Name: ....

SCEGGS Darlinghurst

2004
Preliminary Course
Semester 2 Examination

Mathematics (Extension)

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- This paper has **four** questions
- Attempt **all** questions on the pad paper provided
- Write your name on every page
- Marks will be deducted for careless or badly arranged work
- Approved calculators may be used

Total marks – 60

- Attempt Question 1–4

Question	Communication	Reasoning	Total
1	/3	/3	/15
2	/4	/5	/15
3	/3	/6	/15
4		/11	/15
TOTAL	**/10**	**/25**	**/60**

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Instructions

- Attempt **all** questions on the pad paper provided
- Write your name at the top of each page
- Show all necessary working
- Marks may be deducted for careless or badly arranged work
- Approved calculators may be used
- Begin each question on a new page

Question 1 (15 marks) **Marks**

(a) Find the acute angle between the two lines $3x - 2y + 7 = 0$ and $7x + 5y - 6 = 0$. Answer correct to the nearest minute. **3**

(b) Give α, β and γ are the roots of the equation $2x^3 - 3x^2 - 5x + 1 = 0$, find:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

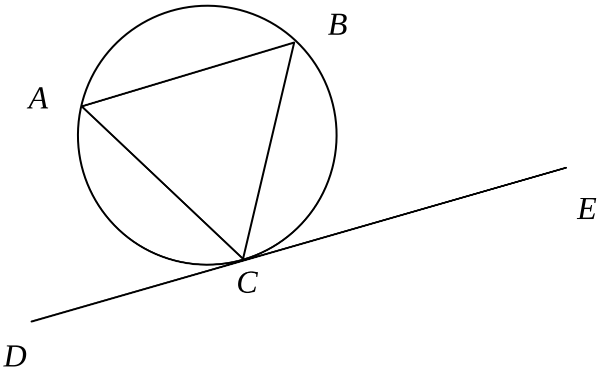
(iii) $(\alpha + 1)\ (\beta + 1)\ (\gamma + 1)$ **2**

(c) Find the co-ordinates of the point P which divides the interval joining $A(2,\ -3)$ and $B(4, 5)$ externally in the ratio 1 : 3. **2**

(d) Solve $\dfrac{3}{1 - x} \leq 2$. **3**

Question 1 continues on the next page

(e) **3**



DE is a tangent to the circle at C.
$\angle BCE = \angle ACD$

Prove that *AB* is parallel to *DE*.

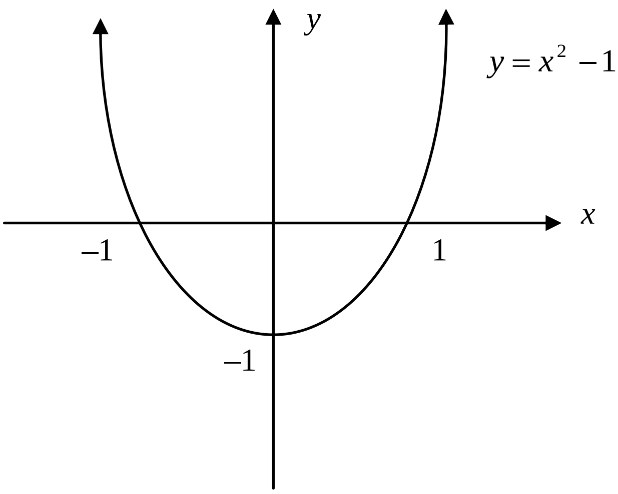
■ **Start a new page**

Question 2 (15 marks) **Marks**

(a) (i) Express $3\sin\theta - 2\cos\theta$ in the form $A\sin(\theta - \alpha)$ where $A > 0$ and α is an acute angle correct to the nearest minute. **3**

(ii) Hence state the exact maximum value of $12 - 3\sin\theta + 2\cos\theta$. **1**

(b)



A student was given the curve $y = x^2 - 1$ as shown above. She was then asked to draw

$$y = \frac{1}{x^2 - 1}$$

(i) Explain why she knew that asymptotes occur at $x = -1$ and $x = 1$. **1**

(ii) Draw the curve $y = \frac{1}{x^2 - 1}$. **2**

(c) (i) Prove that $x + 1$ is a factor of $P(x) = x^3 + 3x^2 - x - 3$. **1**

(ii) Hence factorise $P(x)$ fully. **2**

(iii) Solve $P(x) > 0$ **1**

Question 2 continues on the next page

(d) Use the method of Mathematical Induction to prove that **4**

$$9^{n+2} - 4^{n}$$

is divisible by 5 for all positive integers n.

■ **Start a new page**

Question 3 (15 marks) **Marks**

(a) Consider the polynomial

$$P(x) = x^3 - 3x^2 - 2x + 4$$

(i) Show that it has a zero between $x = 3$ and $x = 4$. **2**

(ii) Taking an initial approximation of $x = 3.5$ and one application of Newton's Method, find a more accurate approximation to the zero correct to 3 significant figures. **2**

(b) Use the method of Mathematical Induction to prove that **5**

$$2 + 5 + 8 + \dots + 3n - 1 = \frac{n(3n+1)}{2}$$

for all positive integers *n*.

(c) (i) Find the centre and radius of the circle whose equation is: **2**

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

(ii) Find, in terms of the constant *k*, the perpendicular distance from the centre to the line whose equation is $3x + 4y = k$. **2**

(iii) Hence find any values of *k* for which the line is a tangent to the circle. **2**

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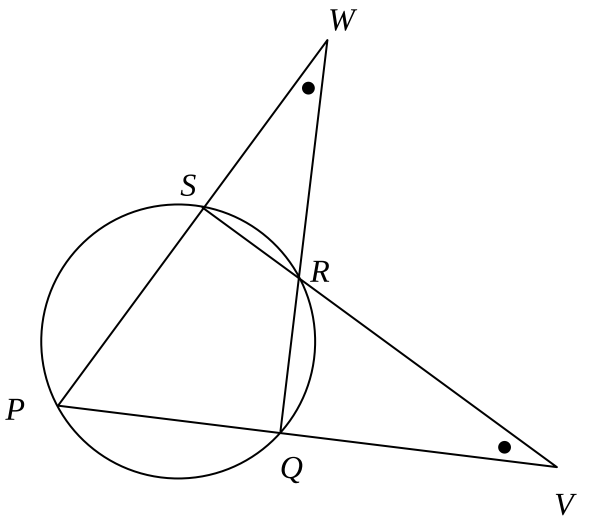
Question 4 (15 marks) **Marks**

(a)



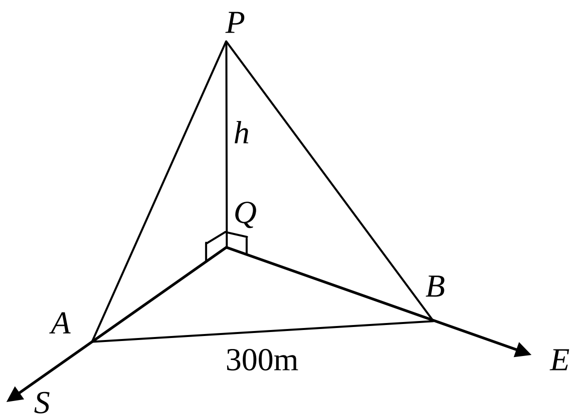
The points P, Q, R and S lie on the circumference of a circle. PS and QR are extended to W while PQ and SR are extended to V.

$$\angle PWQ = \angle SVP$$

(i) Prove that $\angle PQR = \angle PSR$ **3**

(ii) Hence prove that PR is a diameter of the circle. **2**

(b)



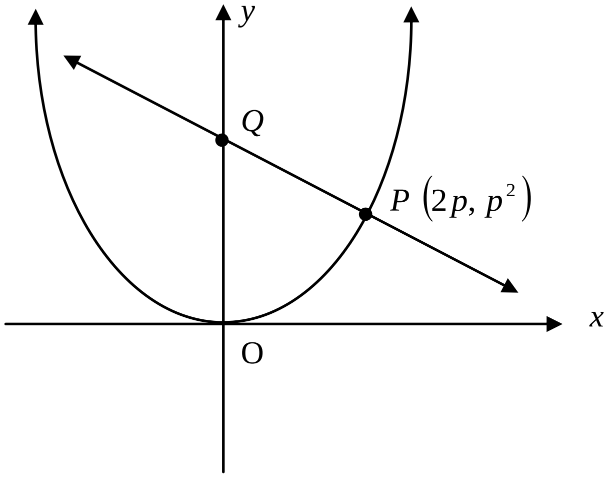
The points A and B are due South and East of a hill of height h m shown as PQ. A and B are 300 m apart. The angle of elevation of the top of the hill P from A is $30°$ and from B is $60°$.

(i) Prove that $AQ = \sqrt{3}\,h$. **1**

(ii) Hence prove that $h = 30\sqrt{30}$ metres. **3**

Question 4 continues on the next page

(c)



$P\left(2p,\ p^2\right)$ is a point on the parabola $4y = x^2$.

(i) Prove that the equation of the normal to the parabola at P has the equation **2**

$$x + py = p^3 + 2p$$

(ii) The normal meets the axis of the parabola at Q. Find the co-ordinates of Q. **1**

(iii) Find the co-ordinates of R, the midpoint of PQ. **1**

(iv) Show that the locus of R is a parabola and find its vertex. **2**

End of paper

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① a) $2y = 3x + 7$ $\qquad$ $5y = -7x + 6$

$m_1 = \frac{3}{2}$ $\qquad$ $m_2 = \frac{-7}{5}$. $\qquad$ 1

$\tan\theta = \left| \dfrac{\frac{3}{2} + \frac{7}{5}}{1 - \frac{3}{2} \times \frac{7}{5}} \right| = \left| \dfrac{\frac{29}{10}}{\frac{-11}{10}} \right| = \dfrac{29}{11}$

$\theta = 69°14'$ (nearest minute) $\qquad$ 1 $\qquad$ /3

b) (i) $\alpha + \beta + \gamma = \frac{3}{2}$ $\qquad$ 1

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-5}{2}$ $\qquad$ 1 $\qquad$ $\alpha\beta\gamma = -\frac{1}{2}$

(iii) $(\alpha+1)(\beta+1)(\gamma+1) = (\alpha+1)(\beta\gamma + \beta + \gamma + 1)$

$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$ $\qquad$ 1

$= -\frac{1}{2} - \frac{5}{2} + \frac{3}{2} + 1$

$= -\frac{1}{2}$ $\qquad$ 1 $\qquad$ /2

c) $(2, -3)$ $(4, 5)$ $\qquad$ $-1 : 3$

$x = \dfrac{2 \times 3 + 4 \times -1}{2} = 1$ $\qquad$ 1

$y = \dfrac{-3 \times 3 + 5 \times -1}{2} = -7$ $\qquad$ 1

P is $(1, -7)$ $\qquad$ /2

d) $\dfrac{3}{1-x} \times (1-x)^2 \leq 2(1-x)^2$

$3(1-x) \leq 2(1 - 2x + x^2)$

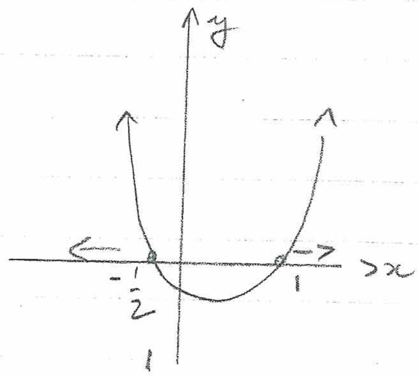
$3 - 3x \leq 2 - 4x + 2x^2$

$\therefore 2x^2 - x - 1 \geq 0$ $\qquad$ 1

$(2x+1)(x-1) \geq 0$

$\therefore x \leq -\frac{1}{2}, \quad x \geq 1$ $\qquad$ but $x \neq 1$

$\therefore x \leq -\frac{1}{2}, \quad x > 1$ is solution $\qquad$ 1



(Reas) $\qquad$ /3

e) $\angle BCE = \angle CAB$ (angle between tangent and chord equals angle in alternate segment) 1

But $\angle BCE = \angle ACD$ (given)

$\therefore \angle ACD = \angle CAB$ 1

$\therefore AB \parallel DE$ (alternate angles are equal) 1 /3

(Comm)

② a) (i) $3\sin\theta - 2\cos\theta = A\sin\theta\cos\alpha - A\cos\theta\sin\alpha$.

$\therefore A\cos\alpha = 3$ and $A\sin\alpha = 2$ 1

$\therefore A = \sqrt{13}$ 1

$\cos\alpha = \frac{3}{\sqrt{13}}$ /3

$\alpha = 33°41'$

$\therefore 3\sin\theta - 2\cos\theta = \sqrt{13}\sin(\theta - 33°24')$ 1

(ii) $-\sqrt{13} \le \sqrt{13}\sin(\theta - 33°24') \le \sqrt{13}$ 1 (Reas)

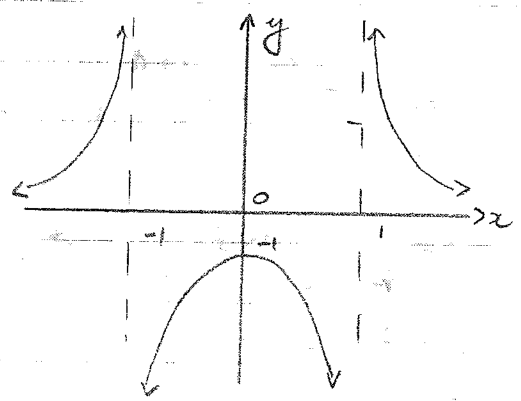
$\therefore$ maximum value is $12 + \sqrt{13}$.

b) (i) For the curve $y = x^2 - 1$, $y = 0$ at $x = 1$ and $x = -1$.

$\therefore$ For the curve $y = \frac{1}{x^2-1}$, y is undefined at $x = 1$, $x = -1$

$\therefore$ asymptotes occur on $y = \frac{1}{x^2-1}$ at $x = 1$, $x = -1$. 1 (Comm.)

(ii)



2 (Comm)

c) (i) $P(-1) = (-1)^3 + 3(-1)^2 + 1 - 3 = -1 + 3 + 1 - 3 = 0$

$\therefore x + 1$ is a factor of $P(x)$ 1

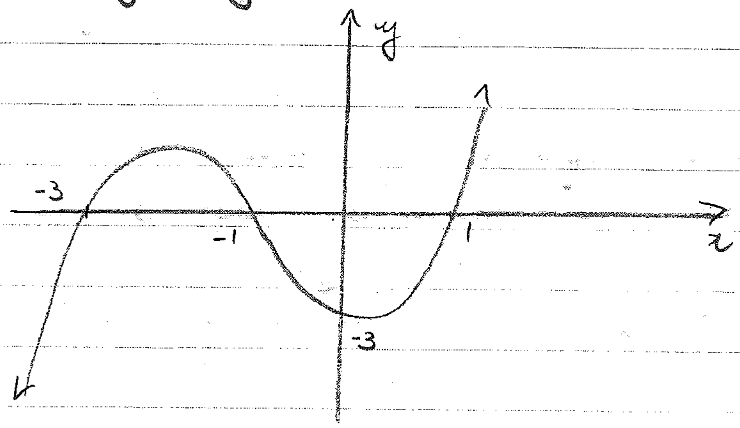
(ii)

$$\begin{array}{r} x^2+2x-3 \\ x+1 \overline{) x^3+3x^2-x-3} \\ \underline{x^3+x^2} \\ 2x^2-x-3 \\ \underline{2x^2+2x} \\ -3x-3 \end{array}$$

$\therefore \quad P(x) = (x+1)(x^2+2x-3)$

$\quad = (x+1)(x+3)(x-1)$ 2

(iii) Graphing $y = P(x)$



$P(x) > 0$ for $-3 < x < -1$ and $x > 1$ 1(com)

d) Assume true for $n = k$

i.e. that $9^{k+2} - 4^k$ is divisible by 5.

$\therefore$ Assume $9^{k+2} - 4^k = 5N$ where N is an integer.

Consider $n = k+1$

$$\begin{aligned} 9^{k+3} - 4^{k+1} &= 9^{k+2} \times 9 - 4^{k+1} \\ &= 9(5N + 4^k) - 4^{k+1} \\ &= 45N + 9.4^k - 4.4^k \\ &= 45 + 5.4^k \\ &= 5[9 + 4^k] \end{aligned}$$

which is divisible by 5.

consider $n=1$, $9^3-4=729-4=725$ which is divisible by 5.

If the statement is true for $n=k$, it is also true for $n=k+1$. It is true for $n=1$ and thus is true for $n=2, 3 \ldots$ etc. Thus true for all n positive integers.

4 (Reas)

(3) a) (i) $P(x) = x^3-3x^2-2x+4$

$P(3) = 3^3-3\times 3^2-2\times 3+4$

$= -2 < 0$

$P(4) = 4^3-3\times 4^2-2\times 4+4$

$= 12 > 0$

since the sign has changed and $P(x)$ is continuous, there is a zero between $x=3$ and $x=4$. ✓2 (Comm)

(ii) $P(3\cdot 5) = 3\cdot 5^3 - 3\times 3\cdot 5^2 - 2\times 3\cdot 5 + 4$

$= 3\cdot 125$

$P'(3\cdot 5) = 3\times 3\cdot 5^2 - 6\times 3\cdot 5 - 2$

$= 13\cdot 75$.

$\therefore a_2 = 3\cdot 5 - \frac{3\cdot 125}{13\cdot 75}$

$= 3\cdot 27$ (3 s.f.)

b) Assume true for $n=k$

i.e. assume $2+5+8+\cdots+3k-1 = \frac{k(3k+1)}{2}$

consider $n=k+1$

$L.H.S = 2+5+8+\cdots+3k-1+3k+2$

$= \frac{k(3k+1)}{2} + 3k+2$

$= \frac{3k^2+k+6k+4}{2}$

$= \frac{3k^2+7k+4}{2}$

$$L.H.S. = \frac{(k+1)(3k+4)}{2}$$

$$= \frac{(k+1)(3(k+1)+1)}{2}$$

$$= R.H.S. \text{ if } n = k+1$$

If $n=1$, $L.H.S. = 2$ 4 (Reas)

$R.H.S. = \qquad = 2$

$\therefore$ If true for $n=k$, the statement is also true for $n=k+1$. It is true for $n=1$ and thus is true for $n=2,3\ldots$ $\therefore$ True for all n positive integers. 1 (Com)

c) (i) $x^2 - 4x + 4 + y^2 + 6y + 9 = 12 + 4 + 9$

$(x-2)^2 + (y+3)^2 = 25$

centre is $(2,-3)$ radius is 5 units.

(ii) $$\text{distance} = \left|\frac{3\times 2 + 4\times -3 - k}{5}\right|$$

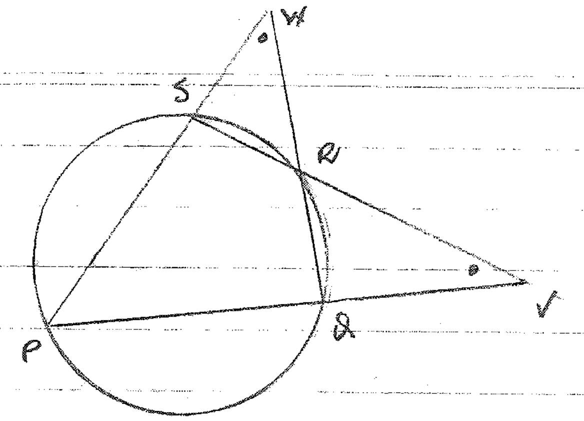
$$= \left|\frac{-6-k}{5}\right| \text{ units.}$$

(iii) if $\left|\frac{-6-k}{5}\right| = 5$. ✓

$-6-k = 25$ or $6+k = 25$

$k = -31$ or $k = 19$. ✓ 2 (Reas)

④ a)



(i) $\angle QRV = \angle SRW$ (vertically opposite angles equal)

$\angle SWR = \angle RVQ$ (given)

$\therefore$ $\Delta SWR \parallel\!| \Delta RQV$ (equiangular)

$\therefore$ $\angle RQV = \angle WSR$ (angles in similar Δ's are equal)

$\angle PQR = 180° - \angle RQV$ (straight angle is 180°)

$\angle PSR = 180° - \angle WSR$ (" " " ")

$\therefore$ $\angle PQR = \angle PSR$.

(ii) $\angle PQR + \angle PSR = 180°$ (opp. angles in a cyclic quadrilateral are supplementary)

$\therefore$ $2\angle PQR = 180°$

$\angle PQR = 90°$

$\therefore$ PR is a diameter (if the angle is 90° it must be a semi circle)

(Reas) 5.

b) (i) $\tan 30° = \frac{h}{AQ}$

$\therefore AQ = \frac{h}{\frac{1}{\sqrt{3}}} = \sqrt{3}h$. (Reas) 1.

(ii) $\tan 60° = \frac{h}{BQ}$

$BQ = \frac{h}{\sqrt{3}}$

using Pythagoras' Theorem,

$300^2 = 3h^2 + \frac{h^2}{3}$

$$90000 = \frac{10h^2}{3}$$

$$h^2 = 27000$$

$$h = \sqrt{27000} = \sqrt{900 \times 30}$$
$$= 30\sqrt{30}\ m$$

3 Reas

c) (i) $y = \frac{x^2}{4}$

$$y' = \frac{2x}{4} = \frac{x}{2}$$

if $x = 2p$, $y' = \frac{2p}{2} = p$

equation of normal: $y - p^2 = \frac{-1}{p}(x - 2p)$

$$py - p^3 = -x + 2p$$

$$x + py = p^3 + 2p.$$

(2)

(ii) if $x = 0$, $py = p^3 + 2p$

$$y = p^2 + 2$$

$\therefore$ Q is $(0, p^2+2)$ (1)

(iii) midpoint : $(p, \frac{2p^2+2}{2})$ which is (p, p^2+1) (1)

(iv) Let $x = p$, $y = p^2+1$

$$\therefore y = x^2 + 1$$

which is of the form of a parabola vertex $(0, 1)$

2 (Reas)