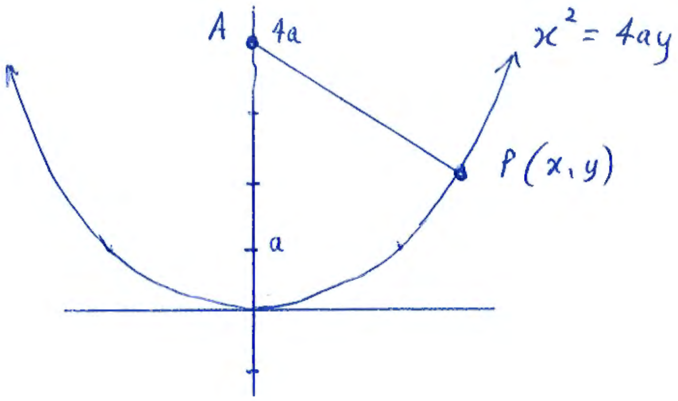
$\longrightarrow$ $x = 2$ gives a max. volume

of $V = 2 \times 10 \times 5$

$V = 100 \text{ cm}^3$

(iii) $0 < x < \frac{9}{2}$

(b) (i)



$$D^2 = (x-0)^2 + (y-4a)^2$$

$$= x^2 + y^2 - 8ay + 16a^2$$

But $x^2 = 4ay$

$$\therefore D^2 = 4ay + y^2 - 8ay + 16a^2$$ → sufficient for full marks

$$D^2 = y^2 - 4ay + 16a^2$$

(ii) D^2 is a quadratic function in y (ie y is the variable)

$$\frac{d(D^2)}{dy} = 2y - 4a$$

solving $2y - 4a = 0 \longrightarrow y = 2a$

Since D^2 is concave up, $y = 2a$ gives a minimum value of D^2 and hence D

Now when $y = 2a$, $x^2 = 4a(2a)$

$$x^2 = 8a^2$$

$$x = \pm 2\sqrt{2}a$$

$\therefore$ P is $(\pm 2\sqrt{2}a, 2a)$