SCEGGS Darlinghurst

2007

Preliminary Course
Semester 2 Examination

Mathematics Extension 1

Outcomes Assessed: P1–P8, PE1, PE2, PE3 and PE6
Task Weighting: 40%

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- This paper has **four** questions
- Write using blue or black pen
- Answer all questions on the pad paper provided
- Write your Student Number at the top of each page
- Attempt **all** questions and show all necessary working
- **Start each question on a new page**
- Marks will be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used
- **Do not** attach all questions together in one bundle.

Total marks – 60

- Attempt Questions 1 – 4

Question	Comm	Reason	Calc	TOTAL
1	/3	/2	/3	**/15**
2	/4	/3	/2	**/14**
3	/3	/7		**/15**
4		/10		**/16**
TOTAL	/10	/22	/5	**/60**

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Marks

Question 1 (15 marks)

(a) Solve the inequality: **2**

$$x(x-2)>3$$

(b) (i) Fully factorise $f(x)=x^4+3x^3+2x^2$ **1**

(ii) Sketch the polynomial $y=f(x)$ showing clearly any x and y intercepts. (You do not have to find the co-ordinates of the turning points.) **2**

(iii) On a new set of axes, sketch the curve $y=f(x-2)$. **1**

(c) A polynomial is given by $P(x)=x^3+Ax^2+Bx-12$. **3**

When $P(x)$ is divided by $(x+5)$ the remainder is -2 and $(x-3)$ is a factor of $P(x)$.

Find the values of A and B.

(d) (i) Write the expansion for $\sin(\theta+\alpha)$. **1**

(ii) If $\tan\theta=\dfrac{1}{2}$ and $0°<\theta<90°$, find the exact value of $\sin(\theta+45°)$ **2**

Give your answer in simplest surd form.

(e) (i) Differentiate $y=\dfrac{1}{x+a}$. **1**

(ii) Find the value(s) of a if $y=\dfrac{1}{x+a}$ has a gradient of -1 when $x=3$. **2**

- Start a new page

Marks

Question 2 (14 marks)

(a) (i) Find the points of intersection of the hyperbola $y = \frac{3}{x}$ and the line $y = 2x - 5$. **2**

(ii) Sketch the graphs of $y = \frac{3}{x}$ and $y = 2x - 5$ on the same set of axes, showing all important features and points of intersection. **2**

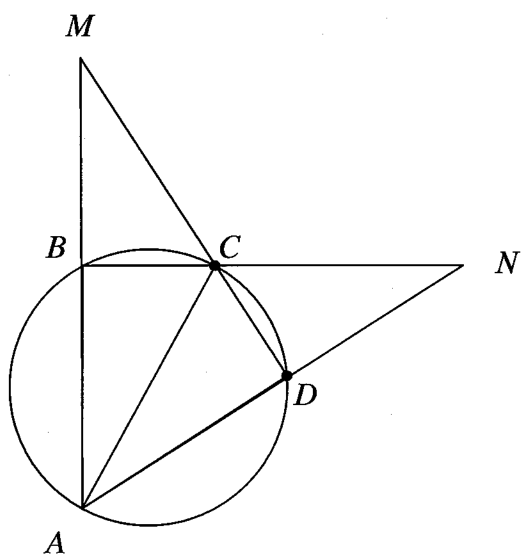
(iii) Let the point where the curves intersect in Quadrant 1 be the point P. Find the gradient of the line and the hyperbola at point P. **2**

(iv) Hence, find the acute angle between the line and the hyperbola at the point P. (Answer correct to the nearest minute.) **2**

(v) Using your graph, or otherwise, find the values of x for which **2**

$$2x - 5 \leq \frac{3}{x}$$

(b)



In the diagram, AM, AN, BN and DM are straight lines and $\angle AMD = \angle ANB$.

(i) By considering ΔAMD and ΔANB or otherwise, explain why $\angle ABC = \angle ADC$. **2**

(ii) Hence prove that AC is a diameter. **2**

• Start a new page

Marks

Question 3 (15marks)

(a) (i) Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. **2**

(ii) Hence, find the general solutions of $\cos\theta + \sqrt{3}\sin\theta = 1$. **2**

(b) Consider the curve $y = \dfrac{x^2 + 6}{x + 2}$.

(i) Use long division to write $\dfrac{x^2 + 6}{x + 2}$ in the form $Q(x) + \dfrac{R(x)}{x + 2}$, where $Q(x)$ is the quotient and $R(x)$ is the remainder. **2**

(ii) Explain why $y = x - 2$ is an oblique asymptote of the graph $y = \dfrac{x^2 + 6}{x + 2}$. **1**

(iii) State any vertical asymptotes. **1**

(iv) Sketch the curve $y = \dfrac{x^2 + 6}{x + 2}$ showing any important features. **2**
(You do not have to find the co-ordinates of any turning points.)

Question 3 continues on the next page

Marks

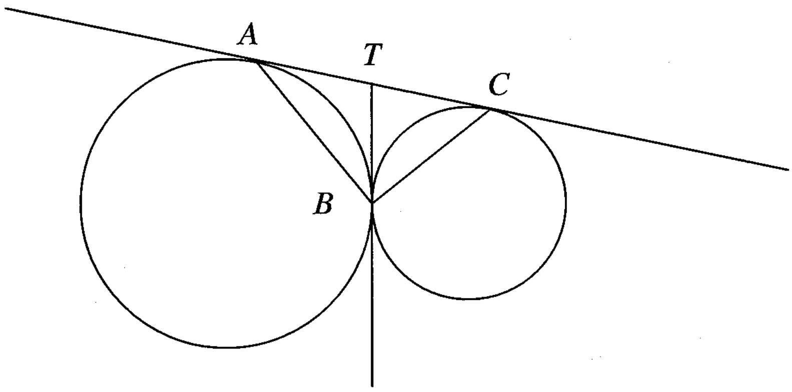
Question 3 (continued)

(c) TA and TB are two tangents drawn to a circle from an external point T. A and B are the points of contact of the tangents with the circle.

Draw a clear diagram on your answer page.

(i) Prove that $TA = TB$. **3**

(ii) AC is an external tangent to two circles that touch at B. The internal tangent at B meets AC at T. **2**



Use part (i) to show T is the centre of the circle passing through A, B and C and hence find the size of $\angle ABC$.

- Start a new page

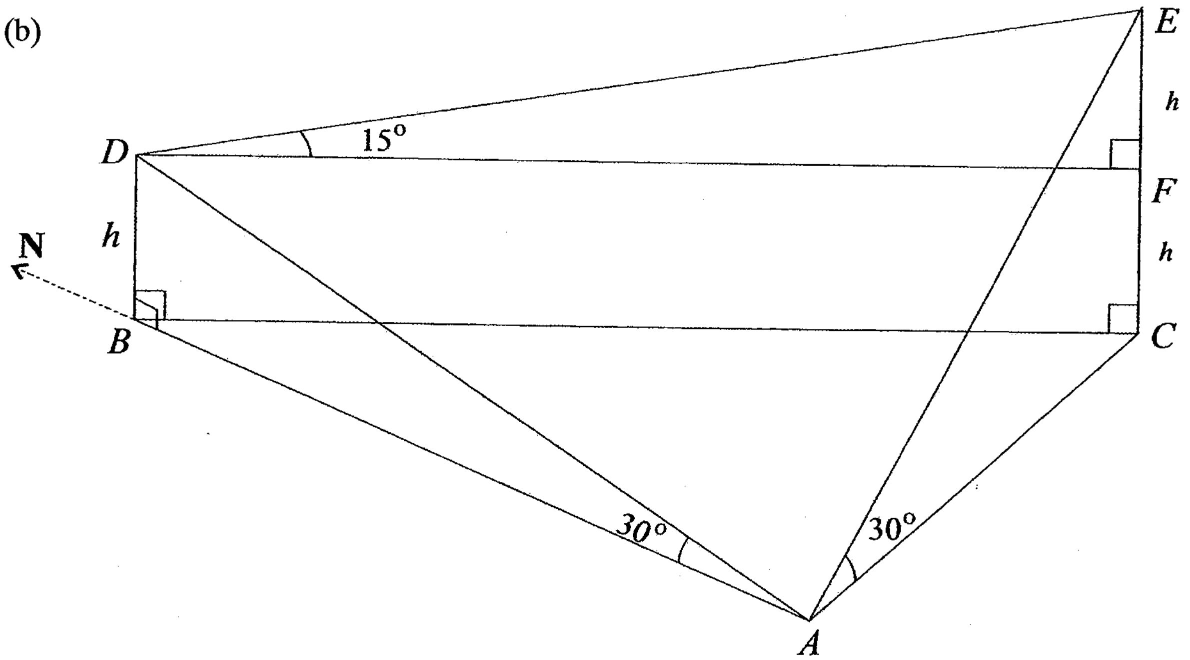
Marks

Question 4 (16 marks)

(a) By making the substitution $t = \tan\dfrac{\theta}{2}$, prove that **2**

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$$

(b)



The diagram above shows two vertical towers BD and CE of heights h and $2h$ respectively on a horizontal plane ABC.

Point A is due south of point B and the angles of elevation of the tops of the towers from A are both $30°$. The angle of elevation from D to E is $15°$.

(i) Show that $AB = \sqrt{3}h$. **1**

(ii) Write down expressions for the lengths of AC and BC. **2**

(iii) Hence, find the bearing of the taller tower CE from point A, correct to the nearest degree. **2**

Question 4 continues on the next page

Marks

Question 4 (continued)

(c)

The diagram shows a straight line segment AC divided by B in the golden ratio $\varphi : 1$.

A divides CB externally in the same ratio that B divides AC internally.

(i) Show that the golden number φ, satisfies the equation **1**

$$\varphi^2 - \varphi - 1 = 0$$

(ii) Hence find the exact value of φ. **1**

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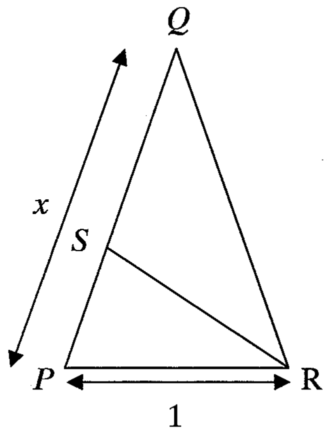
Marks

Question 4 (continued)

(d)



PQR is an isosceles triangle with $\angle QPR = \angle QRP = 72°$.
$PQ = x$ and $PR = 1$.
The angle bisector of $\angle QRP$ meets QP at S.

(i) Show that $PR = SR = SQ = 1$. **2**

(ii) Show that $\Delta PQR \,|||\, \Delta PRS$. **1**

(iii) Hence show that x is the golden number φ (from part (c)). **2**

(iv) By using the sine rule in ΔPQR show that $\cos 36° = \dfrac{\sqrt{5}+1}{4}$. **2**

End of paper

2007
MATHEMATICS EXT. 1
SEMESTER 2 EXAM
(SOLUTIONS)

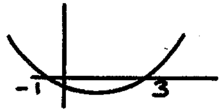
Total /15 Comm /3 Reas /2 Calc /3

Question 1

(a) $x(x-2) > 3$

$x^2 - 2x - 3 > 0$

$(x-3)(x+1) > 0$ ✓



$x > 3, \; x < -1$ ✓

Don't forget to expand first and then form the quadratic inequality.
You can never do $x > 3$, $x - 2 > 3$
Some people got tricked by this part.

(b) i) $f(x) = x^4 + 3x^3 + 2x^2$

$= x^2(x^2 + 3x + 2)$

$= x^2(x+1)(x+2)$ ✓

Look for easy steps such as common factors.
You do not always need to do long division!

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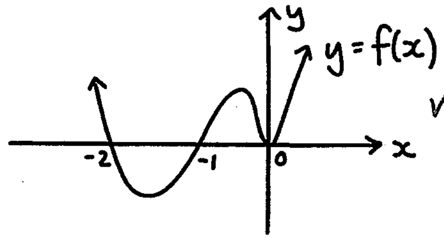
ii)

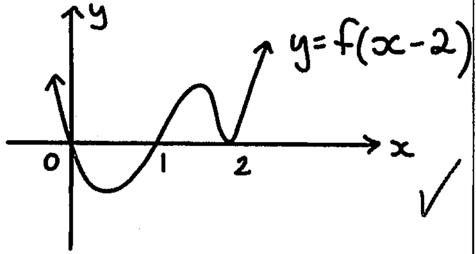


✓✓

(Comm 3)

Well done ü

iii)



✓

Shift your part (i) 2 units to the right.

(c) $P(x) = x^3 + Ax^2 + Bx - 12$

$P(-5) = -2 \Rightarrow$

$-125 + 25A - 5B - 12 = -2$

$25A - 5B = 135$

$5A - B = 27$ ① ✓

An easy question! Learn your theorems!
By the remainder theorem.
$P(-5) = -2$

$P(3) = 0 \Rightarrow$

$27 + 9A + 3B - 12 = 0$

$9A + 3B = -15$

$3A + B = -5$ ② ✓

By the factor theorem.
$P(3) = 0$

Solve Simultaneously

① + ② $\Rightarrow 8A = 22$

$A = 2\frac{3}{4}$

$\Rightarrow B = -13\frac{1}{4}$ ✓

Note: In algebra A is different to a. B is different to b. Do not mix capitals and small letters.

(d) i) $\sin(\theta + \alpha)$

$= \sin\theta\cos\alpha + \cos\theta\sin\alpha$ ✓

Well done. Everyone knew this rule ü

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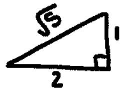
ii) $\tan\theta = \frac{1}{2} \Rightarrow$ 

$0° < \theta < 90°$

Reas. 2 Many students didn't see the link in this question. Why are you given $\tan\theta = \frac{1}{2}$?

$\sin(\theta + 45°)$

$= \sin\theta\cos 45° + \cos\theta\sin 45°$

$= \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{2}}$ ✓

$= \frac{3}{\sqrt{10}}$

$= \frac{3\sqrt{10}}{10}$ ✓

This part stumped a lot of students. It's not really too hard.

→ You must show the substitution for $\sin\theta$ and $\cos\theta$ to get this mark.

Final answer must have rationalised denominator. Read the fine details of the question.

(e) i) $y = \frac{1}{(x+a)}$

$= (x+a)^{-1}$

Calc. 3

Use the Chain rule to differentiate

$\frac{dy}{dx} = -1(x+a)^{-2} \times 1$

$= \frac{-1}{(x+a)^2}$ ✓

Note that a is just a constant. (It works like a number.) Using the quotient rule is open to more errors.

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ii) $\frac{dy}{dx} = -1$ when $x = 3$

$\Rightarrow -1 = \frac{-1}{(3+a)^2}$

$(3+a)^2 = 1$

$3 + a = \pm 1$ ✓

OR you could expand this quadratic and then solve it
$9 + 6a + a^2 = 1$
$a^2 + 6a + 8 = 0$
etc.

$3 + a = 1$	$3 + a = -1$
$a = -2$	$a = -4$

✓

Total /14 Comm /4 Reas /2 Calc /2

Question 2

(a) i) For pts of int. solve simult.

$$\frac{3}{x} = 2x - 5$$
$$3 = 2x^2 - 5x$$
$$2x^2 - 5x - 3 = 0$$
$$2x^2 - 6x + x - 3 = 0$$
$$(2x+1)(x-3) = 0$$
$x = -\frac{1}{2}$ $x = 3$ ✓
$y = -6$ $y = 1$ ✓

$\therefore$ Pts of int: $(-\frac{1}{2}, -6)$, $(3, 1)$

Nicely done

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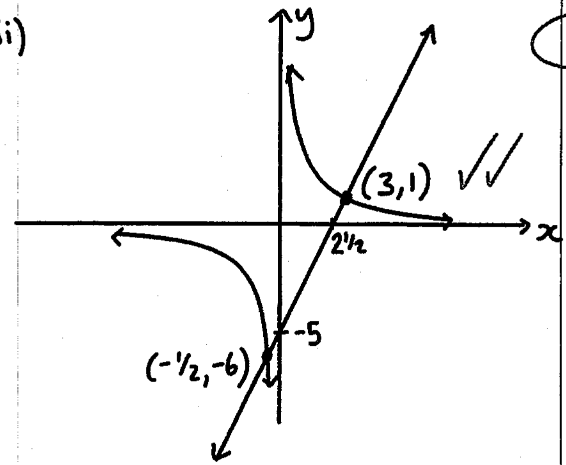
ii)



✓✓

Comm 2.

* All points of intersection and x/y intercepts should be labelled – an indistinct straight line for $y = 2x - 5$ was not good enough.

* Hyperbolas weren't particularly well drawn – it is important they approach the asymptotes

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iii) m of line = 2 ✓

m of hyperbola : $y = 3/x$

$y' = -3/x^2$

$m_T = -3/3^2$

$= -1/3$ ✓

Calc 2.

Differentiation of $\frac{3}{x}$ & subsequent substitution of $x = 3$ was extremely poorly done!

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iv) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - -1/3}{1 + -2/3}\right|$ ✓

$= 7$

$\theta = \tan^{-1} 7$

$= 81°52'$ (to nearest minute) ✓

Most new this formula & how to apply it

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v) $x \le -\frac{1}{2}$ ✓, $0 < x \le 3$ ✓

It is easiest to read the solution off the graph. Most chose to solve 'otherwise' and did so poorly – most remember $x \ne 0$ & also to multiply by the denominator squared

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(b) i) $\angle AMD = \angle ANB$ (given)

$\angle MCB = \angle NCD$

(vertically opposite $\angle$s =)

$\therefore \angle AMD + \angle MCB = \angle ANB + \angle NCD$

$\Rightarrow \angle ABC = \angle ADC$

(exterior $\angle$ Δ = sum of 2 opposite interior $\angle$s)

Comm 2

✓ Many proved similar Δs using just 2 angles then quoted the 3rd angles as equal because of ✓ corresponding angles ... → This shows you don't really understand why only 2 angles are required to prove similarity! The 3rd pair of angles are = because $\angle$ sum Δ = 180°!

ii) $\angle ABC = \angle ADC$ (above)

$\angle ABC + \angle ADC = 180°$

(opposite $\angle$s of a cyclic quad. are supplementary) ✓

$\therefore \angle ABC = \angle ADC = 90°$

$\therefore$ AC is a diameter

($\angle$s in a semicircle = 90°) ✓

Reas 2

Part ii was done well by those who attempted it

Total /15 Comm /3 Reas /7

Question 3

(a) i) $\cos\theta + \sqrt{3}\sin\theta \equiv R\cos(\theta - \alpha)$

$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 1$ ①

$R\sin\alpha = \sqrt{3}$ ②

Find α

$\frac{②}{①}$ $\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{1}$

$\tan\alpha = \sqrt{3}$

$\alpha = 60^\circ$ ✓

Find R

$R^2 = 1^2 + \sqrt{3}^2$

$R = \sqrt{1+3}$

$= \sqrt{4}$

$= 2$ ✓

$\therefore \cos\theta + \sqrt{3}\sin\theta = 2\cos(\theta - 60^\circ)$

This part was very well done. Everyone seems to have mastered this technique.

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ii) $\cos\theta + \sqrt{3}\sin\theta = 1$

$2\cos(\theta - 60^\circ) = 1$

$\cos(\theta - 60^\circ) = \frac{1}{2}$ (Quad 1 & 4)

$\theta - 60^\circ = -60^\circ, 60^\circ, 300^\circ$

$\theta = 0^\circ, 120^\circ, 360^\circ$ ✓

$(\theta - 60^\circ) = 60^\circ + 360^\circ n, -60^\circ + 360^\circ n$

$\theta = 120^\circ + 360^\circ n, 360^\circ n$ ✓

for n an integer

Note the domain

if $0^\circ \leq \theta \leq 360^\circ$

$-60^\circ \leq \theta - 60^\circ \leq 300^\circ$

Reas 2

→ For the general solution look at the pattern. Add revolutions to these angles that you found here.

(*) Don't mix radians and degrees.

(b) i)

$$\begin{array}{r} x - 2 \\ x+2 \overline{) x^2 \quad + 6} \\ x^2 + 2x \\ \hline -2x + 6 \\ -2x - 4 \\ \hline 10 \end{array}$$

$\therefore \frac{x^2+6}{x+2} = (x-2) + \frac{10}{x+2}$ ✓ ✓

Long division was well done except for a few silly addition mistakes.

$$\begin{array}{r} Q(x) \\ A(x) \overline{) P(x)} \\ \vdots \\ \hline R(x) \end{array}$$

Note the correct position of the parts.

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ii) $y = (x-2) + \frac{10}{x+2}$

as $x \to \infty$, $\frac{10}{x+2} \to 0$ ✓

$\therefore y = x - 2$ is an oblique asymptote

Comm 1

You **must** **state** clearly that $\frac{10}{x+2}$ approaches zero to get this mark. Just making it disappear from one line to the next is not sufficient.

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iii) Vertical asymptote: $x = -2$ ✓

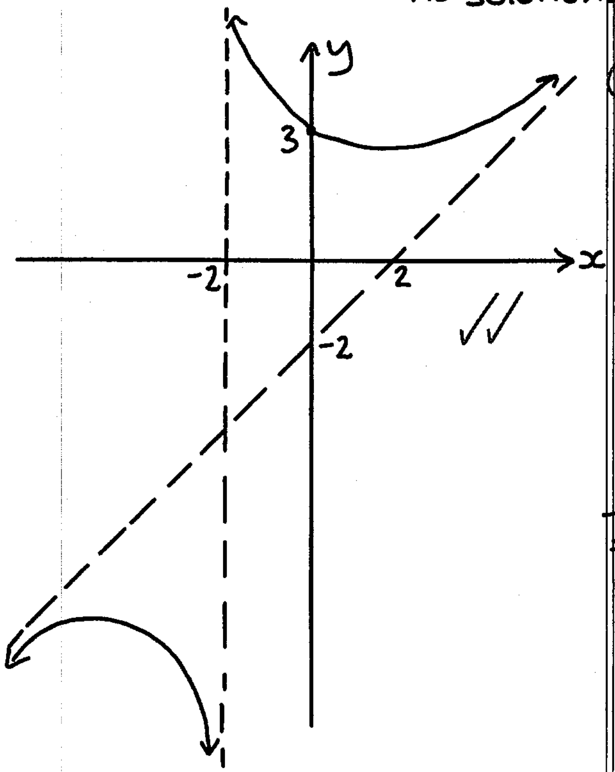
The vertical asymptote is a line. It is incorrect to say that its equation is $x \neq -2$.

iv) y int: $x = 0 \rightarrow y = 3$

x int: $y = 0 \rightarrow \frac{x^2+6}{x+2} = 0$

$x^2 + 6 = 0$

no solutions



Comm 2

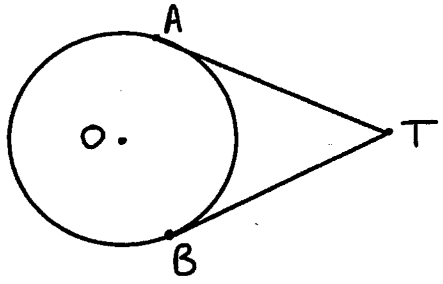
There were some problems with connecting the ideas in the previous parts.

For $y = \frac{x^2+6}{x+2}$

Think!

- there are no possible x intercepts. ($y = 0$, $x^2 + 6 = 0$, No solution)
- The y intercept is (0,3).
- Draw a sign diagram or check some points for various x values to check your graph.

(c)



Reas 3

Very poorly done! You are required to prove why TA = TB. part(i) is worth 3 marks so you have to do more than just quote a theorem.

Here's one method.

i) Let O be the centre of the circle

In ΔOAT & ΔOBT

OA = OB (equal radii of circle)

OT = OT (common)

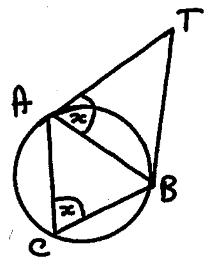
$\angle OAT = \angle OBT = 90°$ (radius ⊥ tangent at the point of contact) ✓

$\therefore \Delta OAT \equiv \Delta OBT$ (RHS) ✓ (proof using congruent Δs)

$\therefore$ TA = TB (matching sides in congruent Δs =) ✓

Here's another method.

Mark in point C on circumference.



$\angle TAB = \angle ACB$ (∠ between a tangent and a chord = ∠ in the alternate segment)

Similarly,

$\angle TBA = \angle ACB$ (∠ between tangent and chord = ∠ in the alternate segment)

$\therefore \angle TAB = \angle TBA$

$\therefore$ TA = TB (sides opposite equal ∠s are equal)

ii) TA = TB = TC (using part i)
$\therefore$ T is the centre of the circle passing through A, B, C ✓

Since T lies on chord AC & TA = TC, AC is the diameter
$\therefore \angle ABC = 90^\circ$
($\angle$ in a semicircle $= 90^\circ$) ✓

(Reas 2)

This part was done better than part (i). Make sure to answer both parts.

Total /16 Reas /10

Question 4

(a) $LHS = \dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{\frac{1+t^2}{1+t^2} + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{\frac{1+t^2}{1+t^2} + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \checkmark$$

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$$

$$= \frac{2t^2+2t}{2+2t}$$

$$= \frac{2t(t+1)}{2(1+t)}$$

$= t$ ✓

$= \tan\theta/2$

$= RHS$

Most substituted the correct t-formulae into LHS however algebra was extremely inefficient and often poorly done.

(b) i) In $\triangle ABD$

$$\tan 30^\circ = \frac{h}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{AB}$$

$AB = \sqrt{3}h$ ✓

An easy mark scored by most.

ii) Similarly $AC = \sqrt{3} \times 2h$
$AC = 2\sqrt{3}h$ ✓

In ΔDEF
$$\tan 75^\circ = \frac{DF}{h}$$
$$DF = h\tan 75^\circ$$
$$\therefore BC = h\tan 75^\circ$$ ✓

Generally done well.

Some tried to find BC using Pythagoras in ΔABC – this does not work because ΔABC is not right angled

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iii) $$\cos(\angle BAC) = \frac{b^2+c^2-a^2}{2bc}$$
$$= \frac{(\sqrt{3}h)^2 + (2\sqrt{3}h)^2 - (h\tan 75)^2}{2(\sqrt{3}h)(2\sqrt{3}h)}$$ ✓
$$= \frac{3h^2 + 12h^2 - h^2\tan^2 75}{12h^2}$$
$$= \frac{15 - \tan^2 75}{12}$$
$$= 0.089\ldots$$
$\angle BAC = 85^\circ$ (to nearest degree) ✓
$\therefore$ bearing of CE from A is 085°T

Reas. 2

Many had the correct idea but a badly executed solution with poor algebra

You must ensure you answer the question!

(c) i) Reas 2
$$\frac{\theta+1}{\theta} = \frac{\theta}{1}$$
$$\theta + 1 = \theta^2$$ ✓
$$\theta^2 - \theta - 1 = 0$$

ii) $$\theta = \frac{1 \pm \sqrt{5}}{2}$$
since θ is a +ve distance
$$\theta = \frac{1+\sqrt{5}}{2}$$ ✓

* A straight forward solution made a mess of by most
* Note that if the word "externally" is used in the question, a negative is not required in the ratio.

Ext. 1 candidates & indeed 2 unit candidates MUST be able to solve a simple quadratic ... blindfolded!

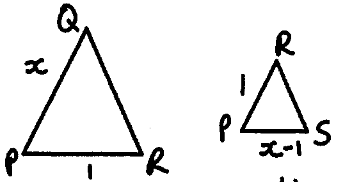
(d) i) Reas 2

$\angle QPR = 72^\circ$ (given)
$\angle SRQ = \angle PRS = 36^\circ$ (RS bisects $\angle QRP$)
$\therefore \angle PSR = 72^\circ$ ($\angle$ sum Δ PSR $= 180^\circ$)
$\therefore SR = PR = 1$ ($\angle$s opposite = sides of ΔPRS =) ✓
$\angle SQR = 36^\circ$ ($\angle$ sum $\Delta PQR = 180^\circ$)
$SQ = SR = 1$ ($\angle$s opposite = sides of ΔSQR =) ✓

Well done, however the proofs could have been more succinct.

ii) In ΔPQR & ΔPRS
$\angle QPR = \angle RPS$ (common)
$\angle QRP = \angle RSP = 72^\circ$ (above)
$\therefore \Delta PQR ||| \Delta PRS$ (AA similarity test) ✓

iii) Reas 2



$$\frac{x}{1} = \frac{1}{x-1}$$ (corresponding sides in similar Δs are in the same ratio) ✓
$$x^2 - x = 1$$
$$x^2 - x - 1 = 0$$ ✓
$$\therefore x = \frac{1+\sqrt{5}}{2}$$ which is φ

This could have easily been done with or with out ii. Not a single student got a mark in iii! "Hence" after a similar Δs proof almost always means use "matching sides in similar Δs in the same ratio"

iv) Reas 2

Sin Rule in ΔPQR
$$\frac{x}{\sin 72^\circ} = \frac{1}{\sin 36^\circ}$$ ✓
$$x\sin 36^\circ = \sin 72^\circ$$
$$x\sin 36^\circ = 2\sin 36^\circ \cos 36^\circ$$
$$\cos 36^\circ = \frac{x}{2}$$
$$= \frac{\sqrt{5}+1}{4}$$ ✓

Once again these 2 marks are accessible even if other parts were skipped. Simply follow the instructions.

Could also get this result using cos Rule in ΔPQR