Name: ______________________

SCEGGS Darlinghurst

Preliminary Year, 2003
Semester 2 Examination

Mathematics (Extension)

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- This paper has **5 questions**
- Attempt **all** questions
- Answer all questions on the pad paper provided
- Write your name on every page
- Total marks for all parts (62)
- Approved calculators may be used

Questions 1-5

Total marks (62)

- Attempt all parts of Questions 1-5

	Communication	**Calculus**	**Reasoning**	**Total**
Question 1	/3		/2	/12
Question 2		/1	/5	/12
Question 3	/3	/2		/13
Question 4			/9	/12
Question 5	/3		/10	/13
TOTAL	/9	/3	/26	**/62**

- Answer the questions on the pad paper provided
- Clearly label each part
- Write your name on the top of each page
- START EACH QUESTION ON A NEW PAGE

Question 1 (12 Marks)

Marks

(a) Solve for x: **3**

$$\frac{3x+1}{2-x} \geq 1$$

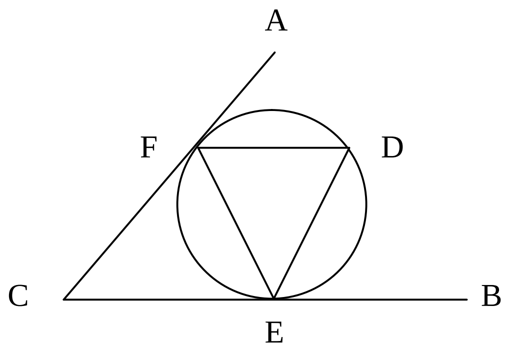
(b) A and B are the points $(-1, 3)$ and $(2, 0)$ respectively:

(i) Find the co-ordinates of the point P which divides AB externally in the ratio $5:2$. **2**

(ii) If the line AB cuts the y axis at Q, find the ratio in which Q divides AB internally without finding the equations of any lines. **2**

(c) In the diagram, AC and BC are tangents to the circle, touching the circle at F and E respectively. $\angle ACB$ equals $50°$. Copy the diagram onto your paper. **3**

Show that $\angle CEF$ is $65°$ and hence find $\angle EDF$.



(d) Prove the identity $\dfrac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A$. **2**

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Question 2 (12 Marks)

Marks

(a) The polynomial $P(x)=x^3+x^2+x-2$ has roots, α, β, γ.

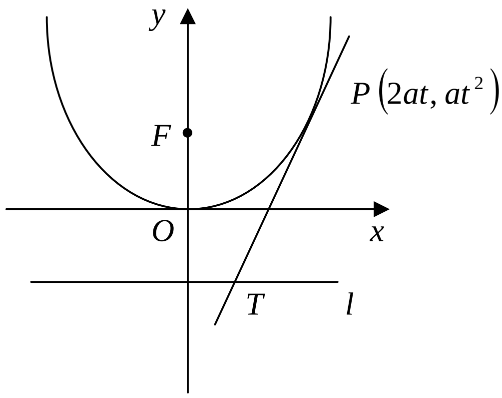
(i) Find the value of $\alpha\beta\gamma$. **1**

(ii) If $\alpha=1$, find the value of $\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(b) (i) Find the gradient of the tangent to the curve $y=x^2+3$ at the point $(1, 4)$. **1**

(ii) Find the acute angle between the line $y=3x+1$ and the curve $y=x^2+3$ at the point of intersection $(1, 4)$. Give your answer to the nearest minute. **2**

(c) The tangent $tx-y-at^2=0$ at the point $P\left(2at, at^2\right)$ on the parabola $x^2=4ay$ cuts the directrix l at T. F is the focus of the parabola.



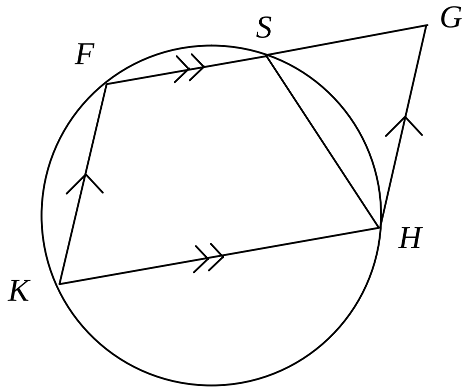
(i) Find the co-ordinates of T. **1**

(ii) Show that TF is perpendicular to PF. **3**

Question 2 continues on the next page

Question 2 (continued) **Marks**

(d) **2**



The figure *FGHK* is a parallelogram. The point *S* lies on *FG*, and *F*,*S*,*H*,*K* lie on a circle.

Copy or trace the diagram onto your answer paper.

Prove that triangle *HGS* is isosceles.

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Question 3 (13 Marks)

Marks

(a) Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$.

(i) Show that 1 is a zero of $P(x)$. **1**

(ii) Express $P(x)$ as a product of 3 linear factors. **2**

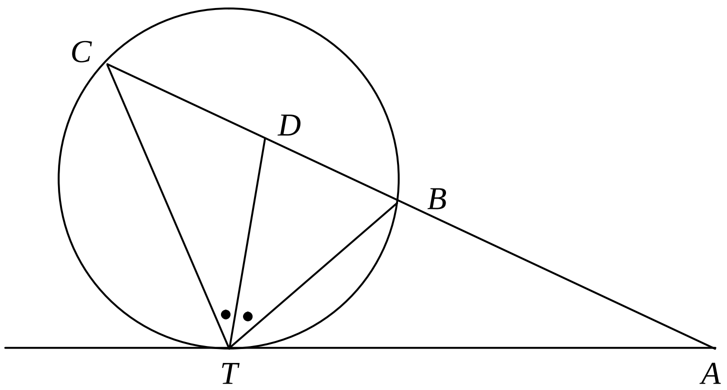
(iii) Solve the inequality $P(x) \le 0$. **1**

(b) Using the substitution $\dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$ find from first principles the derivative of $y = \sqrt{x}$. **2**

(c) (i) Express $4\sin\theta - 3\cos\theta$ in the form $A\sin(\theta - \alpha)$ where $A > 0$ and $0° < \alpha < 90°$. Give α to the nearest degree. **2**

(ii) Find all solutions of $4\sin\theta - 3\cos\theta = 1$ for $0° \le \theta \le 360°$. Give θ to the nearest degree. **2**

(d)



TA is a tangent to a circle. Line *ABDC* intersects the circle at *B* and *C*. Line *TD* bisects angle *BTC*. **3**

Prove *AT* = *AD*.

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Question 4 (12 Marks)

Marks

(a) (i) Using the $\sin(A - B)$ formula, with $A = 60°$ and $B = 45°$ (or otherwise), find the value of $\sin 15°$ in surd form. **2**

(ii) Prove the identity: **2**

$$\tan\alpha = \operatorname{cosec} 2\alpha - \cot 2\alpha$$

(iii) Using the results in part (ii), find a value (in surd form) for $\tan 15°$. **2**

(iv) Using the results in parts (i), (ii) and (iii), (or otherwise) show that:. **3**

$$\tan 7\frac{1}{2}^{\circ} = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$$

(b) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find the values of a and b. **3**

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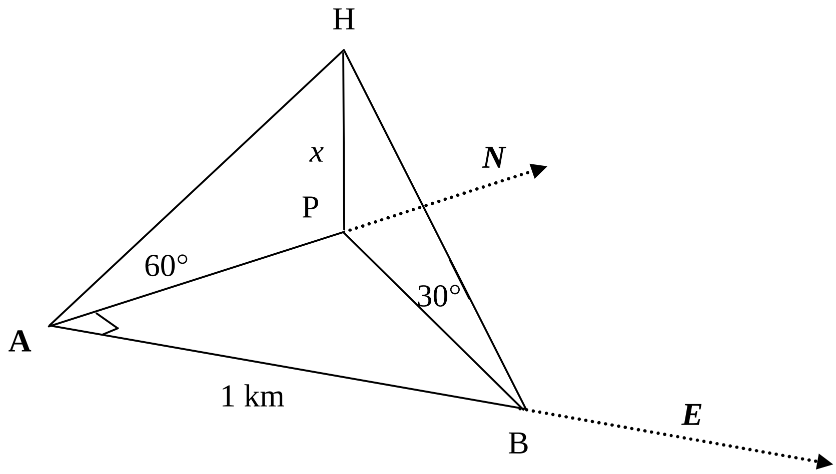
Question 5 (13 Marks)

Marks

(a) On the same set of axes, draw the graphs of $y = |x-2|$ and $y = x+1$. **3**

For what values of x is $|x-2| < x+1$?

(b) Anna (A) is standing due south of Phillip (P) who is assisting an injured bushwalker. A rescue helicopter (H) is hovering directly over P and lowering a stretcher. Anna measures the angle of elevation of the helicopter to be $60°$ from her position. Belinda (B) is 1 kilometre due east of A and measures the angle of elevation of the helicopter to be $30°$. The height of the helicopter above P is x metres.



(i) Write expressions for both AP and BP in terms of x. **1**

(ii) Hence or otherwise, find the height of the helicopter (x), correct to the nearest m. **3**

(c) $P(2ap, ap^2)$ is a variable point on a parabola $x^2 = 4ay$. The line PR is perpendicular to the directrix of the parabola with R on the directrix. The tangent to the parabola at P meets the y-axis at T. If M is the midpoint of the interval RT:

(i) find the locus of M. **4**

(ii) show that the directrix of the locus of M is the x-axis. **2**

End of Paper

Extension 1 Mathematics — Prelim. Year 2003.

① a) $\frac{3x+1}{2-x} \geq 1$

undefined when $x=2$

multiply both sides by $(2-x)^2$

$\frac{3x+1}{\cancel{2-x}} \times (2-x)^{\cancel{2}} \geq 1 \times (2-x)^2$

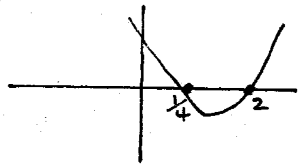
$(3x+1)(2-x) \geq (2-x)^2$

$6x - 3x^2 + 2 - x \geq 4 - 4x + x^2$

$0 \geq 4x^2 - 9x + 2$

$4x^2 - 9x + 2 \leq 0$ ✓

$(4x-1)(x-2) \leq 0$



$\frac{1}{4} \leq x < 2$ ✓✓

b) External division $5:-2$

$A(-1,3)$ $B(2,0)$

$5 : -2$

$x = \frac{-2\times -1 + 5\times 2}{5-2}$

$= \frac{2+10}{3}$

$= \frac{12}{3}$

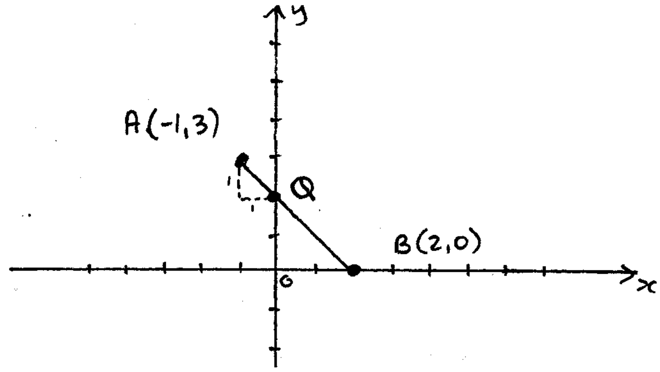
$= 4$

$y = \frac{-2\times 3 + 5\times 0}{5-2}$

$= \frac{-6+0}{3}$

$= -2$

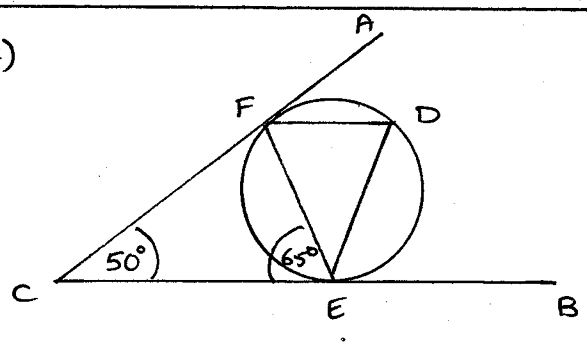
$\therefore P(4,-2)$ ✓✓



By observation

Q divides AB internally in the ratio 1:2 ✓✓

c)



$CF = CE$
since tangents from an exterior point are equal.

$\therefore \triangle CFE$ is isosceles. ✓

$\therefore \angle CEF = \frac{1}{2}(180° - 50°)$ ($\angle$ sum $\triangle = 180°$)

$= \frac{1}{2} \times 130°$

$= 65°$ ✓

$\angle EDF = \angle CEF$

$= 65°$ ✓

(Angles in the alternate segments are equal.)

Com /3

d) LHS $= \dfrac{1-\tan^2 A}{1+\tan^2 A}$

$= \dfrac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}}$

$= \dfrac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}$ ✓

$= \dfrac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{1}{\cos^2 A}}$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A} \times \dfrac{\cos^2 A}{1}$

$= \cos^2 A - \sin^2 A$ ✓

$= \cos 2A$

Reas $\frac{}{2}$

② $P(x) = x^3 + x^2 + x - 2$

Roots α, β, γ

i) Product of roots

$\alpha\beta\gamma = -d/a$

$= --2/1$

$= 2$ ✓

ii) Sum of Roots

$\alpha + \beta + \gamma = -b/a$

$= -1$

Sum two at a time

$\alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$= 1$

$\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓

put $\alpha = 1$

$1 + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{1}{2}$

$\dfrac{1}{\beta} + \dfrac{1}{\gamma} = -\dfrac{1}{2}$ ✓

b) (i) $y = x^2 + 3$

$y = 2x$

At $(1,4)$ gradient tangent ✓

$m_1 = 2$

Calculus $\frac{}{1}$

ii) $y = 3x + 1$

$m_2 = 3$

Acute angle

$\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

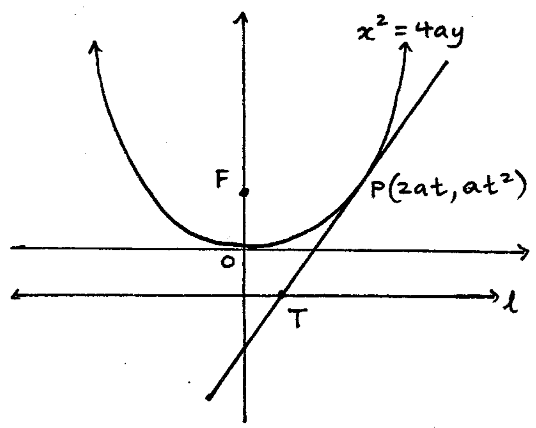
$= \left| \dfrac{2 - 3}{1 + 2 \times 3} \right|$

$= \left| \dfrac{-1}{7} \right|$

$= \dfrac{1}{7}$ ✓

$\theta \doteqdot 8^\circ 8'$ (to nearest minute) ✓

c)



Tangent at $P(2at, at^2)$

$tx - y - at^2 = 0$

cuts the directrix l at T

directrix is $y = -a$

Solve simultaneously

$tx - y - at^2 = 0$ ①

$y = -a$ ②

substitute ② into ①

$tx + a - at^2 = 0$

$tx = at^2 - a$

$x = \dfrac{a(t^2-1)}{t}$

T has coordinates

$\left(\dfrac{a(t^2-1)}{t}, -a\right)$

ii) Focus $F(0,a)$

$P(2at, at^2)$

Gradient TF

$m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-a-a}{\frac{a(t^2-1)}{t} - 0}$

$= \dfrac{-2\cancel{a}}{\frac{\cancel{a}(t^2-1)}{t}}$

$= \dfrac{-2t}{t^2-1}$

Gradient PF

$m_2 = \dfrac{at^2 - a}{2at - 0}$

$= \dfrac{\cancel{a}(t^2-1)}{2\cancel{a}t}$

$= \dfrac{t^2-1}{2t}$

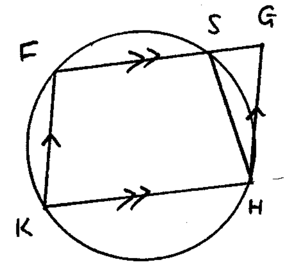
Since $m_1 \times m_2 = \dfrac{-2t}{t^2-1} \times \dfrac{t^2-1}{2t}$

$= -1$

$\therefore$ TF $\perp$ PF

Reas 3

d)



FGHK is a parallelogram

$\therefore \angle FKH = \angle FGH$ (opposite $\angle$s = in a parm.)

$= x$

F, S, H, K lie on a circle

$\therefore$ FSHK is a cyclic quadrilateral

$\therefore \angle FKH + \angle FSH = 180°$

(opp. $\angle$s in a cyclic quad. are supplementary)

$\angle FSH = 180° - \angle FKH$

$= 180° - x$

$\angle GSH = 180° - \angle FSH$ ($\angle$ sum straight $\angle = 180°$)

$= 180° - (180° - x)$

$= x$

Reas 2

$\therefore \angle SGH = \angle GSH = x$

$\therefore \Delta HGS$ is isosceles since it has 2 equal angles.

Q3

a) $P(x) = 6x^3 - 5x^2 - 2x + 1$

i) $P(1) = 6 - 5 - 2 + 1$
$= 0$

$\therefore x = 1$ is a zero of $P(x)$
$\therefore (x-1)$ is a factor of $P(x)$

✓

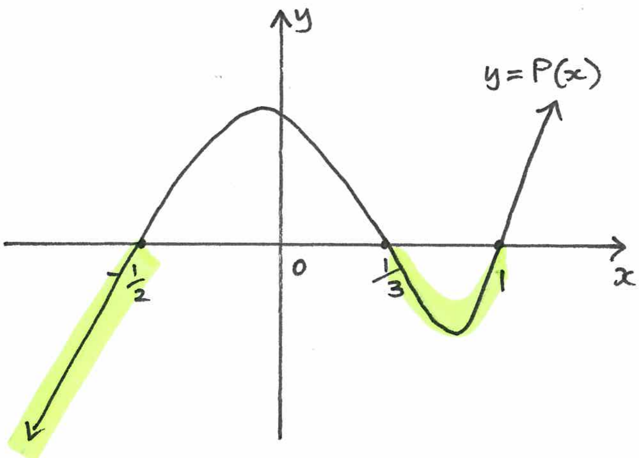
ii)
$$\begin{array}{r|l} & 6x^2 + x - 1 \\ \hline x-1 & 6x^3 - 5x^2 - 2x + 1 \\ & 6x^3 - 6x^2 \\ \hline & \quad\quad x^2 \\ & \quad\quad x^2 - x \\ \hline & \quad\quad\quad -x \\ & \quad\quad\quad -x + 1 \\ \hline & \quad\quad\quad\quad 0 \end{array}$$

✓

$\therefore P(x) = (x-1)(6x^2 + x - 1)$
$= (x-1)(3x-1)(2x+1)$

✓

iii)



$P(x) \le 0$
for

$x \le -\frac{1}{2}$, $\frac{1}{3} \le x \le 1$

✓

b) $y = \sqrt{x}$

$f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

Using first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$ ✓

using the given substitution

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

Calc/2

$$= \frac{1}{2\sqrt{x}}$$ ✓

c) i) $4 \sin\theta - 3\cos\theta$

$= A \sin(\theta - \alpha)$

$= A \sin\theta \cos\alpha - A \cos\theta \sin\alpha$

Match parts

$A \cos\alpha = 4$ ①
$A \sin\alpha = 3$ ②

Find A

$A^2 = 4^2 + 3^2$
$A = \sqrt{16 + 9}$
$= \sqrt{25}$
$= 5$

✓

Find α
divide ②/①

$\tan\alpha = \frac{3}{4}$

$\therefore \alpha \doteq 37°$

✓

$\therefore 4\sin\theta - 3\cos\theta$

$= 5\sin(\theta - 37°)$

ii) $4\sin\theta - 3\cos\theta = 1$

$5\sin(\theta - 37°) = 1$

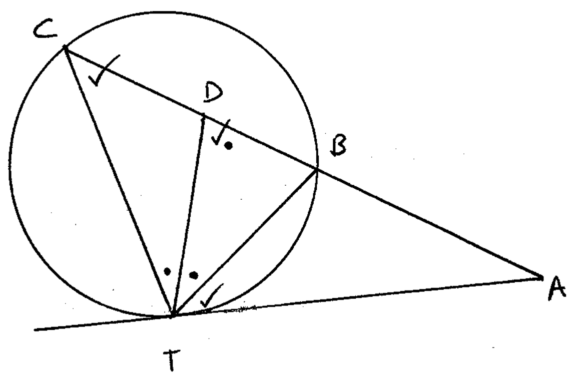
$\sin(\theta - 37°) = \frac{1}{5}$

$\therefore \theta - 37° = 11°32'$ or $168°28'$

$\therefore \theta \doteqdot 48°$ or $205°$

✓✓

d)



$\angle CTD = \angle BTD$ (given)

$\angle ATB = \angle TCD$ ($\angle$ in the alt segment = $\angle$ between chord + tangent) ✓

$\angle TDB = \angle CTD + \angle TCD$ (ext. $\angle$ in Δ) ✓

$\therefore \angle ATD = \angle ADT$

$\therefore AT = AD$ (sides opp = $\angle$ in an isosceles Δ =) ✓

Com
3

4 a) i) $\sin(A-B)$

$= \sin A\cos B - \cos A\sin B$

$\sin 15° = \sin(45° - 30°)$

$= \sin 45°\cos 30° - \cos 45°\sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$ ✓

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$ ✓

$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

ii) Prove $\tan\alpha = \operatorname{cosec}2\alpha - \cot 2\alpha$

RHS $= \operatorname{cosec}2\alpha - \cot 2\alpha$

$= \frac{1}{\sin 2\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha}$

$= \frac{1 - \cos 2\alpha}{\sin 2\alpha}$

$= \frac{1 - (\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha}$ ✓

$= \frac{1 - \cos^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}$

Substituting $1 - \cos^2\alpha = \sin^2\alpha$

$= \frac{\sin^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}$

$= \frac{\cancel{2}\sin^{\cancel{2}}\alpha}{\cancel{2}\cancel{\sin\alpha}\cos\alpha}$

$= \frac{\sin\alpha}{\cos\alpha}$ ✓

$= \tan\alpha$

$= $ LHS

iii) $\tan 15^\circ = \operatorname{cosec} 30^\circ - \cot 30^\circ$

$$= \frac{1}{\sin 30^\circ} - \frac{1}{\tan 30^\circ}$$ ✓

$$= \frac{1}{\frac{1}{2}} - \frac{1}{\frac{1}{\sqrt{3}}}$$

$$= 2 - \sqrt{3}$$ ✓

iv) $\tan 7\tfrac{1}{2}^\circ = \operatorname{cosec} 15^\circ - \cot 15^\circ$

$$= \frac{1}{\sin 15^\circ} - \frac{1}{\tan 15^\circ}$$

$$= \frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}} - \frac{1}{2-\sqrt{3}}$$ ✓

$$= \frac{4}{\sqrt{6}-\sqrt{2}} - \frac{1}{2-\sqrt{3}}$$

Rationalise each denominator

$$\frac{4}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{4(\sqrt{6}+\sqrt{2})}{6-2}$$

$$= \sqrt{6}+\sqrt{2}$$

$$\frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3}$$

$$= 2+\sqrt{3}$$ ✓

$$\therefore \quad = \sqrt{6}+\sqrt{2} - (2+\sqrt{3})$$

$$= \sqrt{6}+\sqrt{2} - \sqrt{3} - 2$$ ✓

Reas 9

b) $P(x) = A(x) . Q(x) + R(x)$

$$x^3 + ax + b = (x-2)(x+3) . Q(x) + 2x + 1$$

Substitute $x = 2$

$$8 + a + b = 0 + 4 + 1$$

$$a + b = -3 \quad (1)$$

substitute $x = -3$

$$-27 - 3a + b = 0 - 6 + 1$$

$$-3a + b = 22 \quad (2)$$ ✓

Solve simultaneously

$$a + b = -3 \quad (1)$$

$$-3a + b = 22 \quad (2)$$

Subtract

$$4a = -25$$

$$a = -25/4$$ ✓

$$a = -6\tfrac{1}{4}$$

subst. into (1)

$$-6\tfrac{1}{4} + b = -3$$

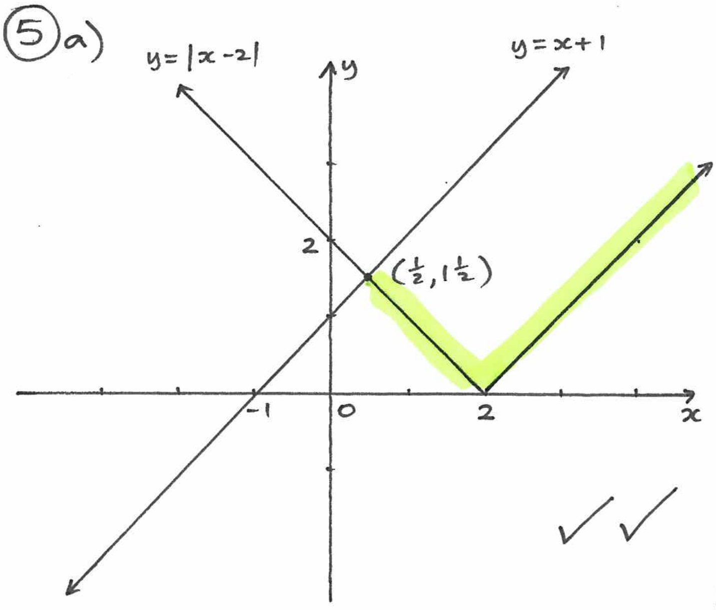
$$b = 3\tfrac{1}{4}$$ ✓

Solution

$$\therefore \quad a = -6\tfrac{1}{4}$$

$$b = 3\tfrac{1}{4}$$

5 a)



One point of intersection only

when $x + 1 = -(x - 2)$

$x + 1 = -x + 2$

$2x = 1$

$x = 1/2$

$(\frac{1}{2}, 1\frac{1}{2})$

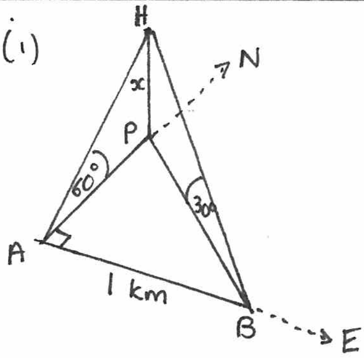
from the graph

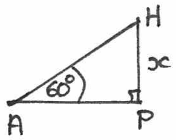
$|x - 2| < x + 1$

for $x > \frac{1}{2}$

6m
3

b) (i)

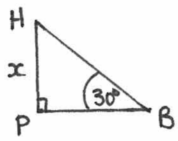




$\tan 60° = \frac{x}{AP}$

$AP = \frac{x}{\tan 60°}$

or $AP = x \tan 30°$

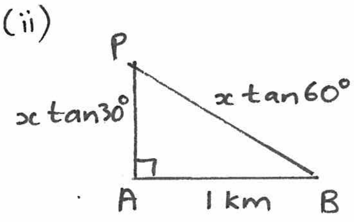


$\tan 30° = \frac{x}{BP}$

$BP = \frac{x}{\tan 30°}$

or $BP = x \tan 60°$

(ii)



Using Pythagoras rule

$BP^2 = AP^2 + AB^2$

$x^2 \tan^2 60° = x^2 \tan^2 30° + 1^2$

$x^2 \tan^2 60° - x^2 \tan^2 30° = 1$

$x^2 (\tan^2 60° - \tan^2 30°) = 1$

$x^2 = \frac{1}{\tan^2 60° - \tan^2 30°}$

$x = \sqrt{\frac{1}{\tan^2 60° - \tan^2 30°}}$

$\doteqdot \sqrt{\frac{1}{3 - 1/3}}$

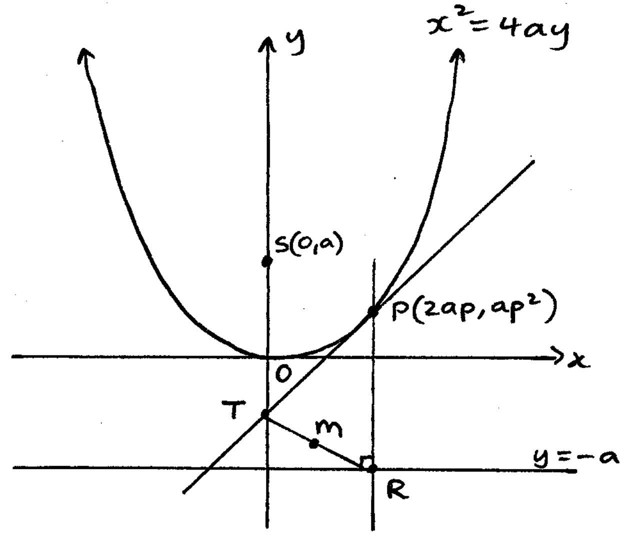
$= \sqrt{\frac{3}{8}}$

$\doteqdot 0.612372\ldots$ km

$\doteqdot 612$ m

(to nearest metre)

c)



$x^2 = 4ay$

directrix $y = -a$

Coordinates of R

$(2ap, -a)$ ✓

Equation of tangent at P

$y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$= \frac{x}{2a}$

At $P(2ap, ap^2)$

tangent $m_1 = \frac{2ap}{2a}$

$= p$

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

Coordinates of T

Tangent cuts y axis when $x = 0$

$y = px - ap^2$

$y = 0 - ap^2$

$y = -ap^2$

$\therefore T(0, -ap^2)$ ✓

midpoint of TR

$T(0, -ap^2)$ $R(2ap, -a)$

$m = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{0+2ap}{2}, \frac{-ap^2-a}{2}\right)$

$= \left(ap, -\frac{a(p^2+1)}{2}\right)$ ✓

Locus of m

$\begin{cases} x = ap \\ y = -\frac{a}{2}(p^2+1) \end{cases}$

substitute $p = x/a$ into y

$y = -\frac{a}{2}\left(\left(\frac{x}{a}\right)^2 + 1\right)$

$y = -\frac{a}{2}\left(\frac{x^2}{a^2} + 1\right)$

$y = -\frac{x^2}{2a} - \frac{a}{2}$ ✓

The locus is a parabola.

(x2a) $2ay = -x^2 - a^2$

$x^2 = -2ay - a^2$

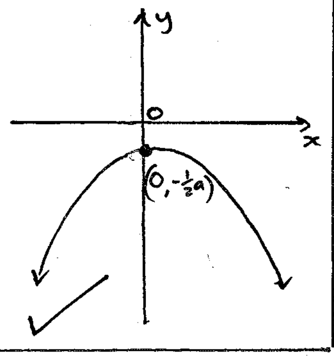
$x^2 = -2a(y + \frac{1}{2}a)$

$x^2 = -4 \cdot \frac{1}{2}a(y + \frac{1}{2}a)$ ✓

The locus is a parabola.

Vertex $(0, -\frac{1}{2}a)$

Focal length $= \frac{1}{2}a$



Directrix $y = 0$

$\therefore$ The directrix is the x-axis. ✓

Rea) 10