SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

Year 11 YEARLY EXAMINATIONS - September 2002

MATHEMATICS

Extension 1

Time allowed — Ninety Minutes
Examiner: A.M.Gainford

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Use a new booklet for each question.
- If required, additional paper may be obtained from the Examination Supervisor upon request.

Question 1. (18 Marks)

(a) Simplify $\frac{x+1}{x}$ where $x = \sqrt{2} + 1$, expressing your answer with rational denominator. **2**

(b) Solve for x: **6**

(i) $|4 - x| = 3$

(ii) $x^2 - 9 < 0$

(iii) $\frac{1}{x-2} < 1$

(c) Find the remainder when the polynomial $P(x) = x^3 - 4x$ is divided by $x+3$. **1**

(d) Simplify $\frac{x^3 + 27}{x^2 - 3x + 9}$ **2**

(e) Give the general solution of the equation $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. **2**

(f) Find the focus and directrix of the parabola $y = \frac{x^2}{8} + x + \frac{1}{4}$. **2**

(g) Express $\cos 4\theta$ as an expression in powers of $\cos\theta$ only. **3**

Question 2. (18 Marks)

(a) Differentiate: **8**

(i) $2x^3 - 4x + 1$

(ii) $\sqrt{x} - \frac{1}{x}$

(iii) $(x^2 - 2)^5 (2x + 1)$

(iv) $\frac{4}{4 - x^2}$

(b) (i) Express $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$, α is acute. **4**

(ii) Hence or otherwise sketch the graph of $y = \sqrt{2}\sin\theta + \sqrt{2}\cos\theta$ in the domain $0 \le \theta \le 2\pi$.

(c) Find the exact value of $\operatorname{cosec}105^{\circ}$. **1**

(d) How many terms of the arithmetic series 96 + 93 + 90 +….. must be taken to give a sum of zero? **2**

(e) Given the polynomial $P(x) = x^3 + 3x^2 - x - 3$. **3**

(i) Use the factor theorem to find a zero of the polynomial.

(ii) Express $P(x)$ as a product of three linear factors.

Question 3. (18 Marks)

(a) Given $\log_a 5 = 0\cdot 827$ and $\log_a 2 = 0\cdot 356$, find $\log_a 50$. **2**

(b) Draw neat sketches of the following functions, showing their principle features: **6**

(i) $y = 1 - |x|$ (ii) $y = \log_2 x$ (iii) $y = \sqrt[3]{x}$

(c) Given the function $f(x) = 2^x + 2^{-x}$ **7**

(i) Find $f(-1)$.

(ii) Show that $f(x)$ is even.

(iii) Find $f(x) = 0$.

(iv) State the domain and range of $f(x)$.

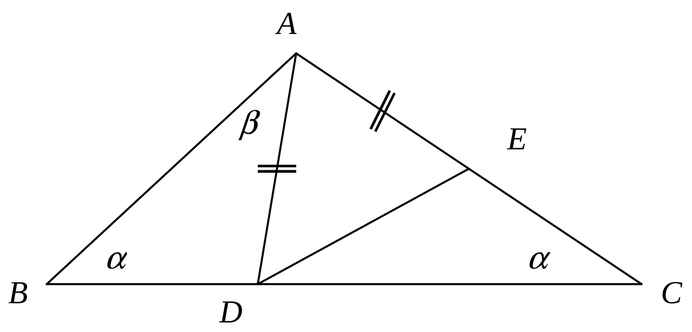
(v) Sketch the function.

(d) A geometric series has 4 as its third term and $-\frac{32}{27}$ as its sixth term. Find the first term and the common ratio. **2**

(e) Calculate to the nearest minute the acute angle between the lines $5x - 4y = 17$ and $3x + 2y = 8$. **1**

Question 4. (18 Marks)

(a) **4**



In the isosceles triangle *ABC*, $\angle ABC = \angle ACB = \alpha$. The points *D* and *E* lie on *BC* and *AC*, so that *AD* = *AE*, as shown in the diagram. Let $\angle BAD = \beta$.

(i) Explain why $\angle ADC = \alpha + \beta$.

(ii) Find $\angle DAC$ in terms of α and β.

(iii) Hence, or otherwise, find $\angle EDC$ in terms of β.

4

(b) Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$.

(i) Write down the equation of the tangent at *P*.

(ii) Find the co-ordinates of *T*, the point where the tangent meets the axis of the parabola.

(iii) Show that the tangent at *P* is equally inclined to the axis of the parabola and to the line joining *P* to the focus *S*.

(c) Find the value of the constants *a* and *b* if $x^2 - 2x - 3$ is a factor of the polynomial $P(x) = x^3 - 3x^2 + ax + b$ **2**

(d) The point $P(11, 7)$ divides *AB* externally in ratio 3:1. If *B* is (6, 5), find the co-ordinates of *A*. **2**

(e) An observer (*O*) in a lighthouse is 180m vertically above a point *B* at sea level on the shore. He observes the frigate HMS Shropshire (*S*) on a bearing of 210°T and an angle of depression of $5^\circ 21'$. He also notes that Wolf Rock (*W*) is on a bearing of 165°T and an angle of depression of $3^\circ 40'$. **6**

(i) Sketch a diagram to represent this situation.

(ii) Calculate the distance of the frigate from the rock, to the nearest metre.

(iii) Calculate the true bearing from the frigate to the rock.

This is the end of the paper.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2002

YEARLY EXAMINATION

YEAR 11

Mathematics Extension

Sample Solutions

Question (1) [18 marks]

(a) $\frac{x+1}{x} = 1 + \frac{1}{x}$

$= 1 + \frac{1}{\sqrt{2}+1}$

$= 1 + \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= 1 + (\sqrt{2} - 1)$

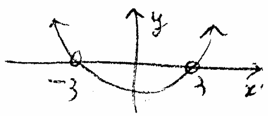
$= \boxed{\sqrt{2}}$ (2)

(b) $|4 - x| = 3$

(i) $4 - x = \pm 3$

$\underline{x = 1, \text{ or } x = 7}$ (2)

(ii) $(x+3)(x-3) < 0$



$\underline{-3 < x < 3}$

(2)

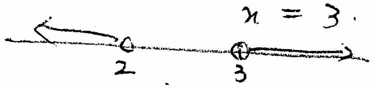
(iii) $\frac{1}{x-2} < 1$

Boundary : $x = 2$

$\frac{1}{x-2} = 1$

$x - 2 = 1$

$x = 3.$



$\underline{x < 2 \text{ or } x > 3.}$ (2)

(c) $P(x) = x^3 - 4x.$

$P(-3) = -27 + 12 = \boxed{-15}$

(1)

(d) $\frac{x^3+27}{x^2-3x+9}$

$= \frac{(x+3)\cancel{(x^2-3x+9)}}{\cancel{(x^2-3x+9)}} = \boxed{x+3}$ (2)

(e) $\cos\left(\theta + \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\theta = 2n\pi \pm \frac{\pi}{4} - \frac{\pi}{4}$ (2)

$$\boxed{\theta = \begin{cases} 2n\pi \\ 2n\pi - \frac{\pi}{2} \end{cases}}$$

(f) $y = \frac{x^2}{8} + x + \frac{1}{4}$

$8y = x^2 + 8x + 2$

$= (x^2 + 8x + 16) - 14$

$\therefore (x+4)^2 = 8y + 14$

$(x+4)^2 = 8\left(y + \frac{7}{4}\right)$

$4a = 8, \ a = 2.$

vertex $\left(-4, -\frac{7}{4}\right)$

focus $\underline{\left(-4, \frac{1}{4}\right)}.$ (2)

directrix $\underline{y = -3\frac{3}{4} \left(-\frac{15}{4}\right)}$

(g) $\cos 4\theta = 2\cos^2 2\theta - 1$ $\boxed{(1)}$

$= 2(\cos 2\theta)^2 - 1$

$= 2[2\cos^2\theta - 1]^2 - 1$ $\boxed{(1)}$

$= 2[4\cos^4\theta - 4\cos^2\theta + 1] - 1$

$= \boxed{8\cos^4\theta - 8\cos^2\theta + 1}$ $\boxed{(1)}$

(3)

Q2 a)

(i) $\frac{d}{dx}(2x^3-4x+1)$

$= 6x^2-4$ (1)

(ii) $\frac{d}{dx}\left(\sqrt{x} - \frac{1}{x}\right)$

$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{x^2}$ (2)

(iii) $\frac{d}{dx}\left((x^2-2)^5(2x+1)\right)$

$u = (x^2-2)^5$

$u' = 5(x^2-2)^4 . 2x$

$= 10x(x^2-2)^4$

$v = 2x+1$

$v' = 2$.

$= (2x+1). 10x(x^2-2)^4 + (x^2-2)^5 . 2$

$= 2(x^2-2)^4\left(5x(2x+1) + x^2-2\right)$

$= 2(x^2-2)^4(10x^2+5x+x^2-2)$

$= 2(x^2-2)^4(11x^2+5x-2)$ (3)

(iv) $\frac{d}{dx}\left(\frac{4}{4-x^2}\right)$

$= \frac{d}{dx}\left(4(4-x^2)^{-1}\right)$

$= 4. -1(4-x^2)^{-2} . -2x$

$= \frac{8x}{(4-x^2)^2}$ (2)

(b) (i) $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta$

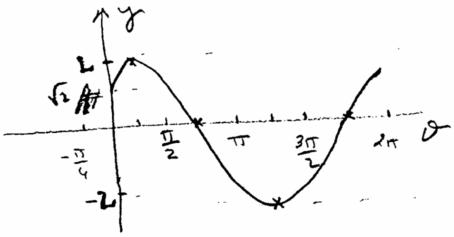
$= 2\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$

$= 2\left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)$

$= 2\sin\left(\theta + \frac{\pi}{4}\right)$

$\boxed{2}$

(ii)



$\sin\theta$

zeroes $\theta = 0, \pi, 2\pi$

$\sin\left(\theta + \frac{\pi}{4}\right)$

zeroes $\theta + \frac{\pi}{4} = 0, \pi, 2\pi$

$\theta = \cancel{-\frac{\pi}{4}}, \frac{3\pi}{4}, \frac{7\pi}{4}$

$\boxed{2}$

(c) $\operatorname{cosec} 105^\circ$

$= \dfrac{1}{\sin(60+45)^\circ}$

$= \dfrac{1}{\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ}$

$= \dfrac{1}{\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}}$

$= \dfrac{2\sqrt{2}}{\sqrt{3} + 1}$

$\boxed{1}$

(d) $S_n = \frac{n}{2}(2a + (n-1)d)$

$0 = \frac{n}{2}(192 + (n-1) \times -3)$

$\therefore 0 = n(192 - 3n + 3)$

$0 = n(195 - 3n)$

$\therefore n = 0$ or $3n = 195$

$\therefore n = 65$ $\boxed{2}$

$\therefore$ 65 terms required

(e) $P(x) = x^3 + 3x^2 - x - 3$

(i) $P(1) = 1 + 3 - 1 - 3 = 0$ $\boxed{1}$

$\therefore x - 1$ is a factor.

(ii) $P(x) = x^2(x+3) - 1(x+3)$

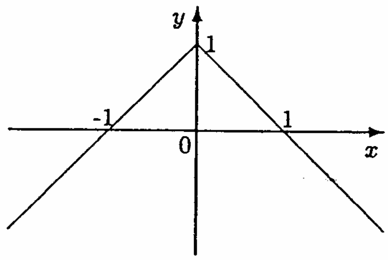
$= (x+3)(x^2-1)$

$= (x+3)(x+1)(x-1)$ $\boxed{2}$

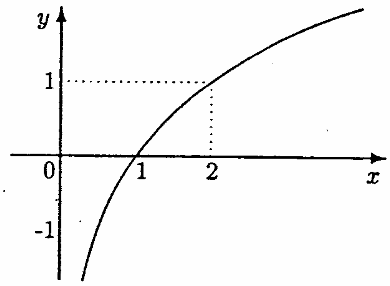
$\boxed{2}$ 3. (a) $\log_a 50 = \log_a 5^2 \times 2,$

$$\begin{aligned} &= 2\log_a 5 + \log_a 2, \\ &= 2 \times 0{\cdot}827 + 0{\cdot}356, \\ &= 2{\cdot}01. \end{aligned}$$

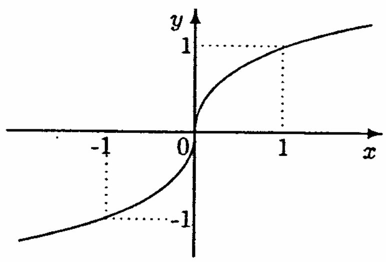
$\boxed{2}$ (b) (i)



$\boxed{2}$ (ii)



$\boxed{2}$ (iii)



$\boxed{1}$ (c) (i) $f(-1) = \frac{1}{2} + 2,$

$$= 2\tfrac{1}{2}.$$

$\boxed{1}$ (ii) $f(-x) = 2^{-x} + 2^{-(-x)},$

$$\begin{aligned} &= 2^{-x} + 2^{x}, \\ &= f(x). \end{aligned}$$

$\therefore$ the function is even.

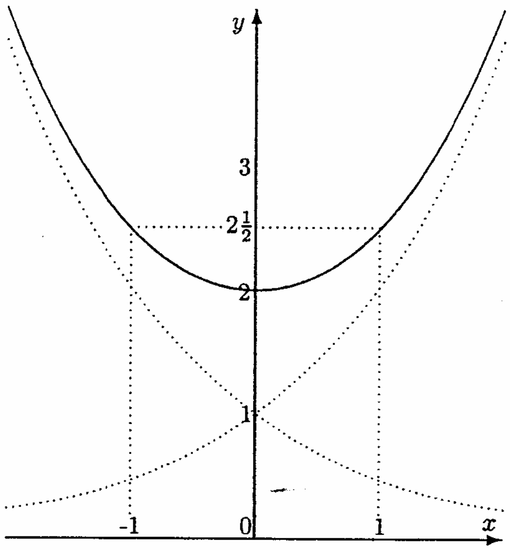
$\boxed{1}$ (iii) $2^x > 0$ and $2^{-x} > 0$,

$\therefore 2^x + 2^{-x} \neq 0$, *i.e.*, there is no solution.

$\boxed{2}$ (iv) Domain is all real x.

Range is $f(x) \geq 2$.

$\boxed{2}$ (v)



$\boxed{2}$ (d) With the usual notation:

$$
\begin{aligned}
ar^2 &= 4,\\
ar^5 &= \frac{-32}{27}.\\
\text{Now, } \frac{ar^5}{ar^2} &= \frac{-32}{27} \times \frac{1}{4},\\
&= \frac{-8}{27},\\
&= r^3.\\
\text{So } r &= -\tfrac{2}{3}.\\
a \times \left(-\tfrac{2}{3}\right)^2 &= 4,\\
\text{Hence } a &= 4 \times \tfrac{9}{4},\\
&= 9.
\end{aligned}
$$

$\boxed{1}$ (e) $m_1 = \frac{5}{4}$ and $m_2 = -\frac{3}{2}$,

$$
\begin{aligned}
\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| &= \tan^{-1}\left(\frac{22}{7}\right),\\
&\approx 72^\circ 21' \text{ (Nearest minute).}
\end{aligned}
$$

Alternative method:

$$
\begin{aligned}
\tan^{-1}\left(\tfrac{5}{4}\right) &\approx 51^\circ 20' 25'',\\
\tan^{-1}\left(-\tfrac{3}{2}\right) &\approx 123^\circ 41' 24'',\\
123^\circ 41' 24'' - 51^\circ 20' 25'' &\approx 72^\circ 21' \text{ (Also to the nearest minute).}
\end{aligned}
$$

Question 4

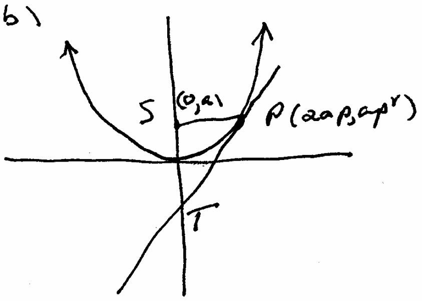
(a) (i) $\angle ADC = \alpha + \beta$ (exterior angle is equal to the sum of the interior opposite angles.) 1

(ii) $\angle DAC = 180 - \alpha - \alpha - \beta$.
$= 180 - 2\alpha - \beta$. 1

(iii) $\therefore \angle ADE = \frac{2\alpha + \beta}{2}$ base angle of isosceles $\triangle$.
$= \alpha + \frac{\beta}{2}$.

$\therefore \angle EDC = \angle ADC - \angle ADE$
$= \alpha + \beta - (\alpha + \frac{\beta}{2})$
$= \frac{\beta}{2}$. 2.

(b)



(i) $y - tx + at^2 = 0$. 1

(ii) T is $(0, -at^2)$ 1

(iii) $SP = \sqrt{(ap^2 - a)^2 + (2ap - 0)^2}$
$= \sqrt{a^2p^4 - 2a^2p^2 + a^2 + 4a^2p^2}$
$= \sqrt{a^2(p^2+1)^2}$
$= a(p^2+1)$

$ST = a + ap^2$
$= a(p^2+1)$ 2.

$\therefore ST = SP$
$\therefore \triangle PST$ is isosceles
$\therefore \angle PTS = \angle SPT$.

(c) If $x^2-2x-3 = (x-3)(x+1)$

is a factor of $P(x) = x^3-3x^2+ax+b$.

then $x-3$ is a factor $\Rightarrow P(3)=0$
& $x+1$ is a factor $\Rightarrow P(-1)=0$ } alle. (1)

Hence. $27-27+3a+b=0$

$3a+b=0$. — (1)

and $-1-3-a+b=0$

$-a+b=4$ — (2)

(1) − (2)

$4a=-4$

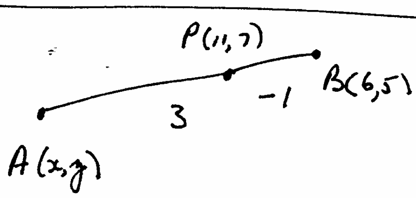
$\boxed{a=-1}$

Subst in (1) $-3+b=0$

$\boxed{b=3}$

(2)

(d)



$11 = \dfrac{3\times 6 + -1\times x}{2}$.

$22 = 18 - x$

$\boxed{x=-4}$

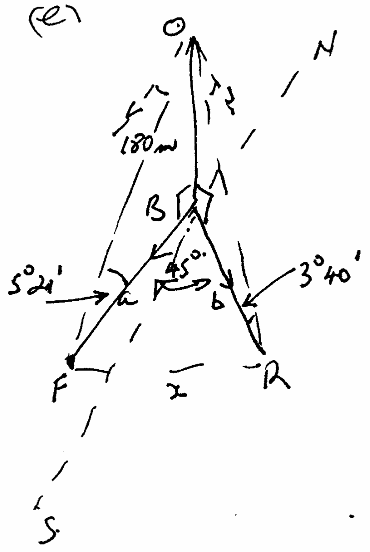
$7 = \dfrac{3\times 5 + -1\times y}{2}$.

$14 = 15 - y$

$\boxed{y=1}$ ∴ Pt $(-4,1)$.

(2)

(e)



(i) Sketch. (1)

(ii) $x^2 = a^2 + b^2 - 2ab\cos 45°$.

where $a = 180 \tan 84° 39'$

and $b = 180 \tan 86° 20'$.

$\therefore x^2 = 180^2 \tan^2 84°39' + 180^2 \tan^2 86°20' - 2 \times 180^2 \tan 84°39' \tan 86°20' \times \frac{1}{\sqrt{2}}$.

$x^2 = 180^2 \left[\tan^2 84°39' + \tan^2 86°20' - \sqrt{2}\tan 84°39' \times \tan 86°20'\right]$

$x = 180\sqrt{(\qquad)}$ (3)

$\doteqdot 1987$ m

(iii) $\frac{\sin \angle BFR}{180 \tan 86°20'} = \frac{\sin 45°}{1987}$

$\sin \angle BFR = \frac{180 \tan 86°20' \sin 45°}{1987}$

$= 0.99957 \ldots$

$\angle BFR = \sin^{-1}(\qquad)$

$= 88°20'$. (2)

$\therefore$ Bearing of R from S is $118°20'$ T