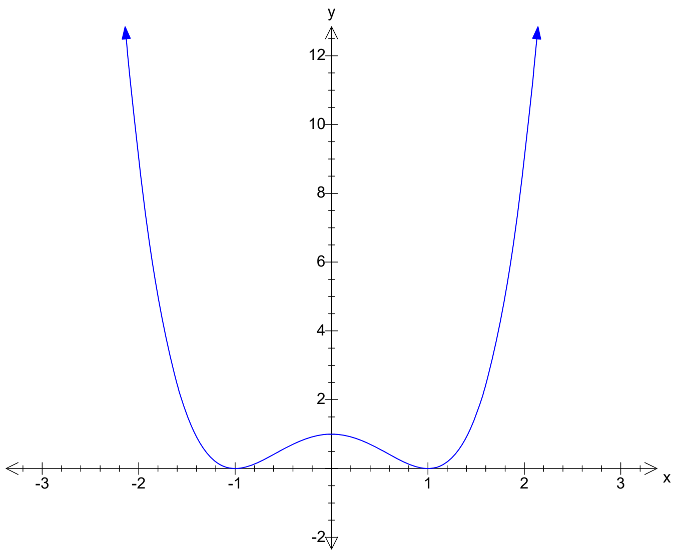
QUESTION 1 *(10 MARKS)* Start this question on a new page.

(a) Simplify $\dfrac{1}{\sqrt{3}+2}-\dfrac{1}{\sqrt{3}-2}$. **2**

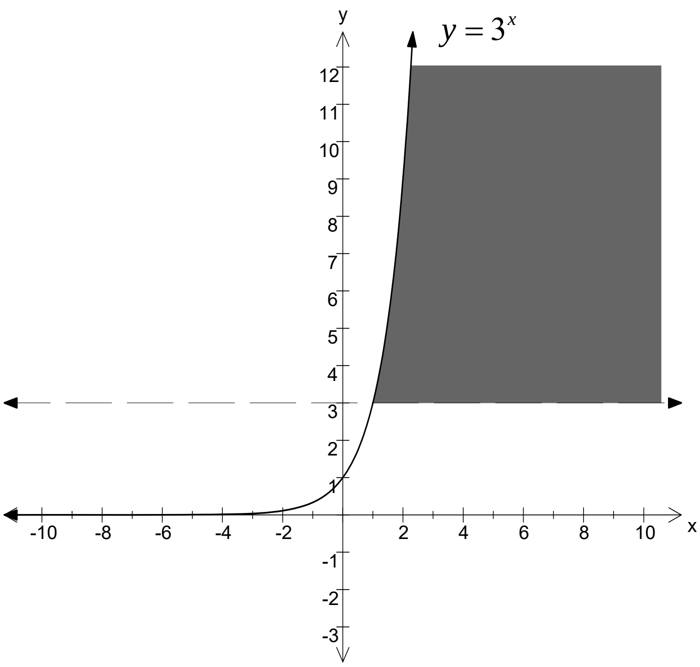
(b) Copy the graph of $f(x)$ below on to your answer paper. On the same axes sketch the graph of $f(x-1)$. **1**



(c) If $g(x)=\dfrac{x}{x-3}$ find the inverse function $g^{-1}(x)$. **2**

(d) Solve the inequation $\dfrac{1}{x-4}\geq 2$. **3**

(e) Define the shaded region below **2**

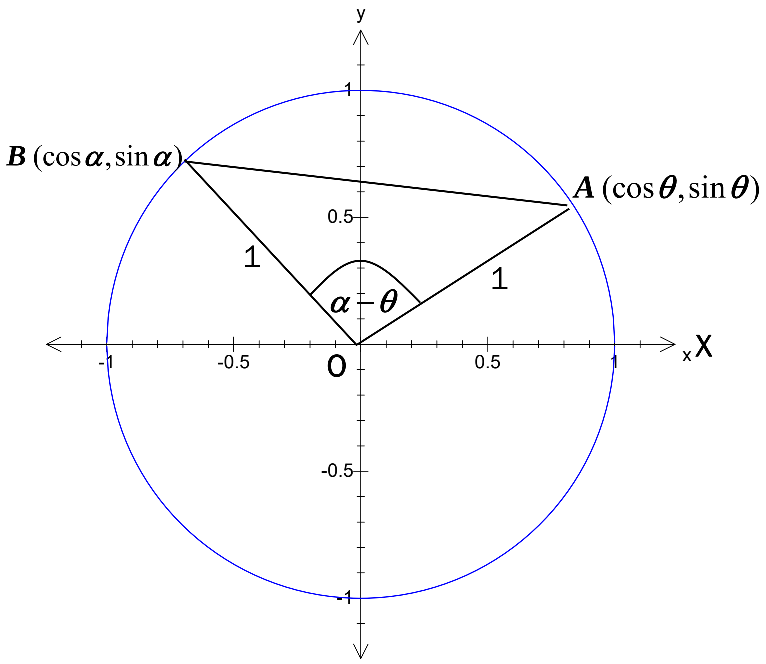


QUESTION 2 *(10 MARKS)* Start this question on a new page.

(a) Solve $\tan 2\theta = \sqrt{3}$ where $0 \le \theta \le 360^\circ$. **3**

(b) Prove the identity $\dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta} = 1$. **2**

(c) Consider the unit circle below. Let $X\hat{O}A = \theta$ and $X\hat{O}B = \alpha$. **5**



(i) By considering the triangle OAB, show that $\cos(\alpha - \theta) = \dfrac{2 - AB^2}{2}$

(ii) Using the distance formula, show that
$AB^2 = 2 - 2\cos\alpha\cos\theta - 2\sin\alpha\sin\theta$

(iii) Hence show that $\cos(\alpha - \theta) = \cos\alpha\cos\theta + \sin\alpha\sin\theta$

QUESTION 3 *(10 MARKS)* Start this question on a new page.

(a) The fourth term of an arithmetic sequence is 30 and the sixth term is 42. **3**

(i) Find the 12^{th} term of the sequence.

(ii) Find the sum of the first 10 terms.

(b) Evaluate $\sum_{r=1}^{5}(-1)^r\, 2r$ **1**

(c) The sum of the first n terms of a sequence is given by $S_n = 32 - 3n^2$. **3**

(i) Show that the n^{th} term of this sequence is given by $T_n = 3 - 6n$.

(ii) Find the value of the first term of the sequence that is less than -100.

(d) Prove by induction that $5^n + 3$ is divisible by 4 for any positive integer n. **3**

QUESTION 4 *(12 MARKS)* Start this question on a new page.

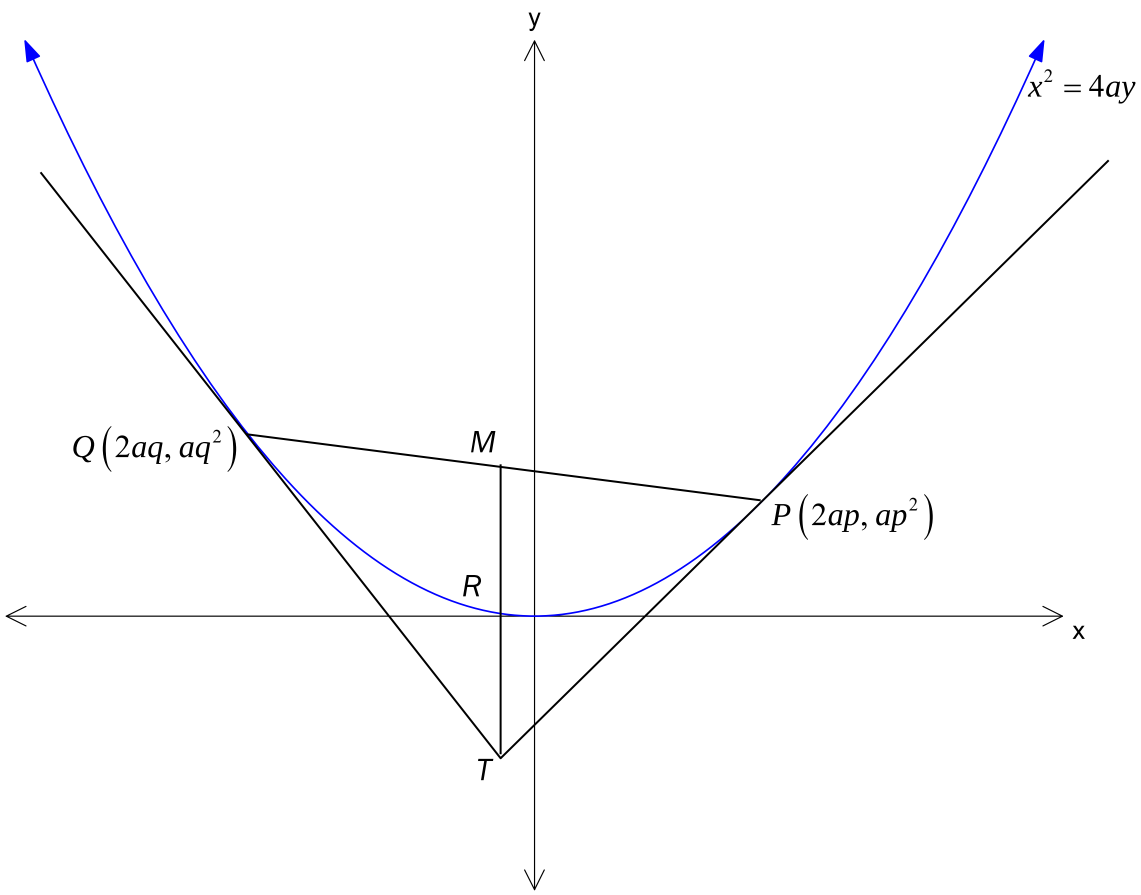
(a) If $x = t + 1$ and $y = \dfrac{1}{t-2}$, eliminate the parameter *t* to find the Cartesian equation. **2**

(b) Consider the parabola $(y-1)^2 = 8(x+2)$. Find **2**

(i) the coordinates of the vertex
(ii) the coordinates of the focus

(c) *P* and *Q* are two points on the parabola $4ay = x^2$ with parameters *p* and *q* respectively. The tangents at *P* and *Q* meet at *T*. *M* is the midpoint of *PQ*. **8**

(i) Find the equation of the tangent at *P* and hence state the equation of the tangent at *Q*.
(ii) Show that these tangents meet at the point *T* $\left(a(p+q), apq\right)$.
(iii) Find the midpoint *M* of the interval *PQ*.
(iv) Show that *TM* is parallel to the axis of the parabola.
(v) Show that the point *R* where *TM* intersects the parabola is the midpoint of *TM*.



QUESTION 5 *(12 MARKS)* Start this question on a new page.

(a) If $f(x)=(2x+1)(x^3-1)^2$ find $f'(-1)$. 2

(b) Using the definition of the derivative $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ 2

find $f'(x)$ where $f(x)=x-2x^2$.

(c) Consider the function $y=\dfrac{x^2}{\sqrt{2x+1}}$. 5

(i) State the domain of the function

(ii) Show that $\dfrac{dy}{dx}=\dfrac{x(3x+2)}{\sqrt{(2x+1)^3}}$

(iii) Hence find the point or points on the curve where the function is horizontal.

(d) An equilateral triangle has sides of length *s cm* and area *A* cm^2. 3

(i) Show that $A=\dfrac{\sqrt{3}s^2}{4}$.

(ii) If the sides are increasing at a constant rate of 2 *cm/second*, find the rate at which the area is increasing when the sides are of length 3 *cm*.

QUESTION 6 *(9 MARKS)* Start this question on a new page.

(a) Solve $x^6 - 7x^3 - 8 = 0$. **2**

(b) Express $2x^2 - 7x - 4$ in the form $a(x+2)^2 + b(x+2) + c$. **3**

(c) Find the values of m if the equation $x^2 - 3mx + (m+3) = 0$ has: **4**

(i) One root double the other (hint: let the roots be α and 2α)

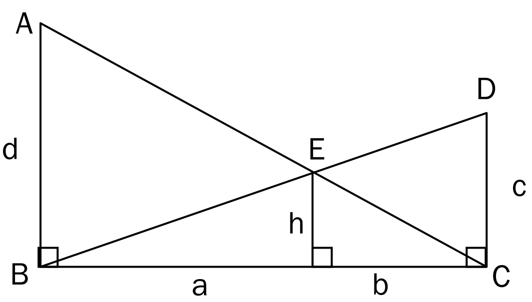
(ii) One root the reciprocal of the other

QUESTION 7 *(10 MARKS)* Start this question on a new page.

(a) If A is the point (10, 2) and B is the point (-2, 6), find the point P dividing AB externally in the ratio 4:3. **2**

(b) The perpendicular distance between the origin and the line $x - 2y + k = 0$ is $\sqrt{5}$. Find two possible values of *k*. **3**

(c) **5**



(i) Show that $\frac{h}{d} = \frac{b}{a+b}$, stating all reasons.

(ii) By stating a similar expression for $\frac{h}{c}$, show that $\frac{1}{c} + \frac{1}{d} = \frac{1}{h}$.

END OF PAPER

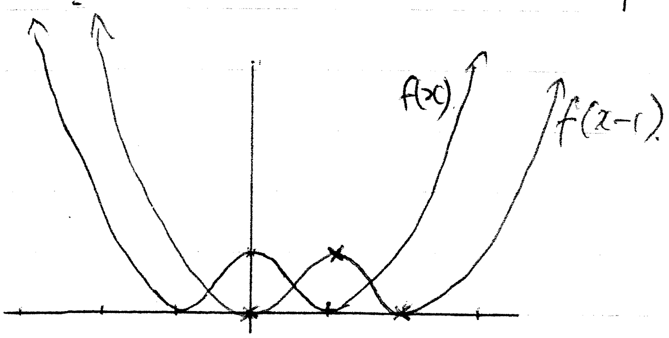
Y11 Ext 1 Prelim 2008

Question 1

(a) $\dfrac{1}{\sqrt{3}+2} - \dfrac{1}{\sqrt{3}-2} = \dfrac{(\sqrt{3}-2)-(\sqrt{3}+2)}{(\sqrt{3}+2)(\sqrt{3}-2)}$

$= \dfrac{-4}{-1} = 4.$

(b)



(c). let $y = \dfrac{x}{x-3}$ for inverse swap x, y

ie $x = \dfrac{y}{y-3}$

$(y-3)x = y$

$xy - 3x = y.$

$xy - y = 3x.$

$y(x-1) = 3x$

$y = \dfrac{3x}{x-1}$

$g^{-1}(x) = \dfrac{3x}{x-1}.$

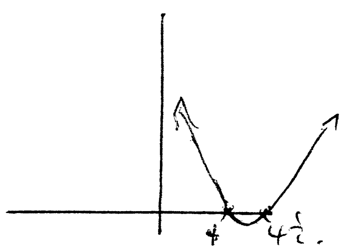
(d) $\times (x-4)^2$ $(x-4) \geq 2(x-4)^2$ $(x \neq 4)$

$2(x-4)^2 - (x-4) \leq 0$

$(x-4)[2(x-4)-1] \leq 0$

$(x-4)(2x-9) \leq 0.$

$4 < x \leq \dfrac{9}{2}.$



(e). $y > 3,\ y \leq 3^x$

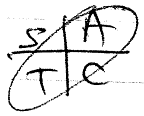
QUESTION 2

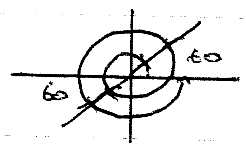
(a) domain $0 \le \theta \le 360$

$0 \le 2\theta \le 720$.

base $\angle$ 60°

Quadrants 1, 3.





$2\theta = 60, 240, 420, 600$

$\theta = 30, 120, 210, 300$.

(b) $LHS = \dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta}$

$= \dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\frac{1}{\sin^2\theta}}$.

$= \dfrac{1}{1+\sin^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta+1}$

$= \dfrac{1+\sin^2\theta}{1+\sin^2\theta} = 1 = RHS.$

(c)(i) $\cos(\alpha-\theta) = \dfrac{1^2+1^2-AB^2}{2\times1\times1}$

$= \dfrac{2-AB^2}{2}.$

(ii) $AB^2 = (\cos\theta - \cos\alpha)^2 + (\sin\theta - \sin\alpha)^2$

$= \cos^2\theta - 2\cos\theta\cos\alpha + \cos^2\alpha + \sin^2\theta - 2\sin\theta\cos\alpha + \sin^2\alpha.$

$= (\sin^2\theta + \cos^2\theta) + (\sin^2\alpha + \cos^2\alpha) - 2\cos\alpha\cos\theta - 2\sin\alpha\sin\theta$

$= 2 - 2\cos\alpha\cos\theta - 2\sin\alpha\sin\theta$ (since $\sin^2x + \cos^2x = 1$)

(iii) From (i) and (ii)

$\cos(\alpha-\theta) = \dfrac{2-[2-2\cos\alpha\cos\theta-2\sin\alpha\sin\theta]}{2}.$

$= \dfrac{2\cos\alpha\cos\theta - 2\sin\alpha\sin\theta}{2}$

$= \cos\alpha\cos\theta + \sin\alpha\sin\theta$.

QUESTION 3

(a) $T_4 = a + 3d = 30$

$T_6 = a + 5d = 42$

$2d = 12$

$d = 6$

$a = 12$

(i) $T_{12} = \quad + 11 \times \quad =$

(ii) $S_{10} = \frac{10}{2}(2\times12 + 11\times6)$

$= 450$

(b) $\sum_{r=1}^{5}(-1)^r 2r = -2 + 4 - 6 + 8 - 10$

$= -6$

(c)(i) $T_n = S_n - S_{n-1}$

$= 32 - 3n^2 - (32 - 3(n-1)^2)$

$= 32 - 3n^2 - 32 + 3(n^2 - 2n + 1)$

$= 32 - 3n^2 - 32 + 3n^2 - 6n + 3$

$= 3 - 6n$

(ii) find n such that $T_n < -100$

ie $3 - 6n < -100$

$-6n < -103$

$n > \frac{103}{6} \quad (17\tfrac{1}{6})$

thus first term < -100 is $T_{18} = 3 - 6\times18$

$= -105$

(d) Prove $5^n + 3 = 4p$ (p integer)

Show true for $n=1$

$LHS = 5^1 + 3$

$= 8$

$= 4\times2$ ie divisible by 4.

Suppose true for $n=k$

ie $5^k + 3 = 4m$ (m integer)

$5^k = 4m - 3 \quad (*)$

Show true for $n=k+1$

ie $5^{k+1} + 3 = 4q$ (q integer)

$LHS = 5\times5^k + 3$

$= 5\times(4m-3) + 3$ from assumption $(*)$.

$= 20m - 15 + 3$
$= 20m - 12$
$= 4(5m - 3)$
$= 4q \quad (q = 4m - 3)$.

Hence by mathematical induction true for all $n \geq 1$.

QUESTION 4.

(a) $x = t + 1$

then $t = x - 1$.

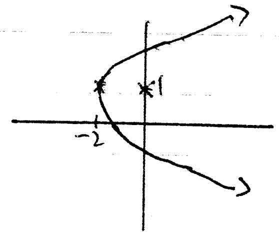
substitute for t in $y = \frac{1}{t-2}$.

$$y = \frac{1}{(x-1)-2}.$$

$$y = \frac{1}{x-3}.$$

(b) (i) vertex is $(-2, 1)$

(ii) focal length is 2.



focus is $(0, 1)$.

(c) (i) $y = \frac{x^2}{4a}$

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}.$$

At $x =$, $\frac{dy}{dx} = p$.

Equation of tangent at P is $y - ap^2 = p(x - 2ap)$

$$y = px - ap^2 \quad (*)$$

Hence tangent at Q is $y = qx - aq^2$.

(ii) these meet when $px - ap^2 = qx - aq^2$

$$px - qx = ap^2 - aq^2.$$

$$x(p-q) = a(p+q)(p-q)$$

$$x = a(p+q).$$

sub in $(*)$ $\quad y = p.a(p+q) - ap^2$.

$$y = apq.$$

Hence point of intersection is $T(a(p+q), apq)$.

(iii) Midpoint of PQ $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$.

$$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right).$$

(iv) Since x-value of M and T are the same equation of MT is $x = a(p+q)$ ie parallel to the axis of the parabola. $= 0$

(v) Midpoint of MT is $R = \left(a(p+q), \dfrac{2apq + ap^2 + aq^2}{4}\right)$.

$$= \left(a(p+q), \frac{a(p+q)^2}{4}\right).$$

If R lies on parabola it must satisfy $x^2 = 4ay$.

$LHS = a^2(p+q)^2$

$RHS = \dfrac{4a \times a(p+q)^2}{4}$

$= a^2(p+q)^2$.

since $LHS = RHS$ R lies on MT.

QUESTION 5

(a) $f(x) = (2x+1)(x^3-1)^2$

$f'(x) = (2x+1)\,2(x^3-1)\,3x^2 + (x^3-1)^2 \times 2$ ✓

$f'(-1) = -1 \times 2 \times -2 \times 3 \times 1 + (-2)^2 \times 2$

$= 20$ ✓ (2)

(b) $$f'(x) = \lim_{h \to 0} \frac{(x+h) - 2(x+h)^2 - (x - 2x^2)}{h}$$ ✓

$$= \lim_{h \to 0} \frac{x + h - 2(x^2 + 2xh + h^2) - x + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{2x^2} - 4xh - 2h^2 - \cancel{x} + \cancel{2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(1 - 4x - 2h)}{\cancel{h}}$$ ✓

$= 1 - 4x$ (2)

(c) (i) domain is $x > -\frac{1}{2}$ ✓

(ii) $y = \dfrac{x^2}{(2x+1)^{\frac{1}{2}}}$

$$\frac{dy}{dx} = \frac{(2x+1)^{\frac{1}{2}} \times 2x - x^2 \times \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2}{(2x+1)}$$ ✓ $\times \dfrac{(2x+1)^{\frac{1}{2}}}{(2x+1)^{\frac{1}{2}}}$

$$= \frac{(2x+1)\,2x - x^2}{(2x+1)^{3/2}}$$

$$= \frac{4x^2 + 2x - x^2}{(2x+1)^{3/2}}$$ ✓

$$= \frac{3x^2 + 2x}{(2x+1)^{3/2}}$$

$$= \frac{x(3x+2)}{(2x+1)^{3/2}}$$ as required

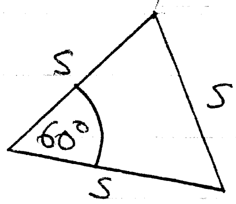
(iii) curve is horizontal when $\dfrac{dy}{dx} = 0$

ie $x(3x+2) = 0$ ✓

$x = 0 \quad x = -\frac{2}{3}$ ✓

but since domain is $x > -\frac{1}{2}$ $\quad x = 0$ ✓ is only solution.

(d)(i)



$$\text{Area} = \frac{1}{2} \times s \times s \times \sin 60$$
$$= \frac{1}{2} s^2 \times \frac{\sqrt{3}}{2}.$$
$$= \frac{\sqrt{3} s^2}{4}.$$

(ii) $A = \frac{\sqrt{3} s^2}{4}$ $\quad \frac{dA}{ds} = \frac{\sqrt{3} s}{2}$

$$\frac{dA}{dt} = \frac{dA}{ds} \times \frac{ds}{dt}$$
$$\frac{dA}{dt} = \frac{\sqrt{3} s}{2} \times 2$$
$$= \sqrt{3} s.$$

when $s = 3$.

$$\frac{dA}{dt} = 3\sqrt{3} \text{ cm}^2/\text{sec}.$$

(3)

QUESTION 6

(a) $(x^3)^2 - 7x^3 - 8 = 0$

$(x^3 + 1)(x^3 - 8) = 0$

$x^3 = -1 \qquad x^3 = 8.$

$x = -1, 2.$

(b) $2x^2 - 7x - 4 \equiv a(x+2)^2 + b(x+2) + c.$

Equating coefficients of x^2 $\quad a = 2.$

Substituting $x = -2 \qquad c = \cdot\cdot \quad .$ 15

substituting $x = 0 \qquad -4 = 4a + 2b + c.$

$-4 = 8 + 2b + 18.$

$2b = -30$

$b = -15.$

$\therefore\ 2x^2 - 7x - 4 \equiv 2(x+2)^2 - 15(x+2) + 18$

(c) (i) let roots be α and 2α.

then $\alpha + 2\alpha = 3m$

$3\alpha = 3m$

$\alpha = m \quad *$

and $\alpha \times 2\alpha = m + 3.$

$2\alpha^2 = m + 3$

from $*$ $\quad 2m^2 = m + 3$

$2m^2 - m - 3 = 0$

$(2m - 3)(m + 1) = 0$

$m = \frac{3}{2} \quad m = -1.$

(ii) let roots be α and $\frac{1}{\alpha}$.

then $\alpha \times \frac{1}{\alpha} = m + 3.$

$m + 3 = 1$

$m = -2.$

QUESTION 7

(a) $P = \left(\frac{kx_2 + lx_1}{k+l}, \frac{ky_2 + ly_1}{k+l}\right)$ where $k:l$ is $4:-3$ (x_1, y_1) is $(10, 2)$ and (x_2, y_2) is $(-2, 6)$.

$= \left(\frac{4\times -2 + -3\times 10}{4-3}, \frac{4\times 6 - 3\times 2}{4-3}\right)$

$= (-38, 18)$.

(b) ... fd perp distance between $x - 2y + k = 0$ and $(0,0)$

$d = \frac{|0 - 2\times 0 + k|}{\sqrt{1+(-2)^2}} = \sqrt{5}$.

$\frac{|k|}{\sqrt{5}} = \sqrt{5}$

$|k| = 5$

$k = \pm 5$.

(c) Most likely solution by similar $\triangle$s (could be done with intercept theorem)

(i) show $\triangle ABC \parallel\!\mid \triangle EFC$

$\angle ABC = \angle EFC = 90^\circ$ (given)

$\angle ACB = \angle ECF$ (common)

$\therefore \triangle ABC \parallel\!\mid \triangle EFC$ (equiangular).

$\therefore \frac{h}{d} = \frac{b}{a+b}$ ① (ratio of corresponding sides in congruent triangles)

(ii) $\triangle DCB \parallel\!\mid \triangle EFB$ (similarly to part (i)).

$\therefore \frac{h}{c} = \frac{a}{a+b}$ ②

From ① & ②

$\frac{h}{c} + \frac{h}{d} = \frac{a}{a+b} + \frac{b}{a+b}$

$h\left(\frac{1}{c} + \frac{1}{d}\right) = \frac{a+b}{a+b}$.

$\frac{1}{c} + \frac{1}{d} = \frac{1}{h}$ as required.