NEWINGTON COLLEGE

2014

Preliminary examination

Mathematics Extension 1

General Instructions:

- Date of task - Wednesday 22nd August
- Working time – 120 mins
- Board-approved calculators may be used.
- Attempt all questions, start each question in a new booklet.
- Show all relevant mathematical reasoning and/or calculations.

Total marks – 70

Outcome	Marks
Section I – Multiple choice	/10
Section II – Q11 (Trigonometry & Functions)	/12
Section II – Q12 (Geometry of the derivative)	/12
Section II – Q13 (Sequences and series)	/12
Section II – Q14 (Induction & Quadratic functions)	/12
Total	/70

Section I (10 marks)
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1-10

Question 1

The co-ordinates of the focus of the parabola $x^2 = 4ay$ are

(a) $(0,-a)$ (b) $(0,a)$ (c) $(0,1)$ (d) $(0,4a)$

Question 2

If α and β are the zeroes of the quadratic polynomial $y = 3x^2 + x - 9$ then

(a) $\alpha+\beta = -\frac{1}{3}$ (b) $\alpha+\beta = \frac{1}{3}$ (c) $\alpha+\beta = -3$ (d) $\alpha+\beta = 3$

Question 3

The 20th term of the arithmetic series $4 + 5\frac{1}{3} + 6\frac{2}{3}$ is

(a) $4+(20-1)\times 1\frac{1}{3}$ (b) $\frac{20}{2}\left[2\times 4+(20-1)\times 1\frac{1}{3}\right]$ (c) $4+20\times 1\frac{1}{3}$ (d) $4\times\left(\frac{4}{3}\right)^{20-1}$

Question 4

$\sum_{r=1}^{r=5}(-3)^r$ is equal to

(a) 45 (b) -183 (c) -182 (d) -363

Question 5

Given $f(x) = 3x^2 - 5x + 2$, find $f(\alpha+1)$

(a) $3\alpha^2 - 5\alpha + 3$ (b) $3\alpha^2 + 11\alpha$ (c) $3\alpha^2 + \alpha + 1$ (d) $3\alpha^2 + \alpha$

Question 6

The perpendicular distance, in units, from $(1,0)$ to $3x-4y=6$ is

(a) $\frac{9}{5}$ (b) $\frac{3}{5}$ (c) $\frac{7}{5}$ (d) $-\frac{9}{5}$

Question 7

If $y=(3x-1)\sqrt{3x-1}$ then $\frac{dy}{dx}$ is

(a) $2\sqrt{3x-1}$ (b) $\frac{9}{2\sqrt{3x-1}}$ (c) $\frac{9}{2}\sqrt{3x-1}$ (d) $3\sqrt{3x-1}-\frac{3}{\sqrt{3x-1}}$

Question 8

The natural domain of the function $y=\sqrt{5-x^2}$ is

(a) $-5<x<5$ (b) $-5\le x\le 5$ (c) $-\sqrt{5}\le x\le\sqrt{5}$ (d) $0<y<\sqrt{5}$

Question 9

The equation of the line perpendicular to $3x+4y-3=0$, which also passes through the point $(-1,-4)$ is

(a) $4x-3y+13=0$ (b) $3x-4y-13=0$ (c) $4x+3y+16=0$ (d) $4x-3y-8=0$

Question 10

The expression $\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$ may be simplified to

(a) $2\sec^2\theta$ (b) $2\sin^2 x$ (c) $2\cos^2 x$ (d) $2\operatorname{cosec}^2 x$

Section II (60 marks)

Attempt Questions 11 – 15
Allow about 1 hours and 45 minutes for this section

Question 11 (12 marks) – Start a new booklet

Marks

(a) Prove $\left(\cot\theta+\operatorname{cosec}\theta\right)^2=\dfrac{1+\cos\theta}{1-\cos\theta}$ **3**

(b) Solve for $0\le\theta\le 360^\circ$:

(i) $2\cos^2\theta-\cos\theta=0,\quad 0^0\le\theta\le 360^0$ **2**

(ii) $5\sec^2\theta-9\tan\theta=7,\quad 90^0\le\theta\le 180^0$. Give your answer correct to the nearest minute **3**

(c)

(i) Show that $y=\dfrac{2x}{x^2+4}$ is an odd function. **1**

(ii) Find any intercepts. **1**

(iii) Find the **equation** of the horizontal asymptote. **1**

(iv) Sketch the curve. **1**

Question 12 (12 marks) – Start a new booklet

(a) Differentiate with respect to x:

(i) $y=\dfrac{1}{x^2}-4\sqrt{2x-3}$ **2**

(ii) $y=\dfrac{x^2-1}{x^2+1}$ Simplify your answer as much as possible. **2**

(b) Find the co-ordinates of the points on the curve $y=2x^3-9x^2+27$ where the tangent is horizontal. **3**

(c) Find the value of the constant a, if the <u>normal</u> to the curve $y=\dfrac{a}{x+2}$ at the point where $x=2$ has a gradient of -4. **2**

(d) If $f\left(x\right)=\left(x-4\right)^2\times\sqrt[3]{2x+1}$, evaluate $f'\left(13\right)$. **3**

Marks

Question 13 (12 marks) – Start a new booklet

(a) A ball is dropped to the ground from a height of 2 metres. After each bounce it rebounds to half the height from which it fell.

(i) Find the total distance it has travelled when it strikes the ground for the 4^{th} time. **2**

(ii) Using the formula for the sum of an infinite geometric progression, how far will it travel before it comes to rest? **2**

(b) The 21^{st} term of an arithmetic series is zero. The sum of the first 25 terms is 100. Find the first term and the common difference. **4**

(c) The series $1+2+4+7+8+10+11+13+14+16+\ldots+398$ was formed by omitting from the first 400 positive integers all those which are multiples of 3 or 5. Find the sum of the series. **4**

Question 14 (12 marks) – Start a new booklet

(a) **Prove by mathematical induction** that $1+x+x^2+x^3+\ \ldots\ +x^{n-1}=\dfrac{1-x^n}{1-x}$. **4**

(b) Solve the equation $\left(x^2+x\right)^2=5x^2+5x-6$, giving your answers in **exact** form. **4**

(c) If α and β are the roots of the equation $ax^2+bx+c=0$, express $\left(\alpha-2\beta\right)\left(2\alpha-\beta\right)$ in terms of a, b, and c. Hence deduce the condition for one root to be double the other. **4**

Marks

Question 15 (12 marks) – Start a new booklet

(a) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - ap^2$. **3**

(ii) The tangent at P and the tangent at Q intersect at T. Find the coordinates of T. **2**

(iii) If T is a point on the directrix of the parabola $x^2 = 4ay$, show that PQ is a focal chord. **2**

(c) $P(2t, t^2)$ is a point on the parabola $x^2 = 4y$ with focus S. The point M divides the interval SP externally in the ratio $3 : 1$.

(i) Show that the co-ordinates of M are $\left(3t, \frac{3t^2 - 1}{2}\right)$ **3**

(ii) Show that as P moves on the parabola $x^2 = 4y$, the locus of M is given by the parabola $x^2 = 6y + 3$. **2**

END OF EXAMINATION

Section I – Multiple choice answer sheet

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.
Select the alternative: A, B, C or D that best answers the question.
Colour in the response oval completely.

Sample:

2 + 4 = ? (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ⊗ C ○ D ○

....

Year 11 Extension 1 Mathematics

Multiple Choice Answer Sheet

Student number:

Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	○	D	○
2	A	○	B	○	C	○	D	○
3	A	○	B	○	C	○	D	○
4	A	○	B	○	C	○	D	○
5	A	○	B	○	C	○	D	○
6	A	○	B	○	C	○	D	○
7	A	○	B	○	C	○	D	○
8	A	○	B	○	C	○	D	○
9	A	○	B	○	C	○	D	○
10	A	○	B	○	C	○	D	○

Mark: /10

Solutions

Section I

1. B
2. A
3. A
4. B
5. D
6. B
7. C
8. C
9. D
10. A

Section II

Question 11

(a) $LHS=\left(\cot\theta+\text{cosec}\,\theta\right)^2$

$$=\left(\frac{\cos\theta+1}{\sin\theta}\right)^2$$

$$=\frac{\left(\cos\theta+1\right)^2}{\sin^2\theta}$$

$$=\frac{\left(\cos\theta+1\right)^2}{1-\cos^2\theta}$$

$$=\frac{\left(\cos\theta+1\right)^2}{\left(1-\cos\theta\right)\left(1+\cos\theta\right)}$$

$$=\frac{1+\cos\theta}{1-\cos\theta}=RHS$$

(b) (i) $\cos\theta\left(2\cos\theta-1\right)=0$

$\cos\theta=0$ or $\cos\theta=\frac{1}{2}$

$\theta=90^0,270^0,60^0$ or 300^0.

(ii) $5\left(1+\tan^2\theta\right)-9\tan\theta=7$

Let $x = \tan\theta$

$5 + 5x^2 - 9x = 7$

$5x^2 - 9x - 2 = 0$

$(5x+1)(x-2) = 0$

$x = -\frac{1}{5}$ or 2

$\tan\theta = -\frac{1}{5}$ or 2. But $90^0 \le \theta \le 180^0$.

$\theta = 168^0 41'$

(c) (i) $f(-a) = \frac{-2a}{(-a)^2 + 4}$

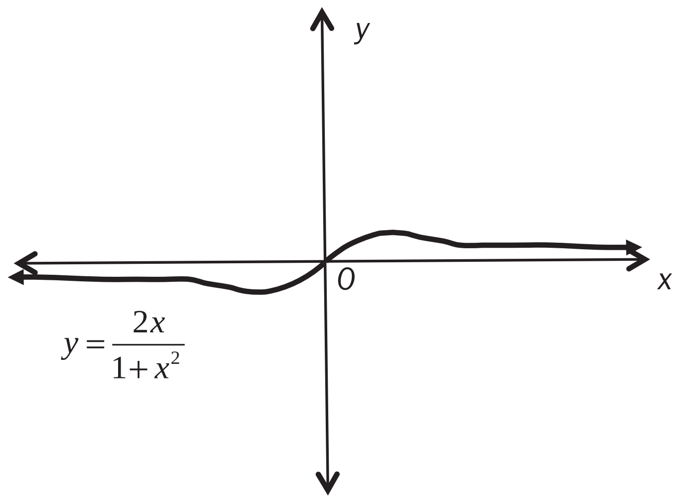
$= \frac{-2a}{a^2 + 4}$

$= -f(a)$

(ii) $(0,0)$

(iii) $y = 0$

(iv)



Question12

(a) (i) $y' = \frac{-2}{x^3} - 4 \times \frac{1}{2}(2x-3)^{-\frac{1}{2}} \times 2$

$= \frac{-2}{x^3} - \frac{4}{\sqrt{2x-3}}$

(ii) $y' = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2}$

$= \frac{4x}{(x^2+1)^2}$

(b) $y = 2x^3 - 9x^2 + 27$

$y' = 6x^2 - 18x = 0$

$6x(x-3) = 0$

$x = 0, x = 3$

Hence, co-ordinates are $(0, 27)$ or $(3, 0)$

(c) $y = \dfrac{a}{x+2}$

Gradient of normal = -4. Hence, gradient of tangent is $\dfrac{1}{4}$.

$y' = \dfrac{-a}{(x+2)^2} = \dfrac{1}{4}$ when $x = 2$

$\dfrac{-a}{(2+2)^2} = \dfrac{1}{4}$

$a = -4$

(d) $f(x) = (x-4)^2 \times \sqrt[3]{2x+1} = (x-4)^2 \times (2x+1)^{1/3}$

$f'(x) = (x-4)^2 \times \dfrac{2}{3}(2x+1)^{-2/3} + (2x+1)^{1/3} \times 2(x-4)$

$f'(13) = (9)^2 \times \dfrac{2}{3}(27)^{-2/3} + (27)^{1/3} \times 2(9)$

$= 60$

Question 13

(a) (i) 2 + 1 + 1 + 0.5 + 0.5 + 0.25 + 0.25 = 5.5 metres

(ii) $= 2 + 2\left(1 + \dfrac{1}{2} + \dfrac{1}{4} + \dots\right)$

$= 2 + 2\left(\dfrac{a}{1-r}\right)$, where $a = 1, r = \dfrac{1}{2}$

$= 2 + 2\left(\dfrac{1}{1-\frac{1}{2}}\right) = 6$ metres

(b) Solve simultaneously:

$a + 20d = 0$ ....1 where a = 1st term and d = common difference

$\dfrac{25}{2}(2a + 24d) = 100$....2

$a = 10, d = -\dfrac{1}{2}$

(c) 1+2+4+7+8+11+13+14+16+...+398

= sum of 1st 400 positive integers – multiples of 3 – multiples of 5 +multiples of 15

$= \dfrac{400}{2}[2\times1 + (400-1)1] - \dfrac{133}{2}[2\times3 + (133-1)3] - \dfrac{80}{2}[2\times5 + (80-1)5] + \dfrac{26}{2}[2\times15 + (26-1)15$

$= 200\times 401 - 133\times 201 - 40\times 405 + 13\times 405$
= 42532

Question 14

(a) Prove $1 + x + x^2 + x^3 + \ ... \ + x^{n-1} = \dfrac{1-x^n}{1-x}$

Step 1 – Prove result true for *n* = 1

LHS= $x^{1-1} = 1$

RHS= $\dfrac{1-x^1}{1-x} = 1$

LHS = RHS

Step 2 – Assume result true for *n* = *k*, where *k* is a positive integer.

i.e. $1 + x + x^2 + x^3 + \ ... \ + x^{k-1} = \dfrac{1-x^k}{1-x}$

Step 3 – Prove result true for *n* = *k* + 1.

i.e. Prove $1 + x + x^2 + x^3 + \ ... \ + x^{k+1-1} = \dfrac{1-x^{k+1}}{1-x}$

Now, LHS = $1 + x + x^2 + x^3 + \ ... \ + x^{k+1-1}$

$= 1 + x + x^2 + x^3 + \ ... \ + x^{k-1} + x^k$

$= \dfrac{1-x^k}{1-x} + x^k$, from step 2.

$= \dfrac{1-x^k + x^k(1-x)}{1-x}$

$= \dfrac{1-x^k + x^k(1-x)}{1-x}$

$= \dfrac{1-x^k + x^k - x^{k+1}}{1-x}$

$= \dfrac{1-x^{k+1}}{1-x}$

= RHS

Step 4 – By the principle of mathematical induction, the result has been proved tru for all positive integers *n.*

(b) Let $y=x^2+x$

Hence, $y^2=5y-6$

$\therefore y^2-5y+6=0$

$(y-2)(y-3)=0$

$y=2 \text{ or } y=3$

$\therefore x^2+x=2 \text{ or } x^2+x=3$

$\therefore x^2+x-2=0 \text{ or } x^2+x-3=0$

$(x+2)(x-1)=0 \text{ or } x=\dfrac{-1\pm\sqrt{1^2-4\times1\times(-3)}}{2}$

$\therefore x=-2,1,\dfrac{-1\pm\sqrt{13}}{2}$

(c) $ax^2+bx+c=0$

$\alpha+\beta=\dfrac{-b}{a}$ $\qquad \alpha\beta=\dfrac{c}{a}$

$(\alpha-2\beta)(2\alpha-\beta)=2\alpha^2-5\alpha\beta+2\beta^2$

$=2(\alpha^2+\beta^2)-5\alpha\beta$

$=2(\alpha^2+2\alpha\beta+\beta^2)-9\alpha\beta$

$=2(\alpha+\beta)^2-9\alpha\beta$

$=2\left(\dfrac{-b}{a}\right)^2-9.\dfrac{c}{a}$

$=\dfrac{2b^2-9ac}{a^2}$

Now, if $\alpha=2\beta$ or $\beta=2\alpha$ then $(\alpha-2\beta)(2\alpha-\beta)=0$

Hence $2b^2=9ac$

<u>Question 15</u>

(a) (i) $x^2=4ay$

$2x=4a\dfrac{dy}{dx}$

$\dfrac{dy}{dx}=\dfrac{x}{2a}$ at $x=2ap$

$\frac{dy}{dx} = p$

The equation of the tangent at *p* is:

$y - ap^2 = p(x - 2ap)$

$= px - 2ap^2$

$\therefore y = px - ap^2$

(ii) Tangent at *P:* $\therefore y = px - ap^2$..........1

Tangent at *Q*: $\therefore y = qx - aq^2$..........2

Sub 1 into 2

$px - ap^2 = qx - aq^2$

$x(p - q) = ap^2 - aq^2$

$x = \frac{a(p^2 - q^2)}{p - q} = a(p + q)$

Sub into 1.

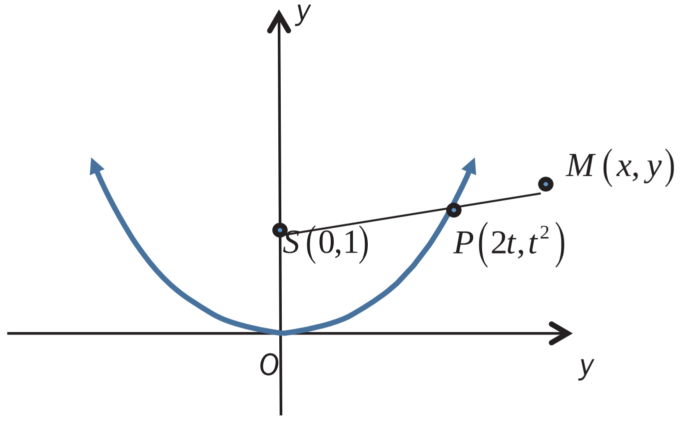
$y = p.a(p + q) - ap^2 = apq$

Hence, $T(a(p + q), apq)$

(iii) *T* on directrix $\rightarrow apq = -a$. Hence, $pq = -1$.

But *p* is the gradient of *PT* and *q* is the gradient of *QT*. Hence the tangents are perpendicular. Hence, $\angle PTQ = 90^0$.

(b) (i)



(i) 3:1 externally

$$x=\frac{3\times 2t-1\times 0}{3-1}=3t$$

$$y=\frac{3\times t^2-1\times 1}{3-1}=\frac{3t^2-1}{2}$$

Coordinates are $\left(3t,\frac{3t^2-1}{2}\right)$

(ii) $t=\frac{x}{3}$. Hence, $y=\frac{3\left(\frac{x}{3}\right)^2-1}{2}$

$$2y=\frac{x^2}{3}-1$$

$$x^2=6y+3$$