NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 1)

2010 Preliminary Course Final Examination

General instructions

- Working time – $2\frac{1}{2}$ hours.
- Commence each new question on a new page.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 11M3A – Mr Trenwith
- ◯ 11M3B – Mr Fletcher
- ◯ 11M3C – Mr Ireland
- ◯ 11M3D – Mr Lam
- ◯ 11M3E – Mr Rezcallah
- ◯ 11M3F – Mr Weiss
- ◯ 11M3G – Mr Berry/Mr Fletcher

NAME: .... **# PAGES USED**: ....

Marker's use only.

QUESTION	1	2	3	4	5	6	7	8	9	**Total**	%
MARKS	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 12	$\overline{\ \ }$ 108	

Question 1 (12 Marks) Commence a NEW page. **Marks**

(a) Evaluate $\dfrac{\sqrt[3]{2.54 \times 6.78}}{\pi^3}$ correct to 3 significant figures. **2**

(b) Find integers a and b such that $\dfrac{\sqrt{3}+1}{2-\sqrt{3}} = a + b\sqrt{3}$. **2**

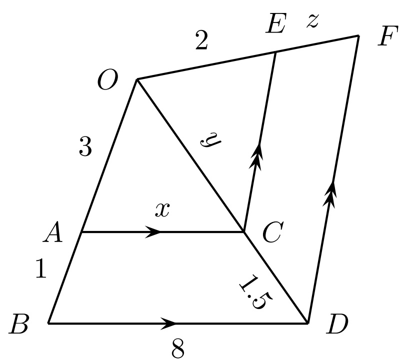
(c) Write the exact value of $2\sin 60^\circ \cos 30^\circ$. **2**

(d) Solve $y = 2x$ and $y = x^2 - 15$ simultaneously. **3**

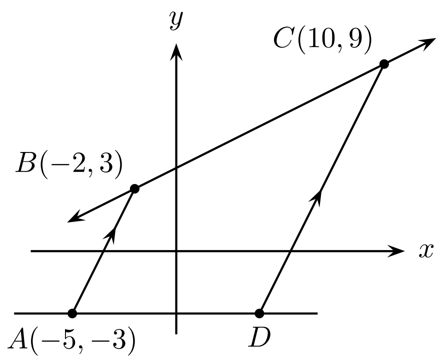
(e) Solve $\dfrac{1}{x} \leq 4x$. **3**

Question 2 (12 Marks) Commence a NEW page. **Marks**

(a) Find x, y and z in the diagram, giving all reasons and showing all working. **3**



(b) On a number plane the points $A(-5,-3)$, $B(-2,3)$, $C(10,9)$ and D form a trapezium, in which AB is parallel to DC. AD is parallel to the x axis.



i. Show that the equation of DC is $2x - y - 11 = 0$. **2**

ii. Find the coordinates of the point D. **2**

iii. Find the angle that the line DC makes with the positive x axis. **1**

iv. Find the equation of the circle centred at C with radius BC. **2**

v. Find the coordinates of the point M that divides BC externally in the ratio $2:3$. **2**

Question 3 (12 Marks) Commence a NEW page. **Marks**

(a) Differentiate with respect to x:

i. $f(x) = x\sqrt{x}$. **1**

ii. $g(x) = \left(3x^2 + 4\right)^3$. **2**

iii. $h(x) = \dfrac{x^2}{x^2 + 9}$. **2**

iv. $y = x^5 \left(3 - x^2\right)^8$. Fully factorise your answer to this part. **3**

(b) Evaluate $\displaystyle\lim_{x \to -3} \frac{x^3 + 27}{x + 3}$. **2**

(c) Sketch the curve with the following properties: **2**

- $f'(x) < 0$ and $f''(x) < 0$ when $-2 < x < 0$.
- $f(0) = 0$.
- $f'(x) > 0$ and $f''(x) < 0$ when $0 < x < 2$.

Question 4 (12 Marks) Commence a NEW page. **Marks**

(a) At what points on the curve $y = x^3 - 4x^2 + 2x$ are tangents parallel to the line $2x + y = 3$? **3**

(b) For the curve $y = x^5 - x^4$

i. Find the x intercepts of the curve. **1**

ii. Find and classify all stationary points of the curve. **3**

iii. Find the x coordinates of all points of inflexion. **3**

iv. Hence, sketch the graph, showing all important features. **2**

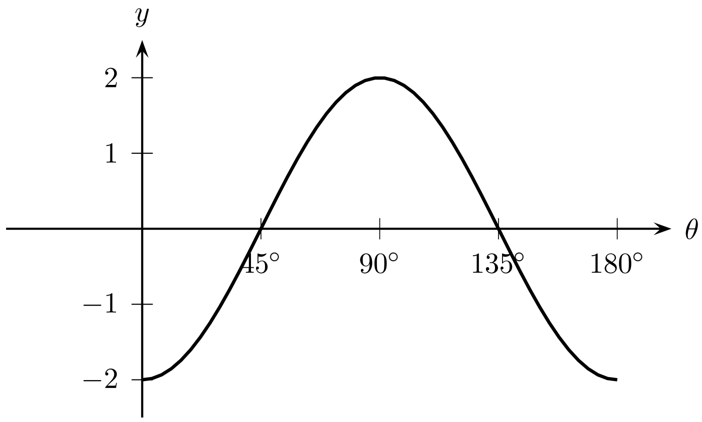
Question 5 (12 Marks) Commence a NEW page. **Marks**

(a) Solve $2\sin^2 x - \sin x = 0$ for $0° \le x \le 360°$. **3**

(b) A ship S sails from a point P on a bearing of N60°E for 56 nm. Ship B leaves port P on a bearing of 110°T for 48 nm. **3**

Calculate the direct distance from S to B, correct to the nearest nautical mile.

(c) The diagram below shows part of a sine or cosine curve between 0° and 180°.

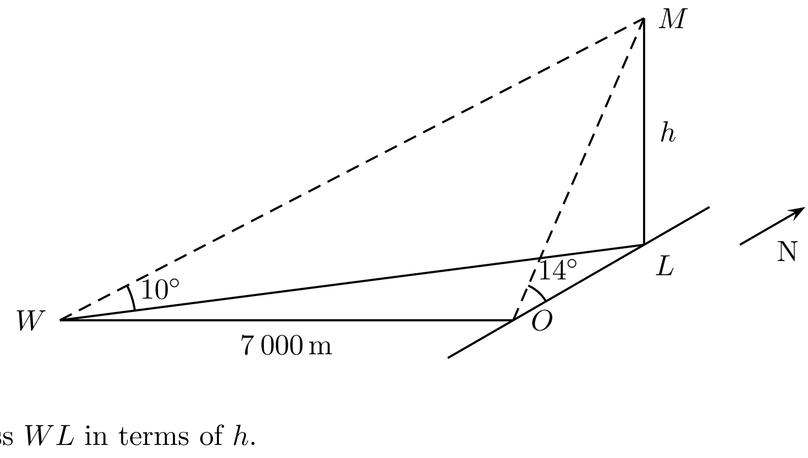


i. State the amplitude a. **1**

ii. State the period T. **1**

iii. Write down the equation of the curve. **1**

(d) The angle of elevation of the summit of a mountain due north of O is 14°. Upon walking 7 000 m due west, it is found to be 10°.

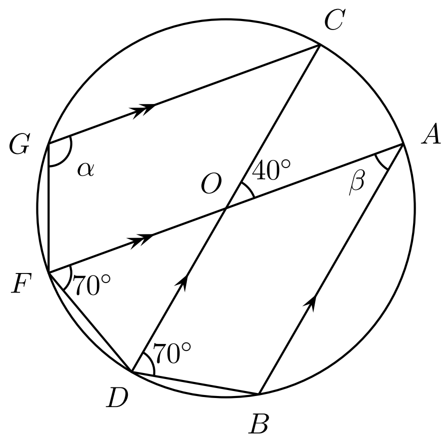


i. Express WL in terms of h. **1**

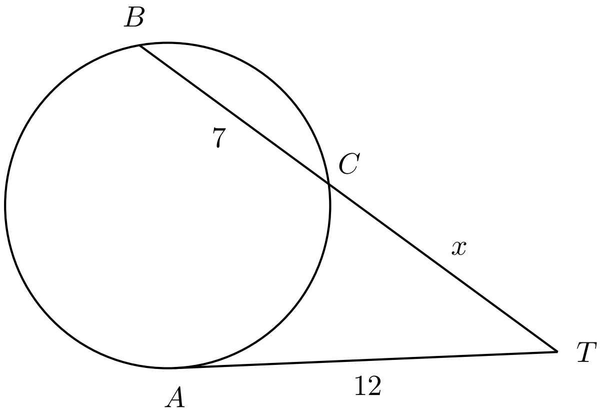
ii. Find the height of the mountain correct to the nearest metre. **2**

Question 6 (12 Marks) Commence a NEW page. **Marks**

(a) Find the value of α and β, giving full reasons. **4**



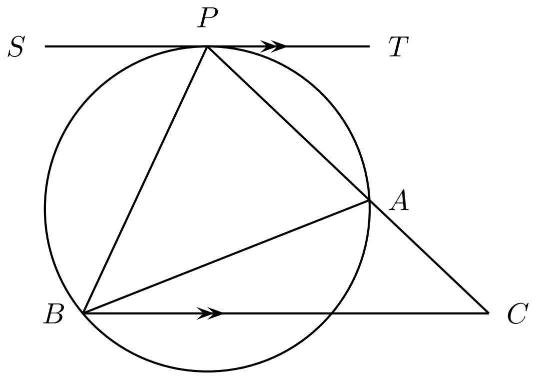
(b) The line AT is the tangent to the circle at A, and BT is a secant meeting the circle at B and C. **2**



Given that $AT = 12$, $BC = 7$ and $CT = x$, find the value of x (giving all reasons).

Question 6 continues on the next page ...

(c) In the diagram, A, P and B are points on the circle. The line PT is tangent to the circle at P, and PA is produced to C so that BC is parallel to PT.



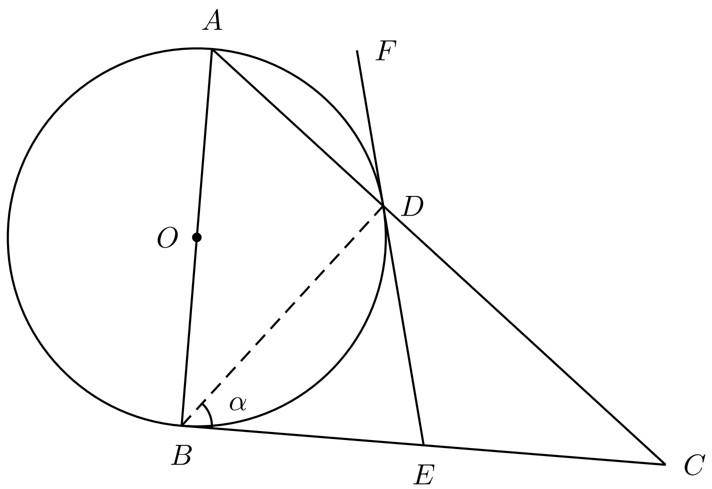
Copy the diagram on to your paper.

i. Show that $\angle PBA = \angle PCB$. **2**

ii. By showing $\triangle PBA \,|||\, \triangle PBC$, deduce that $PB^2 = PA \times PC$. **2**

(d) In the diagram, AB is the diameter of the circle with centre O. BC is a tangent to the circle at B. The line AC is a straight line and intersects the circle at D. The tangent to the circle at D intersects BC at E. Let $\angle EBD = \alpha$. **2**

Reproduce the diagram on to your page.



Prove $\angle EDC = 90^\circ - \alpha$.

Question 7 (12 Marks) Commence a NEW page. **Marks**

(a) i. Show the equation **1**

$$x - 2y - 4 + k(3x - y + 1) = 0$$

(where k is a constant)
can be rewritten as

$$(1 + 3k)x - (2 + k)y + (k - 4) = 0$$

ii. Hence or otherwise, find the equation of a straight line passing through the intersection of $x - 2y - 4 = 0$, $3x - y + 1 = 0$ and perpendicular to the line $4x + 5y + 6 = 0$. **3**

(b) By using the perpendicular distance formula or otherwise, find all possible values of a if $3x + 4y + a = 0$ is a tangent to the circle **3**

$$x^2 + (y + 1)^2 = 9$$

(c) Sketch the region defined by $y \geq \dfrac{1}{x}$. (Hint: be very careful!) **2**

(d) i. Sketch the graph of $y = |x - 2|$. **1**

ii. For what values of x is $|x - 2| < \frac{1}{2}x$? **2**

Question 8 (12 Marks) Commence a NEW page. **Marks**

(a) i. For what values of x does this series have a limiting sum? **2**

$$2x + 3 + (2x + 3)^2 + (2x + 3)^3 + \cdots$$

ii. Find that limiting sum in terms of x. **1**

(b) Solve the equation $2 \log_3 x = \log_3(6 - x)$. Show all necessary working. **3**

(c) i. Rewrite $2^x = 5^y$ with $\frac{x}{y}$ as the subject by taking logarithms to a suitable base. **1**

ii. Hence or otherwise, find the exact value of $8^{\frac{x}{y}}$ if $2^x = 5^y$. **2**

(d) Boxes in a storeroom are stacked in a pile such that there are 25 on the bottom, 22 on the next, 19 afterwards etc, until 116 boxes are on the pile together.

i. How many rows of boxes are there? **2**

ii. How many boxes are there on the top row? **1**

Question 9 (12 Marks) Commence a NEW page. **Marks**

(a) A particle moves in a straight line such that its displacement in meters, x, after t seconds is given by

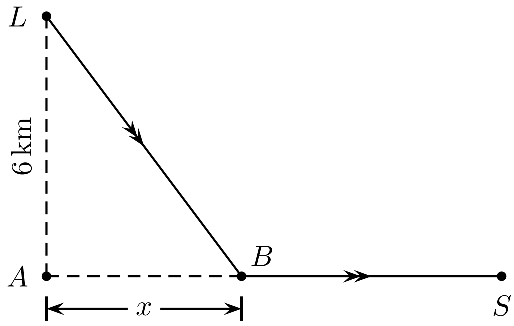
$$x = t^3 + 3t^2 - 9t - 1$$

i. Find the velocity of the particle at any time t. **1**

ii. When and where does the particle first change direction? **2**

iii. What is the average speed of the particle in the first second? **1**

(b) The water's edge is in a straight line ABS which runs east-west. A lighthouse is 6 km due north of A.



10 km due east of A is the general store (S). To travel from the lighthouse (L) to the general store as quickly as possible, the lighthouse keeper rows from L to B, which is x km from A and then jogs to the general store. The lighthouse keeper's rowing speed is 6 km/h and his jogging speed is 10 km/h.

i. Show that it takes the lighthouse keeper $\dfrac{\sqrt{36+x^2}}{6}$ hours to row from the lighthouse to B. **2**

ii. Show that the total time taken for the lighthouse keeper to reach the general store is given by **2**

$$t = \frac{\sqrt{36+x^2}}{6} + \frac{10-x}{10} \text{ hours}$$

iii. Hence by using calculus, show that when $x = 4.5$ km, the time that it takes for the lighthouse keeper to travel from the lighthouse to the general store is a minimum. **3**

iv. Hence find the quickest time it takes to the lighthouse keeper to go to the general store from the lighthouse. Give your answer correct to the nearest minute. **1**

End of paper.

Suggested Solutions

Question 1 (Lam – starts page 2)

(a) (2 marks)

✓ [1] for correct numerical calculation.

✓ [1] for correct rounding.

$$\frac{\sqrt[3]{2.54\times 6.78}}{\pi^3} = 0.083\,3 \text{ (3 s.f.)}$$

(b) (2 marks)

✓ [1] for multiplying by $\frac{\times 2+\sqrt{3}}{\times 2+\sqrt{3}}$.

✓ [1] for correct final answer.

$$\begin{aligned}\frac{\sqrt{3}+1}{2-\sqrt{3}}{}_{\times 2+\sqrt{3}}^{\times 2+\sqrt{3}} &= \frac{(\sqrt{3}+1)(2+\sqrt{3})}{4-3}\\ &= 2\sqrt{2} + \left(\sqrt{3}\right)^2 + 2 + \sqrt{3}\\ &= 5+3\sqrt{3}\\ \therefore a = 5 \qquad & b = 3\end{aligned}$$

(c) (2 marks)

✓ [1] for correct exact ratios of $\sin 60^\circ$ and $\cos 30^\circ$.

✓ [1] for final answer.

$$\begin{aligned}2\sin 60^\circ \cos 30^\circ &= \not{2}\times\frac{\sqrt{3}}{\not{2}}\times\frac{\sqrt{3}}{2}\\ &= \frac{3}{2}\end{aligned}$$

(d) (3 marks)

✓ [1] for equating the given equations.

✓ [1] for x values.

✓ [1] for y values.

$$\begin{cases} y = 2x \\ y = x^2 - 15\end{cases}$$

$$\begin{aligned}x^2 - 15 &= 2x\\ x^2 - 2x - 15 &= 0\\ (x-5)(x+3) &= 0\\ \therefore x &= 5, -3\\ \therefore y &= 10, -6\end{aligned}$$

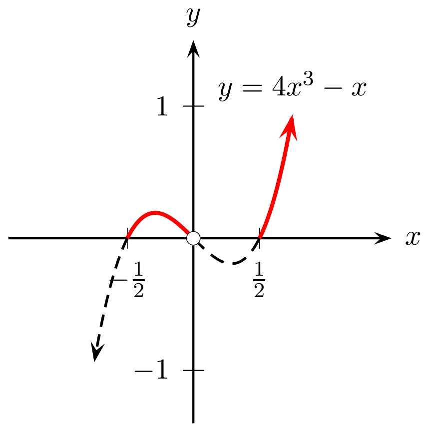
(e) (3 marks)

✓ [1] for multiplying throughout by x^2.

✓ [1] for correct $-\frac{1}{2} \le x < 0$.

✓ [1] for correct $x \ge \frac{1}{2}$.

$$\begin{aligned}\underset{\times x^2}{\frac{1}{x}} &\le \underset{\times x^2}{4x}\\ x &\le 4x^3\\ 4x^3 - x &\ge 0\\ x(4x^2-1) &\ge 0\\ x(2x-1)(2x+1) &\ge 0\end{aligned}$$

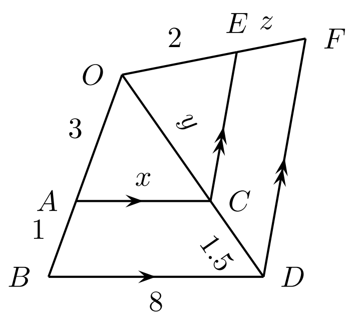


$$\therefore -\frac{1}{2} \le x < 0 \text{ or } x \ge \frac{1}{2}$$

Question 2 (Lam – starts page 2)

(a) (3 marks)

✓ [1] for each correct pronumeral found with reasons. If insufficient reasoning is given, a maximum of one mark will be lost.



- By similar $\triangle OAC$ and $\triangle OBD$,

$$\frac{OA}{OB} = \frac{AC}{BD}$$
$$\frac{3}{4} = \frac{x}{8}$$
$$\therefore x = 6$$

- By the intercepts of transversals,

$$\frac{OA}{OB} = \frac{OC}{OD} = \frac{OE}{EF}$$
$$\therefore \frac{3}{1} = \frac{y}{1.5} = \frac{2}{z}$$

Take the first pair of fractions,

$$y = 1.5 \times 3 = 4.5$$

Take the first and third pair of fractions,

$$\frac{2}{z} = 3$$
$$z = \frac{2}{3}$$

$\therefore x = 6,\ y = 4.5,\ z = \frac{2}{3}$

(b) i. (2 marks)

✓ [1] for realising $m_{DC} = m_{AB}$ and finding m_{AB}.

✓ [1] for showing $y = 2x - 11$ or $2x - y - 11 = 0$.

As $AB \parallel DC$, then $m_{AB} = m_{DC}$.

$$\begin{aligned} m_{DC} &= m_{AB} \\ &= \frac{3-(-3)}{-2-(-5)} = \frac{6}{3} = 2 \end{aligned}$$

Apply the point-gradient formula with $C(10, 9)$:

$$y - y_1 = m(x - x_1)$$
$$y - 9 = 2(x - 10)$$
$$y = 2x - 20 + 9$$
$$y = 2x - 11$$
$$2x - y - 11 = 0$$

ii. (2 marks)

✓ [1] for using y coordinate is -3.

✓ [1] for $D(4, -3)$.

D has the same y coordinate as $A(-5, -3)$. Hence

$$2x - (-3) - 11 = 0$$
$$2x + 3 - 11 = 0$$
$$2x = 8$$
$$x = 4$$
$$\therefore D(4, -3)$$

iii. (1 mark)

✓ [Note:] Accept 63°, 63.43°, $63^\circ 26'$.

$$m_{DC} = 2 = \tan\theta$$
$$\theta = \tan^{-1} 2 = 63.43^\circ$$

iv. (2 marks)

✓ [1] for correct distance of BC.

✓ [1] for final answer.

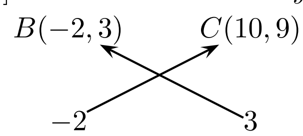
$$\begin{aligned} d_{BC} &= \sqrt{(10-(-2))^2 + (9-3)^2} \\ &= \sqrt{12^2 + 6^2} = \sqrt{180} \end{aligned}$$

$\therefore$ equation of circle is

$$(x - 10)^2 + (y - 9)^2 = 180$$

v. (2 marks)

✓ [1] for each correct x and y value.



$$\begin{aligned} & M\left(\frac{\ell x_1 + k x_2}{k + \ell}, \frac{\ell y_1 + k y_2}{k + \ell}\right) \\ &= \left(\frac{3(-2) + (-2)(10)}{-2+3}, \frac{3(3) + (-2)(9)}{-2+3}\right) \\ &= (-6 - 20, 9 - 18) = (-26, -9) \end{aligned}$$

Question 3 (Ireland – starts on page 3)

(a) i. (1 mark)

$$f(x) = x\sqrt{x} = x^{\frac{3}{2}}$$
$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$$

ii. (2 marks)

✓ [1] for applying chain rule.
✓ [1] for final answer.

$$g(x) = \left(3x^2 + 4\right)^3$$
$$\left|\begin{array}{ll} g(u) = u^3 & u(x) = 3x^2 + 4 \\ g'(u) = 3u^2 & u'(x) = 6x \end{array}\right.$$
$$\begin{aligned} g'(x) &= g'(u) \times u'(x) \\ &= 3u^2 \times 6x \\ &= 3\left(3x^2 + 4\right)^2 \times 6x \\ &= 18x\left(3x^2 + 4\right)^2 \end{aligned}$$

iii. (2 marks)

✓ [1] for applying quotient rule.
✓ [1] for final answer.

$$h(x) = \frac{x^2}{x^2 + 9}$$
$$\left|\begin{array}{ll} u = x^2 & v = x^2 + 9 \\ u' = 2x & v' = 2x \end{array}\right.$$
$$\begin{aligned} h'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{2x(x^2 + 9) - 2x(x^2)}{(x^2 + 9)^2} \\ &= \frac{18x}{(x^2 + 9)^2} \end{aligned}$$

iv. (3 marks)

✓ [1] for applying product rule.
✓ [1] for applying chain rule.
✓ [1] for final fully factorised answer.

$$y = x^5\left(3 - x^2\right)^8$$
$$\left|\begin{array}{ll} u = x^5 & v = \left(3 - x^2\right)^8 \\ u' = 5x^4 & v' = 8 \times (-2x) \times \left(3 - x^2\right)^7 \\ & = -16x\left(3 - x^2\right)^7 \end{array}\right.$$
$$\begin{aligned} \frac{dy}{dx} &= uv' + vu' \\ &= -16x^6\left(3 - x^2\right)^7 + 5x^4\left(3 - x^2\right)^8 \\ &= x^4\left(3 - x^4\right)^7\left(-16x^2 + 5\left(3 - x^2\right)\right) \\ &= x^4\left(3 - x^4\right)^7\left(-21x^2 + 15\right)) \\ &= 3x^4\left(3 - x^4\right)^7\left(-7x^2 + 5\right)) \end{aligned}$$

(b) (2 marks)

✓ [1] for correct factorisation of numerator.
✓ [1] for final answer.

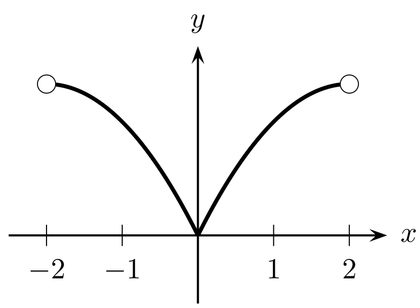
$$\begin{aligned} &\lim_{x \to -3} \frac{x^3 + 27}{x + 3} \\ &= \lim_{x \to -3} \frac{\cancel{(x+3)}(x^2 - 3x + 9)}{\cancel{x+3}} \\ &= \lim_{x \to -3} x^2 - 3x + 9 \\ &= (-3)^2 - 3(-3) + 9 \\ &= 27 \end{aligned}$$

(c) (2 marks)

✓ [1] for shape.
✓ [1] for $f(0) = 0$.

- $f'(x) < 0$ & $f''(x) < 0$ $(-2 < x < 0)$.
- $f(0) = 0$.
- $f'(x) > 0$ & $f''(x) < 0$ $(0 < x < 2)$.

One possibility:



Question 4 (Ireland – starts on page 3)

(a) (3 marks)

- ✓ [1] for correctly differentiating.
- ✓ [1] solve for x; obtaining $x = 2$ & $x = \frac{2}{3}$.
- ✓ [1] for finding corresponding y values.

$$y = x^3 - 4x^2 + 2x$$
$$\frac{dy}{dx} = 3x^2 - 8x + 2$$

The gradient of the line $2x + y = 3$ is -2. The tangent to the curve is parallel to the line when $\frac{dy}{dx} = -2$:

$$3x^2 - 8x + 2 = -2$$
$$3x^2 - 8x + 4 = 0$$
$$(3x - 2)(x - 2) = 0$$
$$\therefore x = 2, \frac{2}{3}$$

When $x = 2$,

$$y = 2^3 - 4\left(2^2\right) + 2(2)$$
$$= 8 - 16 + 4 = -4$$

When $x = \frac{2}{3}$,

$$y = \left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right) = -\frac{4}{27}$$

Hence the points on the curve parallel to $2x + y = 3$ are

$$(2, -4) \quad \& \quad \left(\frac{2}{3}, -\frac{4}{27}\right)$$

(b) i. (1 mark)

$$x^5 - x^4 = 0$$
$$x^4(x - 1) = 0$$
$$\therefore x = 0, 1$$

ii. (3 marks)

- ✓ [1] for correct differentiation.
- ✓ [1] for concluding $(0, 0)$ is a local max.
- ✓ [1] for concluding $\left(\frac{4}{5}, -\frac{256}{3125}\right)$ is a local minimum.
- ✓ [Note:] maximum 1 mark will be lost for not finding the y coordinates.

$$\frac{dy}{dx} = 5x^4 - 4x^3$$

Stationary pts occur when $\frac{dy}{dx} = 0$:

$$5x^4 - 4x^3 = 0$$
$$x^3(5x - 4) = 0$$
$$\therefore x = 0, \frac{4}{5}$$

- At $x = 0$, $y = 0$.
- At $x = \frac{4}{5}$,

$$y = \left(\tfrac{4}{5}\right)^5 - \left(\tfrac{4}{5}\right)^4 = -\frac{256}{3125}$$

x	-1	0	$\frac{1}{2}$	$\frac{4}{5}$	1
y'	$+$	0	$-$	0	$+$
y	↗	0	↘	$-\frac{256}{3125}$	↗

Hence $x = 0$ is a local maximum and $x = \frac{4}{5}$ is a local minimum.

iii. (3 marks)

- ✓ [1] for correct differentiation.
- ✓ [1] for $x = \frac{3}{5}$ being a point of inflexion.
- ✓ [1] testing possible inflexions to eliminate $x = 0$ and confirm $x = \frac{3}{5}$.

$$\frac{d^2y}{dx^2} = 20x^3 - 12x^2$$

Pts of inflexion occur when $\frac{d^2y}{dx^2} = 0$:

$$20x^3 - 12x^2 = 0$$
$$4x^2(5x - 3) = 0$$
$$\therefore x = 0, \frac{3}{5}$$

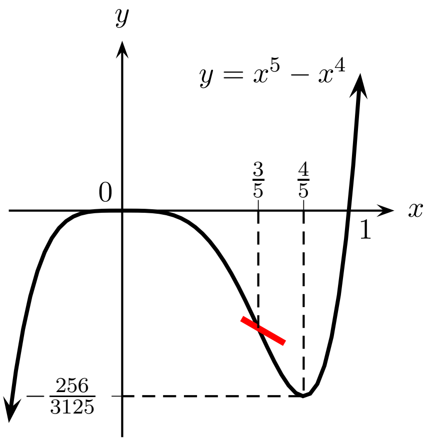
x	-1	0	$\frac{1}{2}$	$\frac{3}{5}$	1
y''	$-$	0	$-$	0	$+$
y'	⌢		⌢		⌣

As the sign of the second derivative does not change around $x = 0$, it is therefore not a point of inflexion despite $f''(0) = 0$. Hence the only point of inflexion occurs at $x = \frac{3}{5}$.

iv. (2 marks)

✓ [2] for correct shape with details, provided it follows on from previous working.

✓ [−1] for each error. If the arrowheads of the end of the curve point in the same direction, deduct [1].



Question 5 (Fletcher – starts on page 4)

(a) (3 marks)

✓ [1] for factorising to $\sin x(2\sin x - 1) = 0$.

✓ [1] each for the solutions: $x = 0°, 180°, 360°$.

✓ [1] each for the solutions: $x = 30°, 150°$.

$$2\sin^2 x - \sin x = 0$$
$$\sin x(2\sin x - 1) = 0$$

$\sin x = 0$	$\sin x = \frac{1}{2}$
$x = 0°, 180°, 360°$	$x = 30°, 150°$

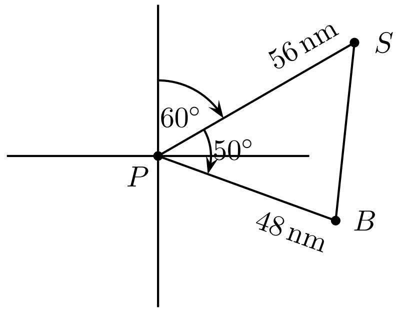
$$\therefore x = 0°, 30°, 150°, 180°, 360°$$

(b) (3 marks)

✓ [1] for correct $\triangle PSB$ with $\angle SPB = 50°$.

✓ [1] for applying cosine rule.

✓ [1] for final answer. Do not penalise for rounding.



$$\angle SPB = 110° - 60° = 50°$$

Apply the cosine rule in $\triangle SPB$,

$$SB^2 = 56^2 + 48^2 - 2(56)(48)\cos 50°$$
$$= 1\,984.37$$
$$\therefore SB = 44.55\cdots = 45\,\text{nm (nearest nm)}$$

(c) i. (1 mark) $a = 2$

ii. (1 mark) $T = 180°$

iii. (1 mark) $y = -2\cos 2\theta$

(d) i. (1 mark)

$$\frac{h}{WL} = \tan 10°$$
$$\therefore WL = \frac{h}{\tan 10°} \qquad (5.1)$$

ii. (2 marks)

✓ [1] for obtaining $OL^2 + 7\,000^2 = WL^2$.

✓ [1] for final answer.

$$\frac{h}{OL} = \tan 14°$$
$$\therefore OL = \frac{h}{\tan 14°} \qquad (5.2)$$

In $\triangle WOL$, $\angle WOL = 90°$ as W is due west of O. Hence

$$OL^2 + 7\,000^2 = WL^2 \qquad (5.3)$$

Substitute (5.1) and (5.2) into (5.3)

$$\begin{aligned} WL^2 - OL^2 &= 7\,000^2 \\ \left(\frac{h}{\tan 10^\circ}\right)^2 - \left(\frac{h}{\tan 14^\circ}\right)^2 &= 7\,000^2 \\ h^2\left(\frac{1}{\tan^2 10^\circ} - \frac{1}{\tan^2 14^\circ}\right) &= 7\,000^2 \\ h^2\left(\frac{\tan^2 14^\circ - \tan^2 10^\circ}{\tan^2 10^\circ \tan^2 14^\circ}\right) &= 7\,000^2 \\ h^2 &= \frac{7\,000^2 \tan^2 10^\circ \tan^2 14^\circ}{\tan^2 14^\circ - \tan^2 10^\circ} \\ h = 1\,745.8\,\text{m} &= 1\,746\,\text{m (nearest m)} \end{aligned}$$

Question 6 (Berry – starts on page 5)

(a) (4 marks)

✓ [1] for each correct pronumeral found.

✓ [1] for working & leading to correct answer.

- To find α:
  - $\angle FOD = 40^\circ$ (vert. opp)
  - $\angle DOA = 180^\circ - 40^\circ = 140^\circ$ (supplementary)
  - $\angle FOC$ (reflex) $= 140^\circ + 80^\circ = 220^\circ$.
  - $\therefore \alpha = 110^\circ$ (angle at the circumference is half the angle at the centre subtended by the same arc)
- To find β:
  - $\beta = 40^\circ$ (alternate $\angle$, $OC \parallel AB$).
  - Alternatively, use $\angle ODF = 70^\circ$ (base $\angle$ of isos $\triangle$ and $\beta = 180^\circ - 140^\circ - 40^\circ$ (opp $\angle$ of cyclic quad).

(b) (2 marks)

✓ [1] for correctly applying tangent-secant theorem.

✓ [1] for solving and obtaining $x = 9$.

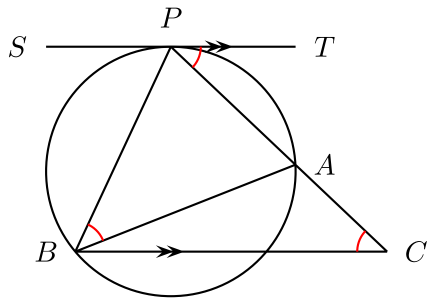
- $x(x+7) = 12^2$ (tangent-secant thm)

$$\begin{aligned} x^2 + 7x - 144 &= 0 \\ (x-9)(x+16) &= 0 \\ x &= 9, -16 \end{aligned}$$

- $x = -16$ is not a valid solution as x is a length. Hence $x = 9$.

(c) i. (2 marks)

✓ [1] for each correct reason.



- $\angle TPC = \angle PCB$ (alternate $\angle$, $PT \parallel BC$)
- $\angle TPC = \angle PBA$ ($\angle$ in alternate segment)

$\therefore \angle PBA = \angle PCB$.

ii. (2 marks)

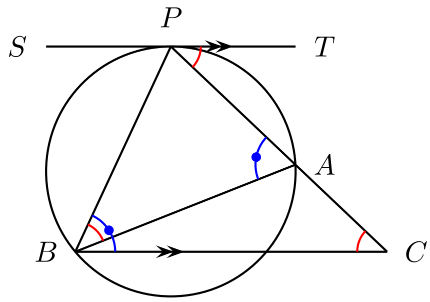
✓ [1] for showing $\triangle PBA \,|||\, \triangle PCB$ correctly.

✓ [1] for $\frac{PB}{PC} = \frac{PA}{PB}$, leading correct solution.

In $\triangle PBA$ & $\triangle PBC$

- $\angle PBA = \angle PBC$ (previously proven)
- $\angle BPA$ is common.
- By the $\angle$ sum of $\triangle$, $\angle PAB = \angle PBC$ (remaining $\angle$)

$\therefore \triangle PBA \,|||\, \triangle PBC$ (equiangular).



Hence the ratio of corresponding sides are equal, i.e.

$$\frac{PB}{PC} = \frac{PA}{PB}$$
$$\therefore PB^2 = PA \times PC$$

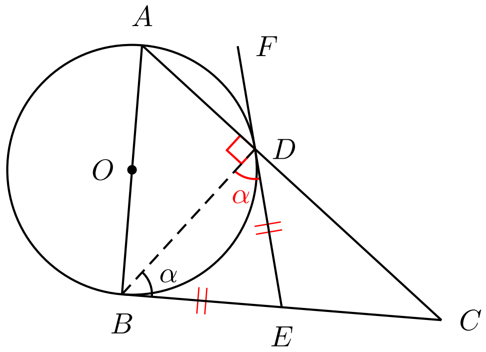
(d) (2 marks)

✓ [1] correct conclusion with proper reasoning.

✓ [1] correctly showing intermediate steps.

✓ [−1] for each important step skipped.

NB. there are a number of methods to prove what is required. These solutions only show one of them.



- $\angle ADB = 90^\circ$ ($\angle$ in a semicircle)
- $\therefore BDC = 90^\circ$ (supplementary)
- $\triangle EBD$ is isosceles as $EB = ED$ (tangents from an external pt)
- $\therefore \angle EDB = \alpha$.
- $\therefore \angle EDC = 90^\circ - \alpha$.

Question 7 (Rezcallah – starts on page 7)

(a) i. (1 mark)

$$\begin{aligned} x - 2y - 4 + k(3x - y + 1) &= 0 \\ x - 2y - 4 + 3kx - ky + k &= 0 \\ x(1 + 3k) - y(2 + k) + (k - 4) &= 0 \quad (7.1) \end{aligned}$$

ii. (3 marks)

✓ [1] for $m_\perp = \frac{5}{4} = \frac{1+3k}{2+k}$.

✓ [1] for $k = \frac{6}{7}$.

✓ [1] for $25x - 20y - 22 = 0$.

- $4x+5y+6 = 0$ has gradient $m = -\frac{4}{5}$. Hence the perpendicular to it will have gradient $m_\perp = \frac{5}{4}$.
- The gradient of $x(1+3k)-y(2+k)+(k-4) = 0$ is

$$m = \frac{1+3k}{2+k}$$

- The line required will have gradient

$$\begin{aligned} m = \frac{1+3k}{2+k} &= \frac{5}{4} \\ \frac{1+3k}{2+k} &= \frac{5}{4} \\ \underset{-4}{4} + \underset{-5k}{12k} &= \underset{-4}{10} + \underset{-5k}{5k} \\ 7k &= 6 \\ k &= \frac{6}{7} \end{aligned}$$

- Substitute $k = \frac{6}{7}$ to (7.1)

$$\begin{aligned} x\left(\tfrac{7}{7} + \tfrac{18}{7}\right) - y\left(\tfrac{14}{7} + \tfrac{6}{7}\right) + \left(\tfrac{6}{7} - \tfrac{28}{7}\right) &= 0 \\ 25x - 20y - 22 &= 0 \end{aligned}$$

Alternative method exists: find points of intersection to obtain $\left(-\frac{6}{5}, -\frac{13}{5}\right)$. $m = \frac{5}{4}$. Apply to point gradient formula.

(b) (3 marks)

✓ [1] for correctly applying $\perp$ dist formula with $d = 3$ and $(x_1, y_1) = (0, -1)$. Max [1] awarded for entire part if incorrect formula used.

✓ [1] for correctly solving absolute value equation (with 2 cases)

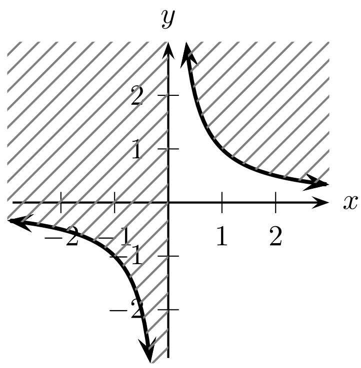
✓ [1] for both values of a.

If $3x + 4y + a = 0$ is a tangent to the circle $x^2 + (y+1)^2 = 9$ (circle with $C(0, -1)$ and $r = 3$), then the radius will be perpendicular to the equation at two possible locations.

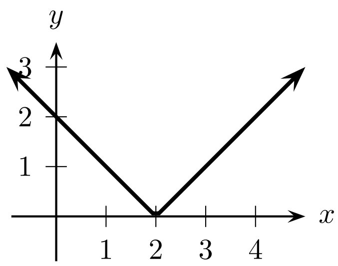
$$\begin{aligned} d_\perp &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ 3 &= \frac{|3(0) + 4(-1) + a|}{\sqrt{3^2 + 4^2}} \\ \frac{|-4 + a|}{5} &= 3 \\ |a - 4| &= 15 \\ a - 4 &= \pm 15 \\ a &= 19, -11 \end{aligned}$$

(c) (2 marks)

✓ [1] for each correct region.



(d) i. (1 mark)



ii. (2 marks)

✓ [1] for correct method leading to correct answer.

✓ [1] for obtaining correct interval.

$$|x-2| < \frac{1}{2}x$$

$x - 2 < \frac{1}{2}x$	$-(x-2) < \frac{1}{2}x$
$\frac{1}{2}x < 2$	$-x + 2 < \frac{1}{2}x$
$x < 4$	$\frac{3}{2}x > 2$
	$x > \frac{4}{3}$

$$\therefore \frac{4}{3} < x < 4$$

Alternative method exists: find the points of intersection and use a sketch of $y = \frac{1}{2}x$ and $y = |x-2|$ to determine the correct interval.

Question 8 (Trenwith – starts on page 7)

(a) i. (2 marks)

✓ [1] for $x > -2$ & $x < -1$.

✓ [1] for combining the inequalities $(-2 < x < -1)$

$$r = 2x + 3$$

A limiting sum exists when $|r| < 1$, i.e.

$$\begin{aligned} \underset{-3}{-1} < 2x + &\underset{-3}{3} < \underset{-3}{1} \\ -4 < 2x &< -2 \\ -2 < x &< -1 \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{2x+3}{1-(2x+3)} \\ &= \frac{2x+3}{-2x-2} \end{aligned}$$

(b) (3 marks)

✓ [2] for $x = 2, -3$.

✓ [1] for discarding $x = -3$ as it is invalid.

$$\begin{aligned} 2\log_3 x &= \log_3(6-x) \\ \log_3 x^2 &= \log_3(6-x) \\ x^2 &= 6-x \\ x^2 + x - 6 &= 0 \\ (x-2)(x+3) &= 0 \\ \therefore x &= 2, -3 \end{aligned}$$

When $x = -3$, $2\log_3 -3$ is not defined.

$$\therefore x = 2$$

(c) i. (1 mark)

$$\begin{aligned} 2^x &= 5^y \\ \underset{\div y}{x} \log_2 2 &= \underset{\div y}{y} \log_2 5 \\ \therefore \frac{x}{y} &= \log_2 5 \end{aligned}$$

ii. (2 marks)

✓ [1] for using $\frac{x}{y}$ as the exponent base 2.

✓ [1] for final answer.

$$\begin{aligned}\frac{x}{y} &= \log_2 5\\ 2^{x/y} &= 2^{\log_2 5} = 5\\ \left(2^3\right)^{\frac{x}{y}} &= \left(2^{\frac{x}{y}}\right)^3 = 5^3 = 125\end{aligned}$$

(d) i. (2 marks)

✓ [1] for $3n^2 - 53n + 232 = 0$.

✓ [1] for $n = 8$ and justifying $n = \frac{29}{3}$ is invalid. NB. This mark not awarded if $n = 8$ is obtained by guessing/with insufficient working/writing out terms of AP to find 8th term.

$$25 + 22 + 19 + \cdots$$
$$a = 25 \quad d = -3 \quad n = ? \quad S_n = 116$$

Sum of AP formula:

$$\begin{aligned}S_n &= \frac{n}{2}(2a + d(n-1))\\ 116 &= \frac{n}{2}(2 \times 50 - 3(n-1))\\ 232 &= n(50 - 3n + 3)\\ 232 &= n(53 - 3n)\\ 3n^2 - 53n + 232 &= 0\end{aligned}$$

Apply the quadratic formula,

$$\begin{aligned}n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{53 \pm \sqrt{53^2 - 4(3)(232)}}{2 \times 3} = \frac{53 \pm \sqrt{25}}{6}\\ &= 8, \frac{29}{3}\end{aligned}$$

As the number of rows an integer, $n = \frac{29}{3}$ is not possible. (This question is not asking what number of rows will exceed a certain sum, but rather asking for the exact number of rows). Hence $n = 8$.

ii. (1 mark)

$$\begin{aligned}T_n &= a + d(n-1)\\ T_8 &= 25 - 3(8-1)\\ &= 25 - 21 = 4\end{aligned}$$

There are 4 boxes on the top row.

Question 9 (Weiss – starts on page 8)

(a) i. (1 mark)

$$\begin{aligned}x(t) &= t^3 + 3t^2 - 9t - 1\\ x'(t) &= 3t^2 + 6t - 9\end{aligned}$$

ii. (2 marks)

✓ [1] for $t = -3, 1$ and discarding $t = -3$.

✓ [1] for finding $x = -6\,\text{m}$.

The particle changes direction when $x'(t) = 0$.

$$\begin{aligned}3t^2 + 6t - 9 &= 0\\ 3(t^2 + 2t - 3) &= 0\\ (t+3)(t-1) &= 0\\ \therefore t &= 1, -3\end{aligned}$$

As $t \geq 0$, then $t = -3$ is not valid.

$$x(1) = 1 + 3 - 9 - 1 = -6\,\text{m}$$

iii. (1 mark)

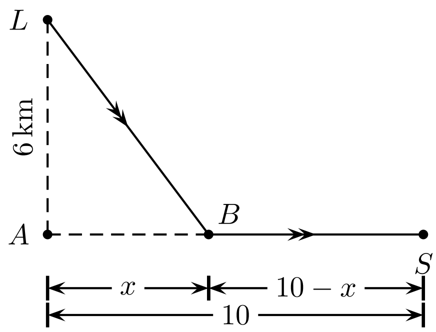
$$x(1) = -6 \qquad x(0) = -1$$

$$\begin{aligned}\text{average speed} &= \left|\frac{x(t_2) - x(t_1)}{t_2 - t_1}\right|\\ &= \left|\frac{(-6) - (-1)}{1 - 0}\right|\\ &= 5\,\text{ms}^{-1}\end{aligned}$$

(b) i. (2 marks)

✓ [1] for $LB = \sqrt{x^2 + 6^2}$.

✓ [1] for $t = \frac{\sqrt{x^2+6^2}}{6}$.



In $\triangle LAB$,

$$AL^2 + AB^2 = LB^2$$
$$LB = \sqrt{x^2 + 6^2}$$

As the keeper's rowing speed is 6 km/h,

$$s = \frac{d}{t}$$
$$6 = \frac{\sqrt{x^2+6^2}}{t} \quad \Rightarrow \quad t = \frac{\sqrt{x^2+6^2}}{6}$$

ii. (2 marks)

- ✓ [1] for time taken from B to S being $\frac{10-x}{10}$.
- ✓ [1] for $t = \frac{\sqrt{x^2+6^2}}{6} + \frac{10-x}{10}$.

Between B and S,

$$s = \frac{d}{t}$$
$$10 = \frac{10-x}{t} \quad \Rightarrow \quad t = \frac{10-x}{10}$$
$$\therefore t_{\text{total}} = \frac{\sqrt{x^2+6^2}}{6} + \frac{10-x}{10}$$

iii. (3 marks)

- ✓ [1] for correctly differentiating.
- ✓ [1] for $x = \pm\frac{9}{2}$.
- ✓ [1] for discarding invalid solution and verify $x = \frac{9}{2}$ gives a minimum. This mark is lost if the solution is not verified.

Need to minimise t w.r.t. x:

$$t(x) = \frac{1}{6}\left(x^2+6^2\right)^{\frac{1}{2}} + \frac{1}{10}(10-x)$$
$$t'(x) = \frac{1}{\not{2}} \times \frac{1}{6} \times \not{2}x\left(x^2+6^2\right)^{-\frac{1}{2}} - \frac{1}{10}$$
$$= \frac{x}{6\sqrt{x^2+6^2}} - \frac{1}{10}$$

Minimum time occurs when $t'(x) = 0$, i.e.

$$\frac{x}{6\sqrt{x^2+6^2}} - \frac{1}{10} = 0$$
$$\frac{x}{6\sqrt{x^2+6^2}} = \frac{1}{10}$$
$$5x = 3\sqrt{x^2+6^2}$$

Square both sides,

$$25x^2 = 9x^2 + 324$$
$$\underset{\div 16}{16x^2} = \underset{\div 16}{324}$$
$$x^2 = \frac{81}{4}$$
$$\therefore x = \pm\frac{9}{2}$$

As x is a distance, then $x = -\frac{9}{2}$ is not valid. Check that $x = \frac{9}{2}$ is a local minimum:

$$t'(4) = \frac{4}{6\sqrt{4^2+6^2}} - \frac{1}{10} \approx -0.0075\cdots$$
$$t'(5) = \frac{5}{6\sqrt{5^2+6^2}} - \frac{1}{10} \approx 0.0066\cdots$$

x	4	$\frac{9}{2}$	5
t'	$-$	0	$+$
t	↘	min	↗

iv. (1 mark)

$$x = \frac{9}{2}$$
$$t = \frac{\sqrt{36+4.5^2}}{6} + \frac{10-4.5}{10}$$
$$= \frac{9}{5} = 1.8\,\text{h}$$

Hence he reaches the store in 1 hr 48 min.