SYDNEY BOYS HIGH SCHOOL

YEAR 11 YEARLY EXAMINATION 2001

MATHEMATICS

EXTENSION

Time allowed – 2 hours

Examiner: C Kourtesis

DIRECTIONS TO CANDIDATES

- **All questions may be attempted.**
- **All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.**
- **Approved calculators may be used.**
- **Use a new booklet for each question.**
  If required, additional booklets may be obtained from the Examination Supervisor upon request.

Question 1 **(16 marks)**

(a) Express 120^o in radians.

(b) Find the value of $19^{-0.5}$ correct to two decimal places.

(c) Factorize $8-x^3$.

(d) Find the gradient of the line $3x+4y=12$.

(e) If $a^b=3$, find the value of $a^{4b}-5$.

(f) Simplify $\dfrac{\log_2 32}{\log_2 16}$.

(g) If $y=\dfrac{1}{x^2+3}$ find $\dfrac{dy}{dx}$.

(h) If $f(\theta)=\theta^4+4\theta^{3/2}$ find the value of $f'(\theta)$.

(i) Find the remainder when the polynomial

$P(x)=x^3-4x$ is divided by $(x+3)$

(j) Given the equation of the parabola

$(x-2)^2=16(y+4)$ find the:

(i) coordinates of the vertex
(ii) coordinates of the focus
(iii) equation of the directrix

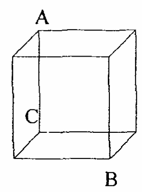
Question 2 (15 marks)

(a) (i) Sketch the graphs of $y = x + 1$ and $y = |x - 2|$ on the same diagram.

(ii) Find the coordinates of any points of intersection.

(b) The graph of $y = 10(x+1)(x-3)$ intersects the x axis at two points A and B. Find the length of the line segment AB.

(c)



AB is a diagonal of a cube of side x *cm* and C is a vertex on the base of the cube. Determine the size of $\angle ABC$ correct to the nearest minute.

(d) The point (x, y) is determined by the parametric equations $x = 3t - 1$ and $y = t^2 - 2t$ where $t \geq 0$.
Find the value of x corresponding to y=3.

(e) Find the values of a, b, c and d if the following equation is satisfied by all real values of x.

$$(ax + b)(x^2 + 3c) = x^3 - 2x^2 + 6x + d.$$

(f) If $\sec\theta = \sqrt{5}$ and $\frac{\pi}{2} < \theta < 2\pi$, find the exact value of $\tan 2\theta$.

Question 3 **(18 marks)**

(a) The interval AB has end points $A(-2,7)$ and $B(8,-8)$.

Find the coordinates of the point which divides AB internally in the ratio $2:3$.

(b) If α, β and γ are the roots of the equation

$$x^3 - 2x^2 + 1 = 0$$

find the:

(i) value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

(ii) equation whose roots are $-\alpha, -\beta, -\gamma$

(c) Solve the inequality

$$\frac{2}{x-1} \leq 1$$

(d) (i) Write down the expansion of $\sin(A+B)$.

(ii) Solve for θ in the interval $0 \leq \theta \leq 2\pi$

$$\sin 2\theta = \frac{1}{2}$$

(iii) Write down the general solution of

$$\sin 2\theta = \frac{1}{2}$$

(e) The polynomial $P(x) = x^3 - 6x^2 + kx + 14$ has a zero at $x = 1$. Find the:

(i) value of k.

(ii) linear factors of $P(x)$.

(iii) roots of the equation $P(x) = 0$.

Question 4 **(15 marks)**

(a) If $Ax - 6y - 6B = 0$ and $7x - 18y + 9 = 0$ represent the same straight line, find the values of A, B and C.

(b) Find the equation of the parabola if its axis is $y = 0$ and the line $y = x + 3$ is a tangent to the curve.

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. M is the midpoint of the chord PQ. The tangents at P and Q meet at the point N.

(i) Show that the equation of the tangent at P is $y - px + ap^2 = 0$ and write down the equation of the tangent at Q.

(ii) Find the coordinates of M and N.

(iii) Show that MN is parallel to the Y axis.

(iv) Find the coordinates of T, the midpoint of MN.

(v) Show that T lies on the parabola.

Question 5 (16 marks)

(a) If $F(x)=\log_2\left(\dfrac{x+1}{x}\right)$ and $F(1)+F(2)+F(3)=\log_2 K$

(K a constant) find the value of K.

(b) The normal to the curve $y=\dfrac{ax+b}{\sqrt{x}}$

(a, b constants) has equation $4a+y=22$ at the point where x=4. Find the values a and b.

(c) The equation $x^3+3Hx+L=0$ has two equal roots. Prove that

$$L^2+4H^3=0$$

(d) A man travelling along a straight flat road passes three points at intervals of 200m. From these points he observes the angle of elevation of the top of a hill to the left of the road to be respectively 30^o, 45^o and again 45^o.

(i) Draw a diagram of the above.

(ii) Find the height of the hill.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2001

YEARLY EXAMINATION

YEAR 11

Mathematics Extension

Sample Solutions

1 (a) $180^\circ = \pi^c$

$1^\circ = \frac{\pi}{180}$

$120 = \frac{\pi}{180} \times 120 = \frac{2\pi}{3}$ 1

(b) $0 \cdot 23$ 1

(c) $8 - x^3 = (2-x)(4+2x+x^2)$ 2

$? + \cancel{4x} + \cancel{2x^2} - \cancel{4x} - \cancel{2x^2} - x^3$

(d) $3x + 4y = 12$

$4y = -3x + 12$

$y = -\frac{3}{4}x + 3$

$m = -\frac{3}{4}$ 1

(e) $a^{4b} - 5 = (a^b)^4 - 5$

$= 3^4 - 5 = 76$ 1

(f) $\frac{\log_2 2^5}{\log_2 2^4} = \frac{5\log_2 2}{4\log_2 2} = \frac{5}{4}$ 2

g) $y = (x^2+3)^{-1}$

$y' = -1(x^2+3)^{-2} \times 2x = \frac{-2x}{(x^2+3)^2}$ 2

(h) $f(\theta) = \theta^4 + 4\theta^{\frac{3}{2}}$

$f'(\theta) = 4\theta^3 + 4 \times \frac{3}{2}\theta^{\frac{1}{2}}$

$= 4\theta^3 + 6\theta^{\frac{1}{2}}$ 1

(i) let $P(x) = x^3 - 4x$.

let $x + 3 = 0$

$x = -3$.

$P(-3) = (-3)^3 - 4(-3)$

$= -27 + 12$

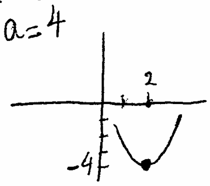
$= -15$ 2

(j) $(x-2)^2 = 16(y+4)$

$(x-h)^2 = 4a(y+k)$

(i) $V(2, -4)$ 1

$a = 4$

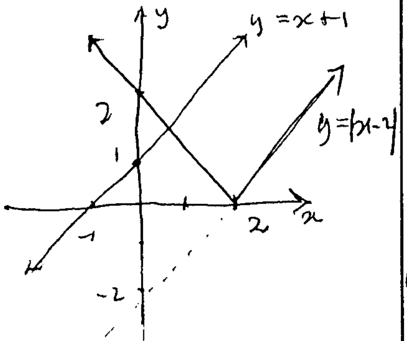


(ii) $F(2, 0)$ 1

(iii) $y = -8$ 1

Question 2

a) (i)



(ii) $|x-2| = x+1$

Square both sides

$x^2 - 4x + 4 = x^2 + 2x + 1$

$3 = 6x$

$x = \frac{1}{2}$

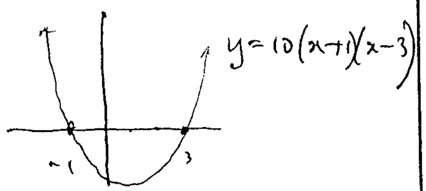
Test: LHS $= |\frac{1}{2} - 2|$

$= \frac{3}{2}$

RHS $= \frac{3}{2}$

$\therefore (\frac{3}{2}, \frac{5}{2})$ is intersection

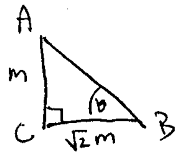
b)



Clearly AB = 4.

c) Base diagonal $BC^2 = m^2 + m^2$

$BC = \sqrt{2}m$



$\tan\theta = \frac{m}{\sqrt{2}m}$

$= \frac{1}{\sqrt{2}}$

$\therefore \theta \doteqdot 35°16'$

(d) $x = 3t-1, \; y = t^2 - 2t \quad t \geq 0$

When $y = 3$ $\quad t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$t = 3$ or $\cancel{-1}$

$\therefore t = 3$

So $x = 3 \times 3 - 1$

$x = 8$

e) $(ax+b)(x^2+3c)$

$= ax^3 + 3acx + bx^2 + 3bc$

$= ax^3 + bx^2 + 3acx + 3bc$

$\equiv x^3 - 2x^2 + 6x + d$

Equating coefficients:

x^3: $a = 1$

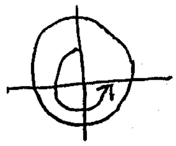
$b = -2$

$3ac = 6 \quad \therefore c = 2$

$3bc = d \quad \therefore d = 3 \times (-2) \times 2$

$d = -12$

(d) $\sec\theta = \sqrt{5}$



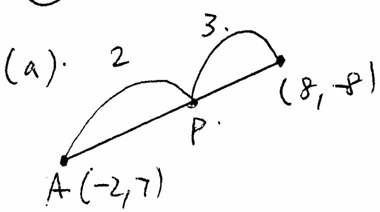
$\cos\theta = \frac{1}{\sqrt{5}}$ $\therefore$ 4th Quad

$\theta \doteq 63.435^\circ$

$\therefore \tan 2\theta \doteq -\frac{4}{3}$

(3)

(a).



$x = \frac{2\times 8 + 3(-2)}{2+3}$

$= 2$

$y = \frac{2(-8)+21}{5}$

$= 1 \qquad (2,1).$

(b). $x^3 - 2x^2 + 1 = 0$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= 0$

(ii) $x^3 + (\sum \alpha_i) x^2 + (\sum \alpha_i \alpha_j) x + (-\alpha\beta\gamma) = 0$

Let $y = -x$.

$x = -y$.

$(-y)^3 - 2(-y)^2 + 1 = 0$

$-y^3 - 2y^2 + 1 = 0$

$\therefore y^3 + 2y^2 - 1 = 0$

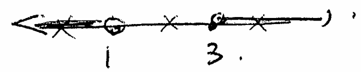
is the equation.

(c) $\frac{2}{x-1} \leq 1$.

this inequation has boundaries given by.

(i) $x = 1$

(ii) $2 = x - 1, \quad x = 3$



$x = 0, \quad -2 \leq 1$

$x = 2, \quad 2 > 1.$

$x = 4, \quad 2/3 < 1.$

$\therefore \{x : x < 1 \text{ or } x \geq 3\}$

(d) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}.$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}.$

(iii) $\sin 2\theta = \sin\frac{\pi}{6}.$

$2\theta = n\pi + (-1)^n \frac{\pi}{6}.$

$\therefore \theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.$

(e).

$P(1) = 0$

$1 - 6 + k + 14 = 0$

$k = -9.$

(ii)

$$
\begin{array}{r}
x^2 - 5x - 14 \\
x-1 \overline{) x^3 - 6x^2 - 9x + 14} \\
-(x^3 - x^2) \\
\hline
-5x^2 - 9x \\
-(-5x^2 + 5x) \\
\hline
-14x + 14
\end{array}
$$

$\therefore P(x) = (x-1)(x-7)(x+2).$

(iii) $P(x) = 0, \quad x = 1, 7, -2$

4

C $x^2 = 4ay$

(i) $\frac{x^2}{4a} = y$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$ at $2ap$

$\therefore y - ap^2 = p(x - 2ap)$ $m = p$

~~$2apx 2ap^2$~~

$y - ap^2 = px - 2ap^2$ (2)

$y - px + ap^2 = 0$

at Q: $y - qx + aq^2 = 0$

(ii) $\boxed{M}$: $\left[a(p+q), \frac{a(p^2+q^2)}{2}\right]$ (1)

N: $y - px + ap^2 = y - qx + aq^2$

$qx - px = aq^2 - ap^2$

$x(q-p) = a(q+p)(q-p)$

$x = a(q+p)$.

So $y - ap(q+p) + ap^2 = 0$

$y - apq - ap^2 + ap^2 = 0$ (2)

$y = apq$

$\boxed{N}$: $[a(q+p), apq]$

(iii) x coord of M & N $a(p+q)$ (1)

$\therefore$ MN is parallel to axis

(iv) of Coords T

Mid MN: $\left[a(p+q), \frac{\frac{a(p^2+q^2)}{2} + apq}{2}\right]$ (1)

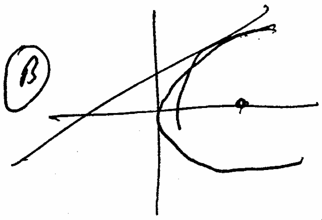
(v) $\left[a(p+q), \frac{ap^2 + aq^2 + 2apq}{4}\right]$

(3)

A $A = 7/3$ (1)

$B = -3/2$ (1)

B



$y^2 = 4ax$

$y = x + 3$

Solve

$(x+3)^2 = 4ax$

$x^2 + 6x + 9 = 4ax$

$x^2 + (6-4a)x + 9 = 0$

$\Delta = 0$ $(6-4a)^2 - 36 = 0$

$16a^2 - 48a = 0$

$a = 0$ or 3.

$\boxed{y^2 = 12x}$ (3)

$\boxed{\text{Coords of T}}$

$\left[a(p+q), \frac{a(p+q)^2}{4}\right]$

$x = a(p+q)$

$x/a = p+q$

$\therefore y = \frac{a(p+q)^2}{4}$

$y = \frac{a\left(\frac{x}{a}\right)^2}{4}$

$4y = a \cdot \frac{x^2}{a^2}$

$4y = \frac{x^2}{a}$

$x^2 = 4ay$

$\therefore$ T lies on parabola

q5

$\boxed{\log m + \log n = \log(m \times n)}$

(a) $F(x) = \log_2\left(\frac{x+1}{x}\right)$

$F(1) + F(2) + F(3) + F(4) = \log_2\left(\frac{2}{1}\right) + \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{4}{3}\right)$

$= \log_2\left(\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3}\right)$

$= \log_2 4$

$\therefore k = 4$

(b) $y = \frac{ax+b}{\sqrt{x}} = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-1/2}$

$\therefore y' = \frac{1}{2}ax^{-1/2} - \frac{b}{2}x^{-3/2}$

$x = 4,\ y' = \frac{1}{2}a \times 4^{-1/2} - \frac{b}{2}(4)^{-3/2}$

$= \frac{a}{4} - \frac{b}{16}$

$= \frac{4a-b}{16}$

$y = \frac{4a+b}{2}$

$\therefore m_{\perp} = \frac{-16}{4a-b} = \frac{16}{b-4a}$ = gradient of normal

$4x + y = 22 \quad \therefore m = -4$

$\boxed{x = 4,\ y = 6}$

$\therefore \frac{16}{b-4a} = -4$

&

$\frac{4a+b}{2} = 6.$

$\therefore 16 = -4b + 16a \quad -(1) \Rightarrow 4 = -b + 4a.$

$12 = b + 4a \quad -(2) \quad 12 = b + 4a.$ +

$\therefore 8a = 16$

$a = 2 \Rightarrow b = 4$

$\boxed{a = 2, b = 4}$

(c) $x^3 + 3Hx + L$

Let the roots be α, α, β

$\therefore\ \alpha + \alpha + \beta = 0$

$\alpha^2 + \alpha\beta + \alpha\beta = 3H$

$\alpha^2\beta = -L$

$2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha$

$\alpha^2 + 2\alpha\beta = 3H \Rightarrow \alpha^2 + 2\alpha(-2\alpha) = 3H \Rightarrow -3\alpha^2 = 3H \Rightarrow \alpha^2 = -H$

$\alpha^2\beta = -L \Rightarrow \alpha^2(-2\alpha) = -L \Rightarrow -2\alpha^3 = -L$

$\Rightarrow \alpha^3 = \frac{L}{2} \Rightarrow \alpha = \left(\frac{L}{2}\right)^{\frac{1}{3}}$

$\therefore \left[\left(\frac{L}{2}\right)^{\frac{1}{3}}\right]^2 = -H$

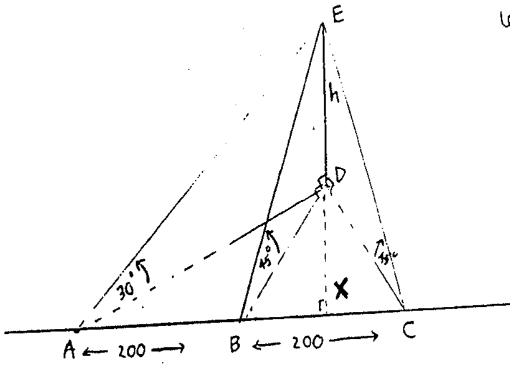
$\therefore \left(\frac{L}{2}\right)^{2/3} = -H$

$\therefore \left(\frac{L}{2}\right)^2 = (-H)^3 = -H^3$

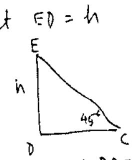
$\therefore L^2 = -4H^3$

$\therefore \underline{L^2 + 4H^3 = 0}$ QED

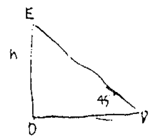
(d)



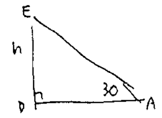
let $ED = h$



$\therefore DC = h$.



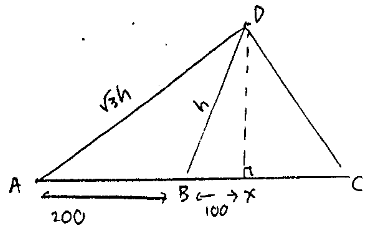
$\therefore DB = h$



$\therefore DA = h \cot 30$
$= \sqrt{3}h$

$\therefore$ ΔDBC is isosceles.

$\therefore BX = 100$



$$\left.\begin{aligned} \therefore DX^2 + 100^2 &= h^2 \quad -① \\ DX^2 + 300^2 &= 3h^2 \quad -② \end{aligned}\right\} -$$

(Pythagoras' Theorem)

$$\therefore \quad 2h^2 = 300^2 - 100^2$$

$$h^2 = \frac{300^2 - 100^2}{2}$$

$$\therefore h = \sqrt{}$$

$$\therefore = 200m$$

$\therefore$ The height of the hill is 200 m