Mrs Greenberg
Mrs Israel
Mrs Kerr
Ms Lau
Mrs Millar
Mrs Squires

Name: ……………………………….………..

Teacher:……………………………………….

Pymble Ladies' College

PRELIMINARY YEARLY EXAMINATION
2017

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using pencil for Questions 1-10.
- Write using black or blue pen for Questions 11-14. Black pen is preferred.
- Board approved calculators may be used.
- Diagrams are not drawn to scale.
- A reference sheet is provided.
- In Questions 11-14, show relevant mathematical reasoning and/or calculations.

Total Marks – 61

Section I Pages 1-4

10 marks

- Attempt all Questions 1-10
- Allow about 15 mins for this section

Section II Pages 5-9

51 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Mark	**/61**
Highest Mark	**/61**

Section I
10 Marks
Attempt all Questions
Use the multiple-choice answer sheet for Questions 1 – 10

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1 Which of the following is equal to $3^x \times 2^x$?

(A) 5^x

(B) 5^{2x}

(C) 6^{2x}

(D) 6^x

2 Which of the following is the correct factorisation of $10y^2 - 19y + 6$?

(A) $(5y-2)(2y-3)$

(B) $(5y-3)(2y-2)$

(C) $(5y-2)(3-2y)$

(D) $(3-5y)(2-2y)$

3 If $f(x) = \dfrac{x+1}{x}$, which of the following is equal to $f\left(\dfrac{1}{\alpha}\right)$?

(A) $1-\alpha$

(B) $\dfrac{\alpha}{\alpha - 1}$

(C) $1+\alpha$

(D) $\dfrac{\alpha - 1}{\alpha}$

4 Which of the following gives the equation of the line perpendicular to $y = 5 - 2x$ and passing through the point $(1, -3)$?

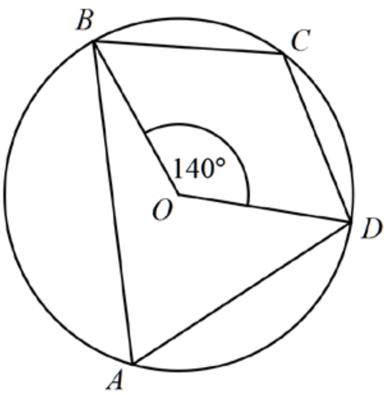
(A) $x - 2y - 7 = 0$

(B) $2x + y + 1 = 0$

(C) $x - 2y + 5 = 0$

(D) $2y + x + 5 = 0$

5 $ABCD$ is a cyclic quadrilateral inscribed in a circle with centre O such that $\angle BOD = 140°$.



Not to Scale

What is the size of $\angle BCD$?

(A) 100°

(B) 110°

(C) 120°

(D) 130°

6 What is the value of $\lim_{a \to 4} \dfrac{a^2 - 16}{a + 4}$?

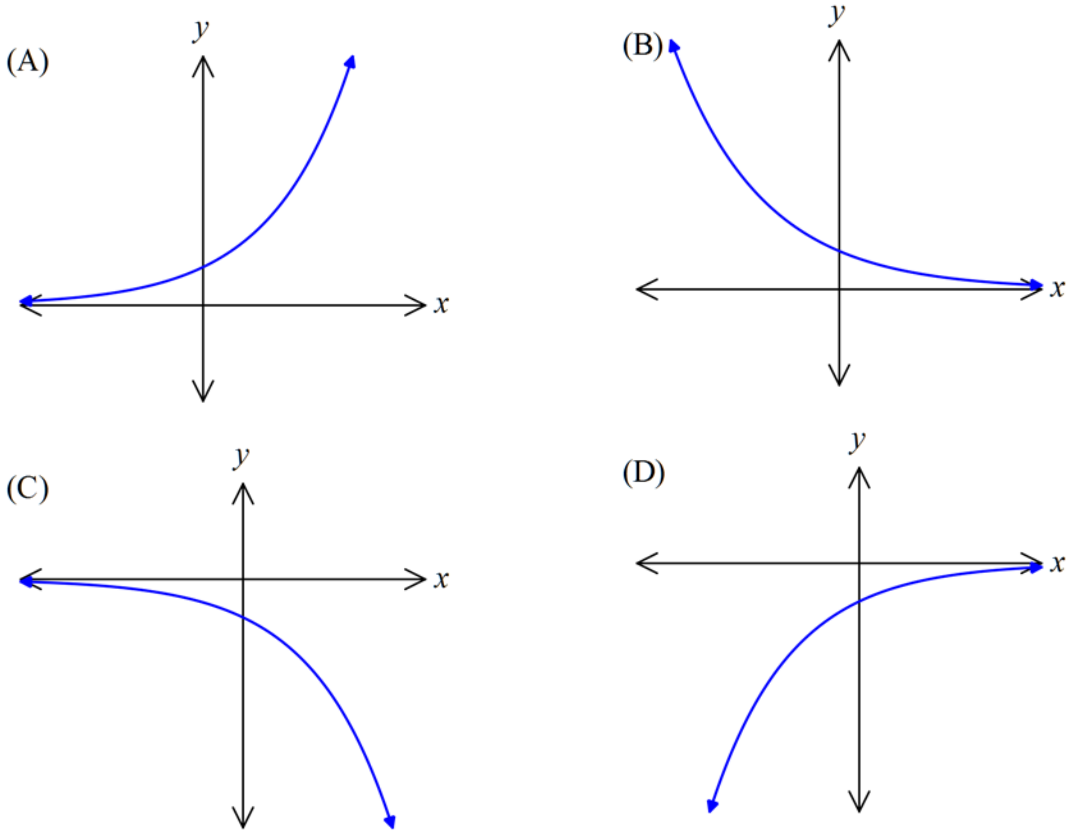
(A) 0

(B) 8

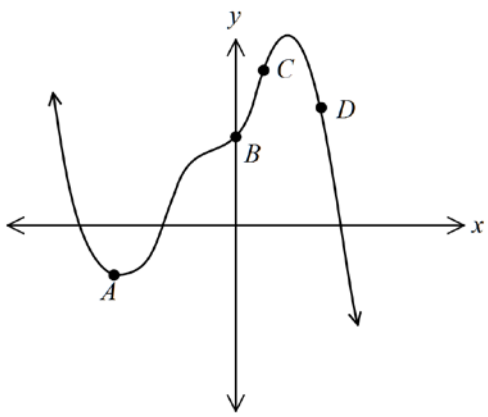
(C) −4

(D) 4

7 Suppose that $f'(x) > 0$ and $f''(x) < 0$ for all real values of x. Which of the following graphs best represents $y = f(x)$?



8 For which point on the graph below is $f(x) > 0$, $f'(x) > 0$ and $f''(x) < 0$?



(A) A

(B) B

(C) C

(D) D

9 It is known that $f''(x) = (x+4)^2(x-3)$.

How many inflexion points does the graph of $y = f(x)$ have?

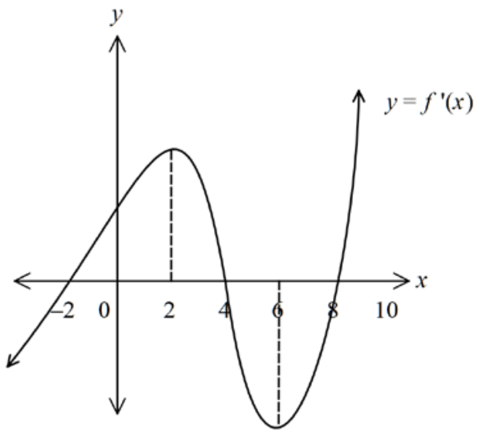
(A) 0

(B) 1

(C) 2

(D) 3

10 The graph of the derivative $y = f'(x)$ is drawn below.



Not to Scale

Where does a maximum turning point occur on $y = f(x)$?

(A) $x = -2$

(B) $x = 2$

(C) $x = 4$

(D) $x = 6$

Section II
51 Marks

Attempt Questions 11 – 14
Allow about 1 hour and 40 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 11 (13 marks)

(a) Simplify $\dfrac{1}{p^2 - pq} - \dfrac{1}{pq - q^2}$. **2**

(b) Find, correct to the nearest degree, the acute angle between the lines $2x + y - 3 = 0$ and $x - 3y + 2 = 0$. **3**

(c) Find the *x* coordinate of the point on the curve $y = x\sqrt{x+3}$ where the tangent is parallel to the *x*-axis. **3**

(d) $A\left(8, \sqrt{200}\right)$ and $B\left(1, \sqrt{72}\right)$ are two points. Find, in simplest exact form, the coordinates of the point *P* which divides the interval *AB* in the ratio 3:1. **3**

(e) Find the domain and range of the function $f\left(x\right) = 4\sqrt{25 - x^2}$. **2**

End of Question 11

Question 12 (11 marks)

(a) Solve the inequality $\frac{2x-1}{x} \geq x$. **3**

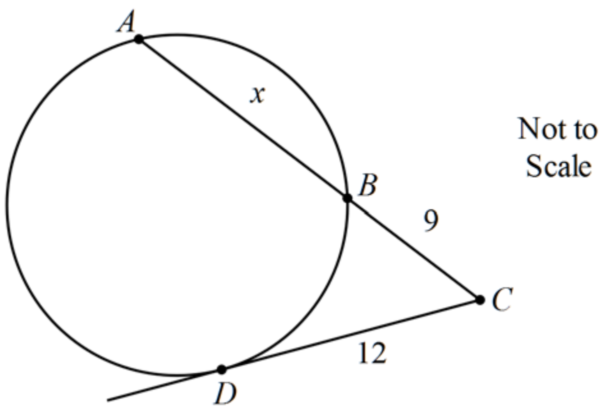
(b) Given the sequence $\sqrt{2}, \sqrt{18}, \sqrt{50}, \ldots\ldots$

(i) Show that the sequence is arithmetic. **1**

(ii) Find the value of the hundredth term. **1**

(iii) Find the sum of the first hundred terms. **1**

(c) **2**



CD is a tangent at D to the circle. $AB = x$, $BC = 9$ and $CD = 12$.

Find the value of x, giving reasons.

(d) Prove by mathematical induction **3**

$$1 + 5 + 9 + \ldots. + (4n - 3) = n(2n - 1) \quad \text{for} \quad n \geq 1.$$

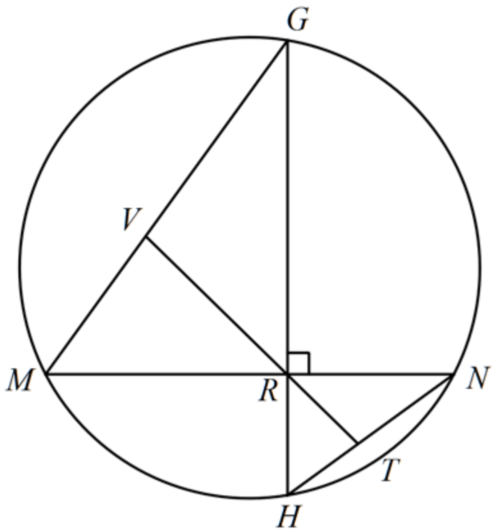
End of Question 12

Question 13 (13 marks)

(a) (i) Find the vertical and horizontal asymptotes of the graph $y = \frac{x-1}{3-x}$ and hence, sketch the graph of $y = \frac{x-1}{3-x}$, showing intercepts with the axes. **4**

(ii) Hence, or otherwise, find the values of x for which $\frac{x-1}{3-x} \le 1$. **2**

(b)



Not to Scale

MN and *GH* are two chords of a circle at right angles to each other, intersecting at *R*. From *R*, a line perpendicular to *HN* is drawn, meeting it at *T*. *TR* produced meets *MG* at *V*.

Prove that:

(i) ΔMVR is isosceles. **2**

(ii) *V* is the mid-point of *MG*. **2**

Note: You do not have to copy the diagram above. It has been reproduced for you on a separate page. Insert this extra page into your answer booklet.

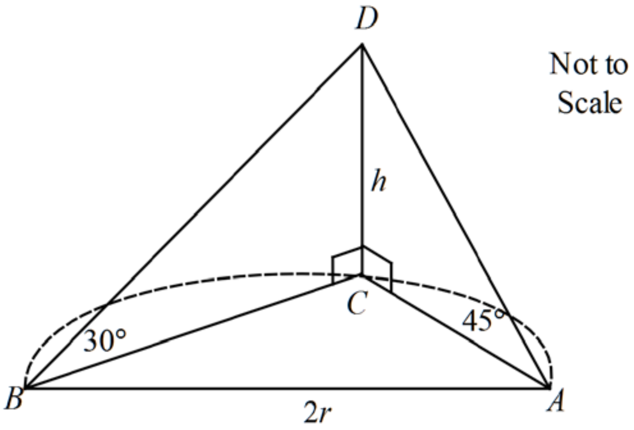
(c) Prove by mathematical induction that $7^n - 1$ is divisible by 3 for all positive integers *n*. **3**

End of Question 13

Question 14 (14 marks)

(a) Solve $4\cos\left(2\alpha - 45°\right) - 2\sqrt{3} = 0$ for the domain $0° \le \alpha \le 360°$. **3**

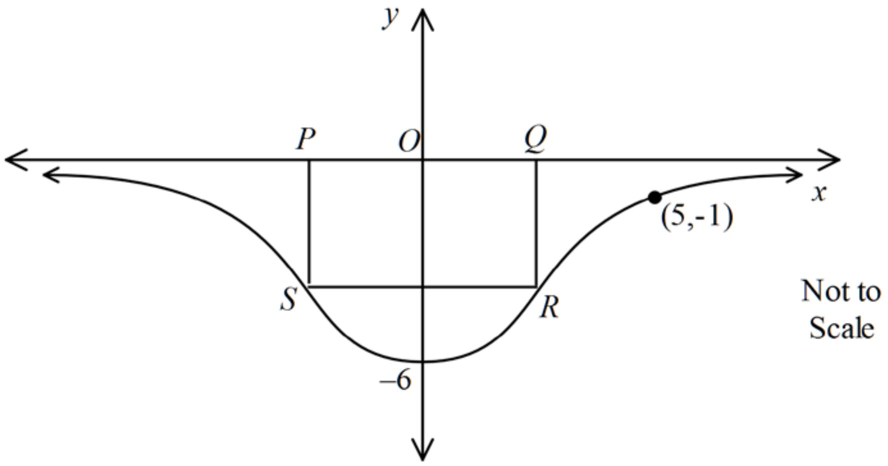
(b) **4**



AB is a diameter of a semicircular piece of horizontal ground with radius *r* metres. *CD* is a vertical flagpole of height *h* metres standing with its base *C* on the arc *AB*. From *A* and *B* the angles of elevation of the top *D* of the flagpole are $45°$ and $30°$ respectively. Show that $h = r$.

Question 14 continues on page 9

(c)



In the diagram above, the curve $y = \dfrac{h}{x^2 + k}$, where h and k are constants, has a minimum turning point at $(0, -6)$ and passes through the point $(5, -1)$. A rectangle $PQRS$ is inscribed within the curve as shown with its axis of symmetry $x = 0$.

(i) Find the values of h and k. **2**

(ii) If Q has coordinates $(\alpha, 0)$ find the coordinates of R in terms of α. **1**

(iii) Show that the area, A, of the rectangle $PQRS$ is $A = \dfrac{60\alpha}{\alpha^2 + 5}$. **1**

(iv) Hence, show that the maximum area of the rectangle $PQRS$ is $6\sqrt{5}$ square units. **3**

End of Paper

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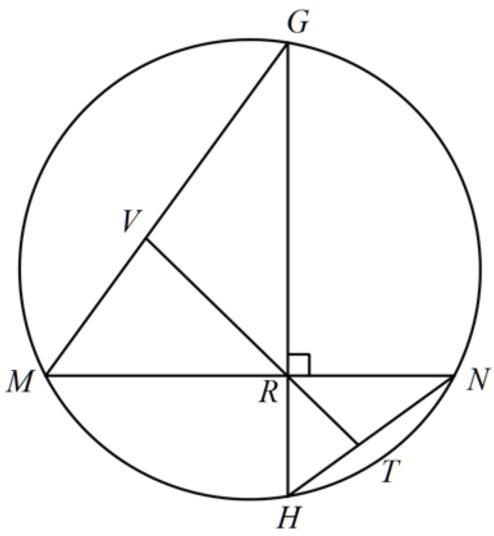
Question 13 (b)

Note: Insert this page into your answer booklet for Question 13.



Not to Scale

2017 : 11 Ext 1 yearly exam : Solutions

multiple choice

1.	D	6.	A.	
2.	A	7.	D.	
3.	C	8.	C	10
4.	A.	9.	B	
5.	B	10.	C	

11 a) $\frac{1}{p(p-q)} - \frac{1}{q(p-q)}$

$= \frac{(q-p)}{pq(p-q)}$

$= \frac{-(p-q)}{pq(p-q)}$

$= \boxed{-\frac{1}{pq}}$

2

factorises denoms	1
correct soln	1

b) $2x+y-3=0 \quad \therefore m_1 = -\frac{a}{b} = -2$

$x-3y+2=0 \quad \therefore m_2 = -\frac{1}{-3} = \frac{1}{3}$

$\therefore \tan\theta = \left| \frac{-2-\frac{1}{3}}{1+(-2)(\frac{1}{3})} \right|$

$= |-7|$

$\therefore \tan\theta = 7$

$\theta = \tan^{-1}7$

$\doteqdot \boxed{82^\circ}$

3

$m_1 = -2$, $m_2 = \frac{1}{3}$	1
correct sub into formula tan θ	1
correct soln	1

question 11

c) $y = x\sqrt{x+3}$

$u = x$
$u' = 1$
$v = \sqrt{x+3}$
$v' = \frac{1}{2}(x+3)^{-\frac{1}{2}}$
$v' = \frac{1}{2\sqrt{x+3}}$

$\therefore \frac{dy}{dx} = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$

$= \frac{2(x+3)+x}{2\sqrt{x+3}}$

$= \frac{3x+6}{2\sqrt{x+3}}$

tang ∥ to x-axis ∴ grad = 0

$\therefore 3x+6=0$

$\boxed{x=-2}$

3	uses prod rule to reach $\frac{dy}{dx} = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$	1
	simplifies to $\frac{dy}{dx} = \frac{3x+6}{2\sqrt{x+3}}$	2
	correct soln	3

d) $(8, \sqrt{200})$ $(1, \sqrt{72})$

3 : 1

$x = \frac{8(1)+3(1)}{4}$

$= \frac{11}{4}$

$y = \frac{\sqrt{200}(1) + 3\sqrt{72}}{4}$

$y = \frac{10\sqrt{2} + 18\sqrt{2}}{4}$

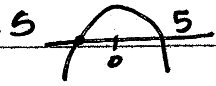
$y = \frac{28\sqrt{2}}{4}$

∴ P is $\left(\frac{11}{4}, 7\sqrt{2}\right)$

3	$x = \frac{8(1)+3(1)}{4}$	1
	$y = \frac{\sqrt{200}(1)+3\sqrt{72}}{4}$	1
	correct simplified soln	1

e) $f(x) = 4\sqrt{25-x^2}$

D: $25 - x^2 \geq 0$



$(5+x)(5-x) \geq 0$

$\therefore \boxed{-5 \leq x \leq 5}$

R: $\boxed{0 \leq y \leq 20}$

2	D: $-5 \leq x \leq 5$	1
	R: $0 \leq y \leq 20$	1

Question 12

a) $\frac{2x-1}{x} \geq x \quad , x \neq 0$

$\times x^2$ $\quad x(2x-1) \geq x^3$ 1

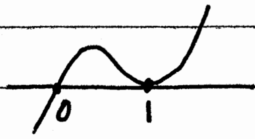
$x(2x-1) - x^3 \geq 0$

$x[2x-1-x^2] \geq 0$

$x(-x^2+2x-1) \geq 0$

$\div -1$ $\quad x(x^2-2x+1) \leq 0$

$x(x-1)^2 \leq 0$ 1



$x \leq 0, \; x = 1$ but $x \neq 0$

$\therefore$ Sol'n: $x < 0, \; x = 1$ 1

~~mult by x^2~~	1
reaches $x(x-1)^2 \leq 0$ or equivalent	1
correct soln	1

Marks: 3

* Alternate sol'n using critical points
① $\times$ by x.
② Eq'n to find 2 sol'ns
③ Test ② correct sol.

b) series $= \sqrt{2} + 3\sqrt{2} + 5\sqrt{2} + \ldots$

* Wrong if straight to decimal → Leave in exact form.

i) $T_2 - T_1 = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$ — 1

$T_3 - T_2 = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

$\therefore$ AP with $a = \sqrt{2}$, $d = 2\sqrt{2}$ — R/W

ii) $T_n = a + (n-1)d$ — 1

$T_{100} = \sqrt{2} + 99(2\sqrt{2})$

$= 199\sqrt{2}$ 1 — R/W

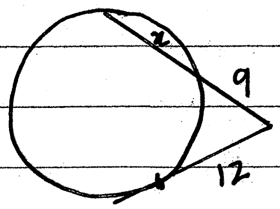
iii) $S_n = \frac{n}{2}[a + l]$

$= \frac{100}{2}[\sqrt{2} + 199\sqrt{2}]$ — 1

$= 50(200\sqrt{2})$ 1

$= 10\,000\sqrt{2}$ — R/W

c)



$9(9+x) = 12^2$
(product of intercepts of secant equals tangent squared)

$81 + 9x = 144$

$\boxed{x = 7}$

2

$9(9-x)=144$	1
reason	1

*Reason must include underlined words or equivalent

d) Let $S(n)$ be the statement

$1+5+9+\ldots+(4n-3) = n(2n-1), \quad n \geq 1$

STEP 1: For $n=1$, LHS $= 1$ RHS $= 1(2(1)-1)$
$= 1$

$\therefore$ LHS = RHS
$\therefore$ $S(1)$ is true

STEP 2: Assume true for $S(k)$
i.e. $1+5+\ldots+(4k-3) = k(2k-1) \quad k \geq 1$ — [1] logical MI statement

STEP 3: Prove true for $S(k+1)$
i.e. RTP:
$1+5+\ldots+(4k-3)+(4k+1) = (k+1)(2k+1)$
LHS $= 1+5+\ldots+(4k-3)+(4k+1)$
$= k(2k-1)+(4k+1)$ by induction hypothesis [1]
$= 2k^2-k+4k+1$
$= 2k^2+3k+1$
$= (k+1)(2k+1)$
$=$ RHS

STEP 4: $\therefore$ if $S(k)$ is true, then $S(k+1)$ is true
since $S(1)$ is true, it follows by mathematical induction that $S(n)$ is true for $n \geq 1$. QED.

$S(1)$ true	1
$S(k)$ and $S(k+1)$ statements correct	1
correct proof	1
logical MI statement (be v. generous	1

Question 13

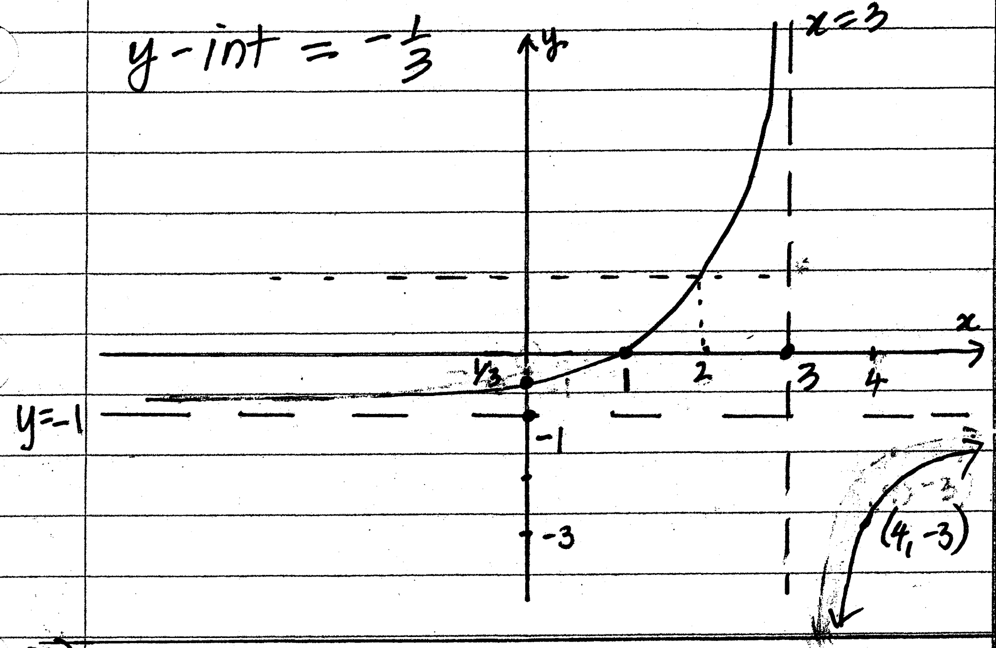
a) i) $y = \dfrac{x-1}{3-x}$

zeroes: $x = 1$
discontinuities: $x = 3$
$\therefore$ vertical assymptote at $x = 3$

$\div x$ $\therefore y = \dfrac{1 - \frac{1}{x}}{\frac{3}{x} - 1}$ as $x \to \infty, y \to -1$

$\therefore$ $y = -1$ is horizontal assymp.

$y\text{-int} = -\frac{1}{3}$



ii) soln: $\dfrac{x-1}{3-x} = 1$

$x - 1 = 3 - x$

$2x = 4$

$x = 2$ $\therefore$ from graph

$\therefore$ soln: $x \le 2$, $x > 3$

Criteria	Mark
vert. assymptote $x = 3$	1
$y=-1$ horiz ass with correct limit notation	1
y-int, x-int, $x=3$, $y=-1$	1
correct shape & correct quads + point on lower arm.	1

2

$x \le 2$	1
$x > 3$	1

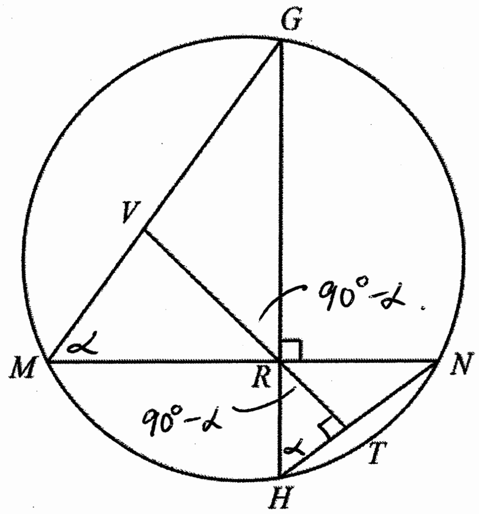
6

Name...

Teacher's Name.......................................

Question 13 (b)

Note: Insert this page into your answer booklet for Question 13.



Not to Scale

(i) Let ∠GMN = α

∠GHN = ∠GMN (angles at circumference standing on same arc GN) (1)
= α

∠RHT = ∠GHN = α (common)

∠VMR = ∠GMN = α (common)

(1)

∠HTR = 90° (RT ⊥ HN)

∠HRT + α + 90° = 180° (angle sum of Δ HRT)
∠HRT = 90° − α

∠GRV = ∠HRT (vertically opposite)
= 90° − α

∠VRM + (90° − α) = 90° (adjacent complementary angles where ∠MRG = 90°)
∠VRM = α

∴ ∠VMR = ∠VRM = α

∴ ΔMVR is isosceles (two equal angles)

13 b)(ii) In ΔVRG

$\angle GRV = 90° - \alpha$ (from (i))

$\angle GVR = \alpha + \alpha$ (exterior angle of ΔMVR)

$= 2\alpha$

(1) $\angle VGR + 2\alpha + (90° - \alpha) = 180°$ (angle sum of ΔVRG)

$\therefore \angle VGR = 90° - \alpha$

$\therefore \angle VGR = \angle GRV = 90° - \alpha$

$\therefore VR = VG$ (sides opposite equal angles)

In ΔMVR

$VR = VM$ (sides opposite equal angles)

(1) $\therefore VM = VG$

$\therefore V$ is midpoint of MG.

Question 13

c) Let the statement be

$7^n - 1$ is divisible by 3 for integer $n \geq 1$

statement true for $n=1$	1
correctly states k hypothesis and $k+1$ statement	2
correct solution	3

Prove statement true for $n=1$

$7^1 - 1 = 6$

$= 3 \times 2$ which is divisible by 3

$\therefore$ true for $n=1$

Assume statement true for $n=k$, for integer $k \geq 1$

ie assume $7^k - 1 = 3P$ for integer P

ie $7^k = 3P + 1$

Prove statement true for $n=k+1$

ie prove $7^{k+1} - 1 = 3Q$ for integer Q

$$7^{k+1} - 1 = 7 \times 7^k - 1$$
$$= 7 \times (3P+1) - 1 \quad \text{(by assumption)}$$
$$= 21P + 7 - 1$$
$$= 21P + 6$$
$$= 3(7P+2)$$
$$= 3Q$$

where $Q = 7P + 2$ is an integer if P is an integer

If statement is true for $n=k$, then it is also true for $n=k+1$

Since statement is true for $n=1$, then it is also true for $n=1+1=2$, $n=2+1=3$, and so on for all integers $n \geq 1$

Question 14

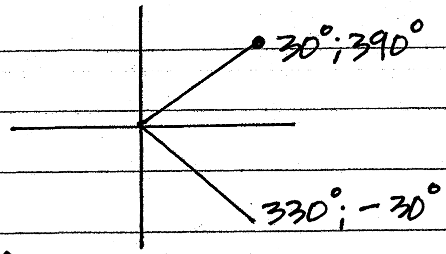
a) $4\cos(2a-45) = 2\sqrt{3}$

$\cos(2a-45) = \frac{\sqrt{3}}{2}$ $\quad 0° \le a \le 360°$

let $u = 2a - 45°$ $\quad -45° \le 2a-45 \le 675$ (3)

$\cos u = \frac{\sqrt{3}}{2}$ $\quad \therefore -45° \le u \le 675°$ ✓

related $u = 30°$ ✓



$\therefore 2a - 45 = -30°;\ 30°;\ 330°;\ 390$

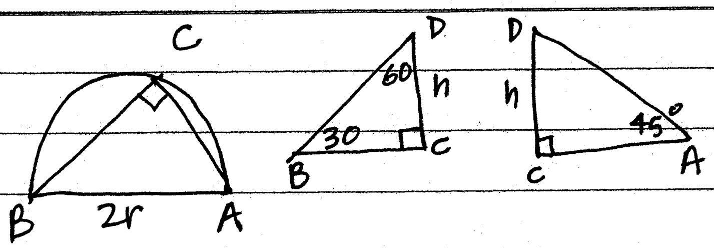
$2a = 15;\ 75;\ 375;\ 435$

$\therefore$ $a = 7.5°;\ 37.5°;\ 187.5°;\ 217.5°$ ✓

recognises domain $-45° \le u \le 675$	1
related angle 30°	2
correct soln	1

• poorly done.
• few did necessary domain amendment

b) (4)



in $\triangle BDC$: $\frac{BC}{h} = \tan 60$

$BC = h\tan 60$

$BC = \sqrt{3}h$ —— [1] ✓

in $\triangle DCA$: $DC = AC$ (sides opposite equal angles: isosceles $\triangle$)

$\therefore AC = h$ —— [2] ✓

in $\triangle ABC$: $\angle BCA = 90°$ (angle in semicircle)

$\therefore (2r)^2 = (\sqrt{3}h)^2 + (h^2)$ ✓

$4r^2 = 3h^2 + h^2$

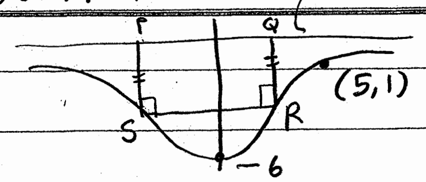
$r^2 = h^2$

$\therefore$ $r = h$ ✓ since $h > 0$, $r > 0$

$BC = \sqrt{3}h$ or $AC = h$	1
$\angle ACB = 90°$ + reason	1
realises eqn $BA^2 = BC^2 + AC^2$ or uses cos rule correctly	1
corr soln	1

Question 14

(c)



i) $y = \frac{h}{x^2+k}$ sub $(0,-6)$ | 2

$\therefore -6 = \frac{h}{k}$ $\therefore h = -6k$ — [1]

sub $(5,-1)$ $\therefore -1 = \frac{h}{25+k}$

$h = -25-k$ — [2]

[1] → [2] $-6k = -25-k$

$-5k = -25$

$k = 5$

$\therefore h = -6(5)$

$h = -30$

substs either $(0,-6)$ or $(5,-1)$	1 mk
correct soln	2

ii) $y = \frac{-30}{x^2+5}$ $Q(\alpha, 0)$ | 1

$\therefore R(\alpha, y)$

$y = \frac{-30}{\alpha^2+5}$

$\therefore R\left(\alpha, \frac{-30}{\alpha^2+5}\right)$

subst $x = \alpha$ into eqn	1

iii) $PQ = 2\alpha$, $QR = \frac{30}{\alpha^2+5}$ | 1

$\therefore$ area $= 2\alpha \cdot \left(\frac{30}{\alpha^2+5}\right)$

area $= \frac{60\alpha}{\alpha^2+5}$ as req'd

$A = 2\alpha\left(\frac{30}{\alpha^2+5}\right)$	1 mk

Question 14

iv)

$$A = \frac{60\alpha}{\alpha^2+5}$$

For max area, $\frac{dA}{d\alpha} = 0$, $\frac{d^2A}{d\alpha^2} < 0$

(candidates are encouraged to state these conditions)

$u = 60\alpha$

$u' = 60$

$v = \alpha^2 + 5$

$v' = 2\alpha$

(say why we want $\frac{dA}{d\alpha}$ = to zero.)

$$\therefore \frac{dA}{d\alpha} = \frac{60(\alpha^2+5) - 120\alpha^2}{(\alpha^2+5)^2}$$

$$\frac{dA}{d\alpha} = \frac{300 - 60\alpha^2}{(\alpha^2+5)^2}$$

for $\frac{dA}{d\alpha} = 0$ $\therefore$ $60\alpha^2 = 300$

$\alpha^2 = 5$

$\therefore \alpha = \sqrt{5}$, $\alpha \geq 0$

sign of $\frac{dA}{d\alpha}$

α	2	$\sqrt{5}$	3
$\frac{dA}{d\alpha}$	$\frac{20}{27}$	0	$-\frac{60}{49}$
	/	—	\

since grad changes from pos to neg $\therefore$ $\alpha = \sqrt{5}$ gives max area

(or) used $\frac{d^2A}{d\alpha^2}$

$$\frac{d^2A}{d\alpha^2} = \frac{-120\alpha(\alpha^2+5)^2 - 4\alpha(\alpha^2+5)(300-60\alpha^2)}{(\alpha^2+5)^4}$$

when $\alpha = \sqrt{5}$, $\frac{d^2A}{d\alpha^2} = -2.6832815\ldots$ < 0

$\therefore \alpha = \sqrt{5}$ gives max area

$$\therefore A = \frac{60\sqrt{5}}{(\sqrt{5})^2+5}$$

$$= \frac{60\sqrt{5}}{10}$$

$A = 6\sqrt{5}\ u^2$ as required

3		
	gets $\frac{dA}{d\alpha} = \frac{300-60\alpha^2}{(\alpha^2+5)^2}$	1m
	solves $\frac{dA}{d\alpha} = 0$ $\therefore \alpha = \sqrt{5}$	2m
	shows max and subst. into area to get $A = 6\sqrt{5}$	3

must explain how $\alpha = \sqrt{5}$ gives a max area

sign line $\quad$ $\frac{d^2A}{d\alpha^2}$