SCEGGS Darlinghurst

Name: ....

Teacher: ....

2005
Preliminary Course
Semester 2 Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- Write your name and your teacher's name at the top of each page
- Attempt **all** questions and show all necessary working
- Marks will be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used
- Begin each question on a new page
- **Do not** attach all questions together in one bundle

Total marks – 60

- Attempt Questions 1–5

Question	Total
1	/12
2	/12
3	/12
4	/12
5	/12
TOTAL	60
	%

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Instructions

- Write your name and your teacher's name at the top of each page
- Attempt **all** questions on the pad paper provided
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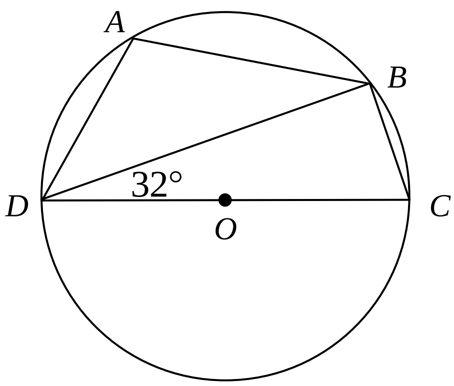
- Begin each question on a new page

Question 1 (12 marks) **Marks**

(a) A and B are the points $(1, 2)$ and $(5, 0)$ respectively. **2**
Find the co-ordinates of the point P which divides AB in the ratio $3:1$.

(b) A, B, C, D lie on the circumference of a circle, centre O. DC is the diameter and $\angle BDC = 32°$. **2**

Find $\angle BCD$ and $\angle BAD$ giving full reasons.



(c) Consider the polynomial $P(x) = 4x^3 - 8x^2 - 3x + 9$.

(i) Show that –1 is a zero of $P(x)$. **1**

(ii) Express $P(x)$ as a product of 3 linear factors. **2**

(iii) Hence, solve the inequality $P(x) \geq 0$. **2**

(d) Solve for x, **3**

$$\frac{2}{x+3} \leq 5$$

■ **Start a new page**

Question 2 (12 marks) **Marks**

(a) The polynomial $P(x) = x^3 + ax^2 + bx - 18$ has $x + 2$ as a factor and leaves a remainder of –24 when divided by $(x - 1)$. **3**

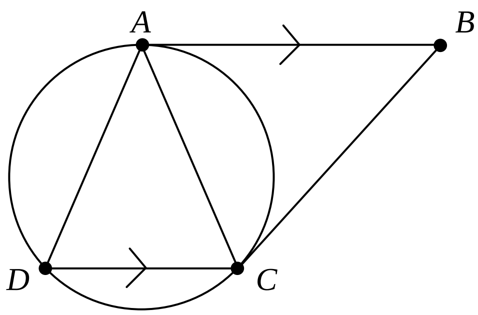
Find the values of *a* and *b*.

(b) (i) Show that the equation of the tangent to the parabola $x = 2at,\ y = at^2$ at the point $t = p$ is given by $y = px - ap^2$. **2**

(ii) Find the co-ordinates of the points where the tangent intersects the *x* and *y* axes. **2**

(iii) Hence find the area of the triangle formed by the two intercepts and the origin. **1**

(c)



A, C, D lie on the circumference of a circle. *AB* and *CB* are tangents at *A* and *C* respectively and *AB* is parallel to *DC*. **4**

Prove that $\Delta ABC \,|||\, \Delta DAC$.

■ **Start a new page**

Question 3 (12 marks) **Marks**

(a) α, β and γ are roots of the equation $x^3 - 3x^2 + 4x + 8 = 0$

(i) Find the values of:

(1) $\alpha + \beta + \gamma$ **1**

(2) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

(3) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Hence write down the cubic equation with roots $2\alpha, 2\beta$ and 2γ. **1**

(b) (i) The equation of the chord of contact from the point $P(-1, -2)$ to the parabola $x^2 = 4y$ is given by $y = -\frac{x}{2} + 2$. Show that the points of contact are $(-4, 4)$ and $(2, 1)$. **2**

(ii) Find the gradients of the tangents at these points of contact. **2**

(iii) Hence determine the size of the acute angle formed between the tangents (to the nearest minute). **2**

■ **Start a new page**

Question 4 (12 marks) **Marks**

(a) (i) Express $\cos x + \sin x$ in the form $R\cos\left(x - \alpha\right)$ where $R > 0$ and $0° < \alpha < 90°$. **2**

(ii) Hence solve $\cos x + \sin x = 1$ for $0° < x \le 360°$. **2**

(b) (i) Write down the expansion of $\tan 2\theta$. **1**

(ii) Given that $t = \tan 22.5°$, show that $t^2 + 2t - 1 = 0$. **2**

(iii) Hence show that $\tan 22.5° = \sqrt{2} - 1$. **2**

(c) Show by substitution that the parametric equation **3**

$$x = \sin\theta + \cos\theta$$
$$y = \tan\theta + \cot\theta$$

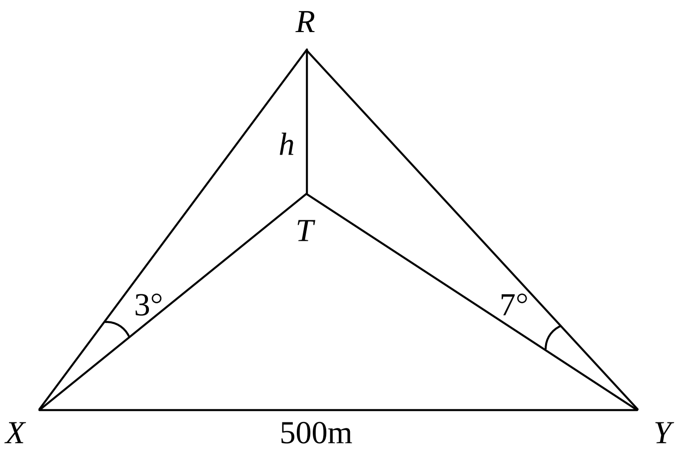
satisfies the cartesian equation

$$x^2 y = y + 2.$$

■ **Start a new page**

Question 5 (12 marks) **Marks**

(a) A knight in shining armour standing at point X observes the base of Rapunzel's tower (T) on a bearing of $035°$ and Rapunzel at the top of her tower (R) at an angle of elevation of $3°$. The knight rides 500m to a point Y from which the tower is on a bearing of $300°$ and Rapunzel is at an angle of elevation of $7°$.



(i) Show that $\angle XTY = 95°$. **1**

(ii) Show that the height, h, of the tower (in metres) is given by **2**

$$h = \frac{500}{\sqrt{\cot^2 3° + \cot^2 7° - 2\cot 3° \cot 7° \cos 95°}}$$

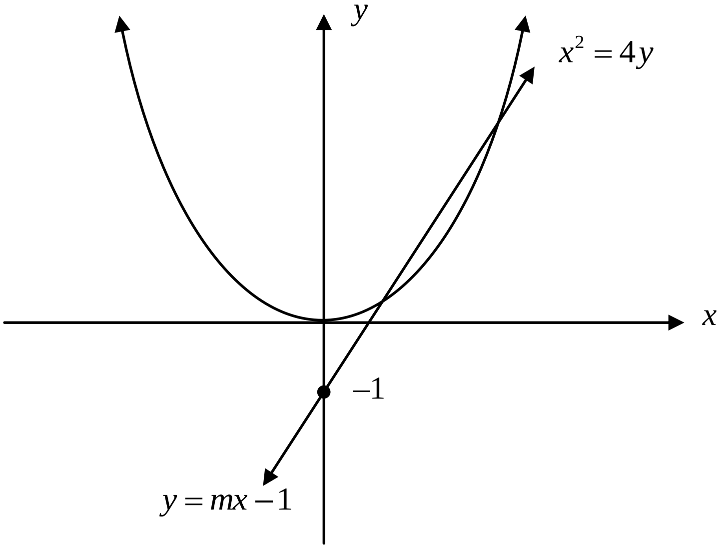
(iii) How long must Rapunzel's hair be to reach the base of the tower? **1**

Question 5 continues on the next page

Marks

Question 5 (continued)

(b)



(i) Show that the x co-ordinates of the points of intersection of the line $y = mx - 1$ and the parabola $x^2 = 4y$ satisfy the equation **1**

$$x^2 - 4mx + 4 = 0$$

(ii) For what values of m does the line $y = mx - 1$ and the parabola $x^2 = 4y$ have one or more points of intersection? **2**

Question 5 continues on the next page

Marks

Question 5 (continued)

(c) $P\left(2p,\ p^2\right)$ and $Q\left(2q, q^2\right)$ are variable points on the parabola $x^2 = 4y$ such that PQ passes through the point $(0, -1)$.

The equation of the chord PQ is given by $y = \left(\dfrac{p+q}{2}\right)\ x - pq$.

(i) Show that $pq = 1$. **1**

(ii) Show that the midpoint, M, of PQ is given by **1**

$$\left(p+q\ ,\ \frac{p^2+q^2}{2}\right)$$

(iii) Hence find the cartesian equation of the locus of M. **2**

(iv) Using the result from part (b), explain why the domain of the locus of M is restricted to $x \geq 2,\ x \leq -2$. **1**

End of paper

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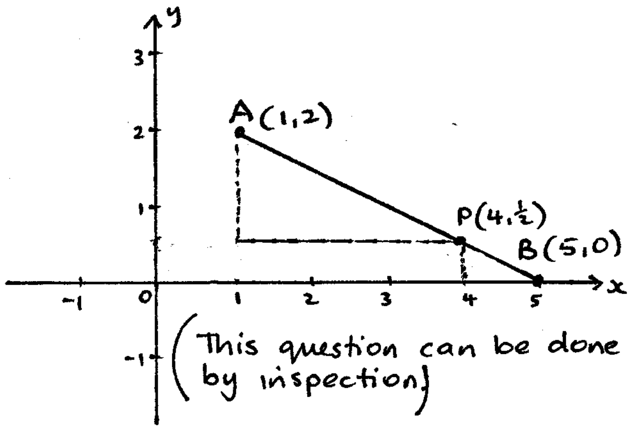
Question 1

a)



A(1,2) B(5,0)

3 : 1

$$P = \left(\frac{3\times5 + 1\times1}{3+1}, \frac{3\times0 + 1\times2}{3+1} \right)$$

$$= \left(\frac{16}{4}, \frac{2}{4} \right)$$

$$= \left(4, \tfrac{1}{2} \right)$$

✓ ✓

The point $P(4, \frac{1}{2})$ divides AB in the ratio 3:1

- This question very well done.

b)

$\angle DBC = 90°$ (Angle in a semicircle is 90° at the circumf.)

$\angle BCD = 180° - (90+32)°$ (Angle sum $\Delta BDC = 180°$)

$= 58°$

✓

$\angle BAD = 180° - \angle BCD$ (opposite angles in a cyclic quad. are supplementary)

$= 180° - 58°$

$= 122°$

✓

Since A,B,C,D lie on the circumference of the circle ABCD is a cyclic quadrilateral.

Comm 2

- Learn the correct wording of circle geometry reasons.
- Include this reason for angle sum of a triangle.

3 reasons → 2 marks.

c) $P(x) = 4x^3 - 8x^2 - 3x + 9$

i) $P(-1) = -4 - 8 + 3 + 9$
$= 0$

$\therefore x = -1$ is a zero ✓

$\therefore (x+1)$ is a factor of $P(x)$

• This part was very well done.

Note $x = -1$ is a zero

$(x+1)$ is a factor

---

ii)

$$\begin{array}{r}
4x^2 - 12x + 9 \\
x+1 \overline{)\, 4x^3 - 8x^2 - 3x + 9} \\
\underline{4x^3 + 4x^2} \qquad\qquad\quad \\
-12x^2 \qquad\quad \\
\underline{-12x^2 - 12x} \qquad \\
9x \quad \\
\underline{9x + 9} \\
0
\end{array}$$

$P(x) = (x+1)(4x^2 - 12x + 9)$ ✓

$= (x+1)(2x-3)^2$ ✓

Since $(x+1)$ is a factor, there should be no remainder. If you have a remainder, something's gone wrong!

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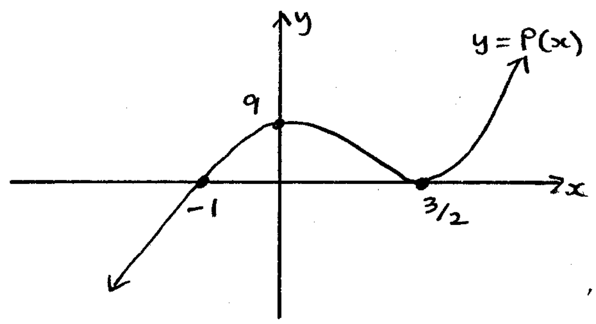
iii)



$P(x) \geq 0$

for $x \geq -1$ ✓ ✓

✓ one mark for correct sketch

✓ one mark for correct solution from your sketch.

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d) $\dfrac{2}{x+3} \leq 5$ undefined for $x = -3$

multiply both sides by $(x+3)^2$

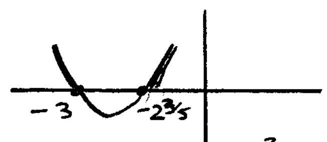
$\dfrac{2}{x+3} \times (x+3)^2 \leq 5(x+3)^2$

$2(x+3) \leq 5(x^2+6x+9)$

$5x^2 + 30x + 45 - 2x - 6 \geq 0$

$5x^2 + 28x + 39 \geq 0$ ✓

$(5x+13)(x+3) \geq 0$



$x \leq -3,\ x \geq -2\tfrac{3}{5}$

$\therefore$ Solution ✓

$x < -3,\ x \geq -2\tfrac{3}{5}$ ✓

• Don't forget that the original is undefined when $x = -3$.
(This is an easy mark.)

✓ one mark $x \neq -3$
✓ correct quadratic
✓ correct solution of your quadratic.

Question 2 Calc /2 Reas /4

(a) $(x+2)$ is a factor $\Rightarrow P(-2)=0$

$$(-2)^3+a(-2)^2+b(-2)-18=0$$
$$-8+4a-2b-18=0$$
$$4a-2b=26$$
$$\boxed{2a-b=13} \quad (1)$$

•Generally people knew the factor/rem. theorems or they didn't. If you are someone who doesn't know it – learn it!

$(x-1)$ leaves rem. of $-24 \Rightarrow P(1)=-24$

$$1^3+a\times1^2+b\times1-18=-24$$
$$1+a+b-18=-24$$
$$\boxed{a+b=-7} \quad (2)$$

Solve simult.

$(1)+(2) \Rightarrow 3a=6$

$a=2$

sub in (2) $\Rightarrow b=-9$

(b) (i) $x=2at \quad y=at^2$

Pt: $t=p \therefore \boxed{P(2ap, ap^2)}$

Gradient: $\underbrace{\frac{dx}{dt}=2a \quad \frac{dy}{dt}=2at}$

$$\frac{dy}{dx}=\frac{2at}{2a}=t$$

$$\therefore \boxed{m_T=p}$$ Calc2

• You needed to show the gradient of the tangent $=p$ by differentiating first.

Eqn tang: $y-ap^2=p(x-2ap)$

$$y-ap^2=px-2ap^2$$
$$y=px-ap^2$$

• If there are no ts in the question & you know $t=p$, you should substitute!

(ii) x int: $y=0$

$0=px-ap^2$

$x=ap$ $(ap,0)$

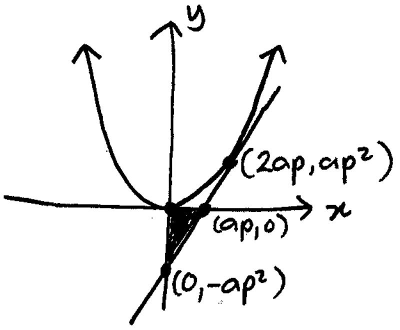
y int: $x=0$

$y=p\times0-ap^2$

$y=-ap^2$ $(0,-ap^2)$

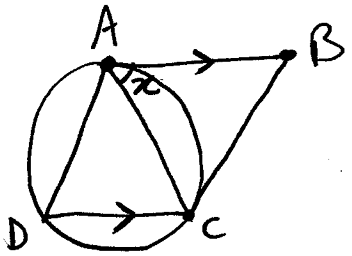
• You could have answered from (ii) $\rightarrow$ even if you didn't understand (i). You need to be able to recognise when you can do this.

(iii)



$$\text{Area } \Delta = \left|\tfrac{1}{2} \times \text{base} \times \text{height}\right|$$
$$= \left|\tfrac{1}{2} \times ap \times ap^2\right|$$
$$= \left|\frac{a^2p^3}{2}\right| \quad ✓$$

(c)



let $\angle BAC = x$

$AB = BC$ (tang. from an ext. pt. are =) ✓

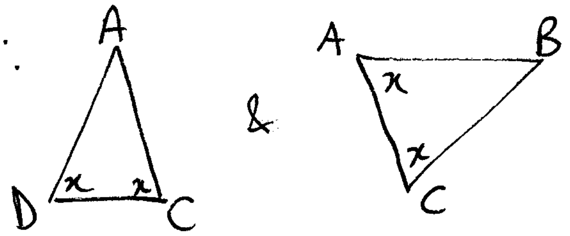
$\therefore \Delta ABC$ is isos.

$\therefore \angle BAC = x = \angle BCA$ ($\angle$s opp. = sides of isos Δ =) ✓

$\angle BAC = x = \angle ACD$ (alt. $\angle$s on // lines AB & DC =) ✓

$\angle BAC = x = \angle ADC$ ($\angle$ bet. tang. & chord = $\angle$ in alt. seg.) ✓

$\therefore$



Reas. 4

$\therefore \Delta ABC \,|||\, \Delta DAC$ (equiangular)

• Technically you need the absolute value signs for the area. 'p' can certainly be −ve.

• There are various ways to prove this question

You need to be careful with the alternate segment theorem eg. $\angle BCA \neq \angle DAC$!

You also need to be careful when naming angles – one incorrect letter can mean a world of difference.

Question 3

a) $x^3 - 3x^2 + 4x + 8 = 0$

i) 1) $\alpha+\beta+\gamma = -b/a$

$= --3/1$

$= 3$ ✓

(2) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ ✓

$= \frac{c/a}{-d/a}$

$= \frac{4/1}{-8/1}$

$= -\frac{1}{2}$ ✓

(3) $\alpha^2+\beta^2+\gamma^2$

Note $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 3^2 - 2\times4$ ✓

$= 9-8$

$= 1$ ✓

ii) The equation has the form

$ax^3+bx^2+cx+d=0$

$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}x$

Sum of roots

$2\alpha+2\beta+2\gamma = -\frac{b}{a}$

$2(\alpha+\beta+\gamma) = -\frac{b}{a}$

$2\times3 = -\frac{b}{a}$

$\therefore \frac{b}{a} = -6$

Sum two at a time

$2\alpha.2\beta+2\alpha.2\gamma+2\beta.2\gamma = \frac{c}{a}$

$4(\alpha\beta+\alpha\gamma+\beta\gamma) = \frac{c}{a}$

$4\times4 = \frac{c}{a}$

$\therefore \frac{c}{a} = 16$

Product.

$2\alpha.2\beta.2\gamma = -\frac{d}{a}$

$8\alpha\beta\gamma = -\frac{d}{a}$

$8\times-8 = -\frac{d}{a}$

$\therefore \frac{d}{a} = 64$

✓ Reas

$\therefore$ Equation of cubic equation

$x^3 - 6x^2 + 16x + 64 = 0$

LEARN THE RULES!!!

$ax^3+bx^2+cx+d=0$

$$\boxed{\begin{aligned}\alpha+\beta+\gamma &= -\frac{b}{a}\\ \alpha\beta+\alpha\gamma+\beta\gamma &= \frac{c}{a}\\ \alpha\beta\gamma &= -\frac{d}{a}\end{aligned}}$$

This question was well done as almost everyone knew their rules. ☺

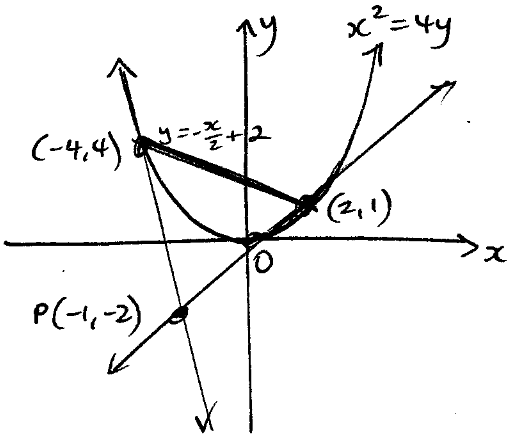
These results came from part(a)

$$\left.\begin{aligned}\alpha+\beta+\gamma &= -\frac{b}{a} = 3\\ \alpha\beta+\alpha\gamma+\beta\gamma &= \frac{c}{a} = 4\\ \alpha\beta\gamma &= -\frac{d}{a} = -8\end{aligned}\right\}$$ use these values in part(ii)

- Another method is to write the equation as
  $(x-2\alpha)(x-2\beta)(x-2\gamma)=0$
  then expand and collect like terms.
  then substitute the values for sum, product etc. found in part(a).

- part(ii) was a bit tricky.
  It had to be perfectly correct to gain one mark.

b) i)



① $y=-\frac{x}{2}+2$
② $x^2=4y$
} Solve simultaneously

substitute ① into ②

$$x^2=4\left(-\frac{x}{2}+2\right)$$
$$x^2=-2x+8$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$

$x=-4$ $\qquad$ $x=2$

$y=\frac{4}{2}+2$ $\qquad$ $y=-1+2$

$=4$ $\qquad$ $=1$

$(-4,4)$ $\qquad$ $(2,1)$ $\qquad$ points of contact

- It is not sufficient to substitute the points $(-4,4)$ and $(2,1)$ into $y=-\frac{x}{2}+2$. This shows only that the points satisfy this line's equation.

ii) $x^2=4y$

$$y=\frac{x^2}{4}$$
$$\frac{dy}{dx}=\frac{2x}{4}$$
$$=\frac{x}{2}$$

At $(-4,4)$ gradient tangent

$$m_1=-\frac{4}{2}$$
$$=-2$$

At $(2,1)$ gradient tangent

$$m_2=\frac{2}{2}$$
$$=1$$

Calc

- Another method is to find gradients using two end points $m=\frac{y_2-y_1}{x_2-x_1}$

iii)
$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$=\left|\frac{-2-1}{1-2\times1}\right|$$
$$=\left|\frac{-3}{-1}\right|$$
$$=3$$

$\therefore\ \theta=71^\circ 34'$ (to nearest minute)

If you're going to use a formula make sure you know it and you get the signs correct!!!!

Question 4 Reas. /7

(a) (i) $\cos x + \sin x$

$R\cos(x-\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

$\therefore \left.\begin{array}{ll} R\cos\alpha = 1 & ① \\ R\sin\alpha = 1 & ② \end{array}\right]$ Solve ✓

$\frac{②}{①} \Rightarrow \tan\alpha = 1$

$\alpha = 45°$

$①^2 + ②^2 \Rightarrow R = \sqrt{2}$ ✓

$\therefore \cos x + \sin x \equiv \sqrt{2}\cos(x - 45°)$

(ii) Instead, solve

$\sqrt{2}\cos(x-45°) = 1$

$\cos(x - 45°) = 1/\sqrt{2}$

S A ✓ T C ✓

$(x - 45°) = 45°, 315°$

$x = 90°, 360°.$ ✓ ✓

(b) (i) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ ✓

(ii) let $\theta = 22{\cdot}5°$ above

$\Rightarrow \tan 45° = \dfrac{2\tan 22{\cdot}5°}{1-\tan^2 22{\cdot}5°}$ ✓

$1 = \dfrac{2t}{1-t^2}$

$1 - t^2 = 2t$ ✓

$t^2 + 2t - 1 = 0$

Reas. 4

(iii) $t = \dfrac{-2 \pm \sqrt{2^2 - 4\times1\times-1}}{2\times1}$

$= \dfrac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$ ✓

$\therefore \tan 22{\cdot}5° = -1+\sqrt{2}$ or $-1-\sqrt{2}$

since $22{\cdot}5°$ is in 1st quad. ✓

$\tan 22{\cdot}5° = -1+\sqrt{2}$

• Auxiliary angle method is a standard method for solving trig equations — & one you need to know!

• Surprising is how many of you knew how to solve the equation (ii) but didn't understand how to answer part (i).

• Watch the domain — there was actually no trick this time, but people assumed there was. $0° < x \leq 360°$.

(i) is a GIVEAWAY! A ridiculous # of people got it wrong.

• They wouldn't give you such ~~an~~ easy mark for part (i) unless they wanted you to use it. So use it!

• (iii) is doable even if you got lost previously. When you get lost you must scan down to see where you can pick up the question again.

(c) $x = \sin\theta + \cos\theta$

$y = \tan\theta + \cot\theta$

Prove $x^2 y = y + 2$

$$\begin{aligned}
LHS &= (\sin\theta + \cos\theta)^2 \times y \\
&= (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\, y \\
&= (1 + 2\sin\theta\cos\theta)\, y \\
&= y + 2\sin\theta\cos\theta\, y \\
&= y + 2\sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \\
&= y + 2\sin^2\theta + 2\cos^2\theta \\
&= y + 2(\sin^2\theta + \cos^2\theta) \\
&= y + 2 \\
&= RHS
\end{aligned}$$

Reas. 3

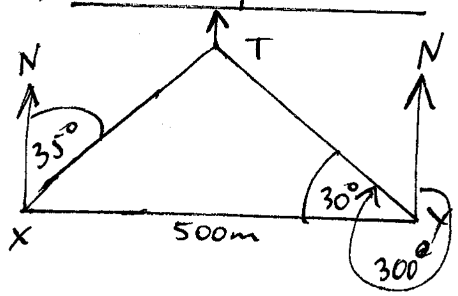
- $x^2 \neq \sin^2\theta + \cos^2\theta$!

$$\begin{aligned}
x^2 &= (\sin\theta + \cos\theta)^2 \\
&= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta
\end{aligned}$$

- Writing up such proofs are easier if you use LHS / RHS - type setting out.

Question 5

a) Bird's Eye View



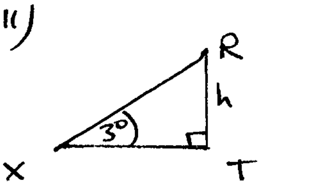
i) $\angle XYT = 300° - 270°$
$= 30°$

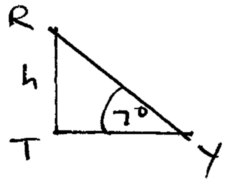
$\angle TXY = 90° - 35°$ (right $\angle = 90°$)
$= 55°$

$\therefore \angle XTY = 180° - (30° + 55°)$ ($\angle$ sum $\Delta = 180°$)
$= 95°$

You should list out steps to show why $\angle XTY = 95°$. Although it is not a full geometric proof it is still not sufficient to just write angles on a diagram with no clear reasoning.

ii)





$\tan 3° = \frac{h}{XT}$

$XT = \frac{h}{\tan 3°}$

$= h \cot 3°$

$\tan 7° = \frac{h}{TY}$

$TY = \frac{h}{\tan 7°}$

$= h \cot 7°$.

① mark for expressions for both XT and TY

① mark for use of the cosine rule to find h → showing clear factorising step. (no fudging!)

Using the cosine rule in ΔXTY

$500^2 = XT^2 + TY^2 - 2XT.TY.\cos 95°$

$250000 = (h\cot 3°)^2 + (h\cot 7°)^2 - 2h\cot 3°.\cot 7°.\cos 95°$

(factorise h^2)

$500^2 = h^2(\cot^2 3° + \cot^2 7° - 2\cot 3° \cot 7° \cos 95°)$

$\therefore h^2 = \frac{500^2}{\cot^2 3° + \cot^2 7° - 2\cot 3° \cot 7° \cos 95°}$

$h = \frac{500}{\sqrt{\cot^2 3° + \cot^2 7° - 2\cot 3°.\cot 7°.\cos 95°}}$

iii) $\doteqdot 23.38$ m

Calculator step. Worth practising! (1 mark)

b) i) Solve simultaneously

$y = mx - 1$ ①

$x^2 = 4y$ ②

Substitute ① into ②

$x^2 = 4(mx-1)$

$x^2 = 4mx - 4$

$x^2 - 4mx + 4 = 0$. ✓

Most students managed this part of part b).

ii) One or more points of intersection means that this quadratic equation has one or more solutions

$\therefore\ \Delta \geq 0$

$\Delta = b^2 - 4ac$

$= (-4m)^2 - 4 \cdot 1 \cdot 4$

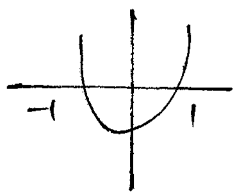
$= 16m^2 - 16$

Solve $\Delta \geq 0$ ✓

$16m^2 - 16 \geq 0$

$16(m^2-1) \geq 0$

$16(m-1)(m+1) \geq 0$



✓

$m \leq -1, \ m \geq 1$

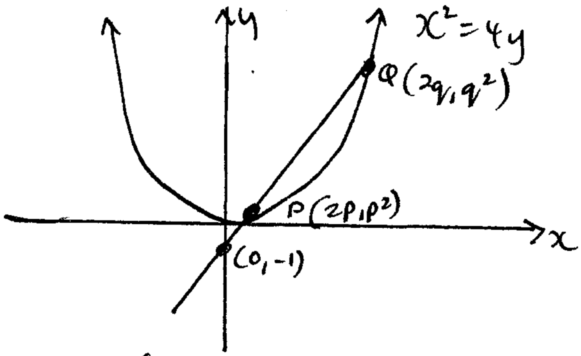
For these values of the gradient, the line and the parabola have one or more points of intersection.

Reas 2

Applications of the Discriminant
For points of intersection between line and a parabola.
Solve simultaneously and then apply to the quadratic equation formed.
$\Delta \geq 0$ one or more solutions
$\Delta > 0$ more than one solution
$\Delta = 0$ one solution → line is a tangent
$\Delta < 0$ no solutions → line and curve have no intersection.

It is always a good idea to be aware of these applications. The use of the discriminant is a technique that appears frequently in harder questions in both the 2U and Ext① exams.

c)



Draw a sketch.
It helps sort out the logic in this question.

Chord PQ $y=\left(\frac{p+q}{2}\right)x-pq$

---

i) Chord PQ passes through the point $(0,-1)$

Substitute this point into PQ

$$y=\left(\frac{p+q}{2}\right)x-pq$$

$$-1 = 0-pq$$

$$\therefore pq=1$$ ✓

Reas 1

---

ii) $P(2p,p^2)$ $Q(2q,q^2)$

midpoint $M=\left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$

$$=\left(p+q, \frac{p^2+q^2}{2}\right)$$ ✓

- Easy one mark here.
- Very well done by everyone who attempted it.

---

iii) Locus of M

$$x=p+q$$

$$y=\tfrac{1}{2}(p^2+q^2)$$

$$=\tfrac{1}{2}\left((p+q)^2-2pq\right)$$ ✓

$$=\tfrac{1}{2}\left(x^2-2\times1\right)$$

$$=\tfrac{1}{2}(x^2-2)$$ ✓

$$2y=x^2-2$$

$$x^2=2y+2$$

$$x^2=2(y+1)$$

$$x^2=4\cdot\tfrac{1}{2}(y+1)$$

the locus of M is a parabola.

Learn this
$$\boxed{p^2+q^2=(p+q)^2-2pq.}$$

This substitution is very common in locus problems.

Reas 2

---

iv) See next page for 3 different solutions that were presented by students.
Well done if you were responsible for any of them! ☺

① iv) Since the locus of M is formed by a chord on the parabola $x^2 = 4y$, this chord has one or more points of intersection when

$m \le -1, \; m \ge 1$ (from part(b))

<u>Since the chord</u> has equation

$$y = \left(\frac{p+q}{2}\right)x - pq$$

(substitute $pq = 1$)

$$= \left(\frac{p+q}{2}\right)x - 1$$

this matches the line's equation given in part (b)

$$y = mx - 1$$

<u>matching the gradients</u>

$$\frac{p+q}{2} = m$$

$m \le -1 \quad , \quad m \ge 1$

$\frac{p+q}{2} \le -1 \quad , \quad \frac{p+q}{2} \ge 1$

$p+q \le -2 \quad , \quad p+q \ge 2$

Since $x = p+q$ for point M

$x \le -2 \quad , \quad x \ge 2$

✓ Reas

② Outside the parabola, there is no chord and hence no midpoint.

The chord only exists inside or on the parabola $x^2 = 4y$.

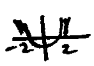
The region inside the parabola is described by $x^2 \le 4y$

<u>the locus of M</u>

$$x^2 = 2y + 2$$
$$2y = x^2 - 2$$

substitute this locus into the region

$$x^2 \le 4y$$
$$x^2 \le 2.2y$$
$$x^2 \le 2.(x^2 - 2)$$
$$x^2 \le 2x^2 - 4$$
$$-x^2 \le -4$$
$$x^2 \ge 4$$
$$(x-2)(x+2) \ge 0$$



$x \le -2, \; x \ge 2$

The locus of M lies within the parabola for this domain.

③ <u>Parabola</u> $x^2 = 4y, \; y = \frac{x^2}{4}$

<u>Chord</u> $y = \left(\frac{p+q}{2}\right)x - 1$

<u>points of intersection</u>

solve simultaneously

$$\frac{x^2}{4} = \left(\frac{p+q}{2}\right)x - 1$$
$$x^2 = 2(p+q)x - 4$$
$$x^2 - 2(p+q)x + 4 = 0$$

$$\Delta = b^2 - 4ac$$
$$= (2(p+q))^2 - 4.1.4$$
$$= 4(p+q)^2 - 16$$

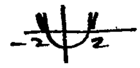
for the chord and parabola to have one or more points of intersection

$\Delta \ge 0$

$$4(p+q)^2 - 16 \ge 0$$
$$(p+q)^2 - 4 \ge 0$$
$$(p+q-2)(p+q+2) \ge 0$$

for the locus of M $\quad x = p+q$

$$(x-2)(x+2) \ge 0$$



$x \le -2, \; x \ge 2.$

<u>Note</u>: this method is using the same logic as in part(b) just a bit more complicated algebra.