SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2003

YEARLY EXAMINATION

YEAR 11

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Questions 1 - 4
- All questions are NOT of equal value.

Examiner: *E. Choy*

Total marks – 120
Attempt Questions 1 – 4
All questions are NOT of equal value

Answer each question in a SEPARATE writing booklet.

Question 1 (35 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the value of x given that $x^3 = 3^6$. **2**

(b) Solve $(x-2)(x+1) \geq 0$. **2**

(c) Sketch $y = |x+1|$, showing the x and y intercepts. **2**

(d) If $\sec\theta = 2$, find the possible values of $\tan\theta$. **2**

(e) Find, in EXACT general form, the equation of the line that passes through the point $(-1,3)$ and has an angle of inclination, to the positive direction of the x axis, of $150°$. **3**

(f) If $f(x) = \dfrac{x}{1-3x}$, find and simplify $f\left(\dfrac{1}{x}\right)$. **2**

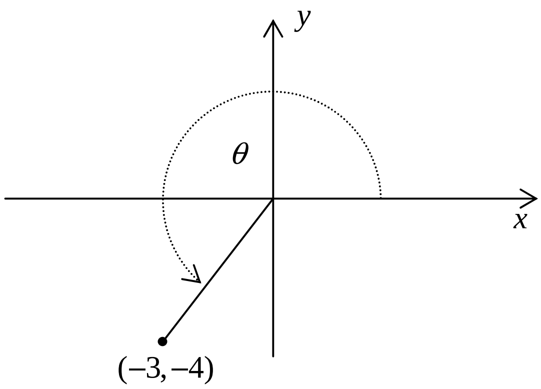
(g) Solve the inequation $|x+1| < 2$ **2**

(h) If $y = 8\sqrt{x}$, find $\dfrac{dy}{dx}$. **2**

(i) Complete the square to find the minimum value of the quadratic function $y = x^2 + 4x - 6$. **2**

Question 1 continued **Marks**

(j) Write down the value of $\sin\theta$ in the diagram below. **2**



(k) Solve $2\sin 2\theta = -1$ where $0 \le \theta \le 2\pi$. **3**

(l) If $(x-1)$ is a factor of $p(x) = x^3 + ax^2 - 2x - 4$, find the value(s) of a. **2**

(m) Simplify $\dfrac{\log_2 32}{\log_2 16}$ **3**

(n) Find A and B if $x^3 - 27 = (x-3)(x^2 + Ax + B)$. **3**

(o) If $f(x) = x^2 - 4$ and $g(x) = x - 2$, find in simplest form $f(g(x))$. **3**

Question 2 (32 marks) Use a SEPARATE writing booklet **Marks**

(a) If $p = 1 + \sqrt{2}$ and $q = 1 - \sqrt{2}$ find

(i) $p - q$ **2**

(ii) pq **2**

(b) Differentiate the following with respect to *x*:

(i) $y = 4x^5 + 2x^2 - 1$ **2**

(ii) $y = \frac{7}{x}$ **2**

(iii) $y = (4x^2 - 3x)^{12}$ **3**

(c) The limiting sum of the geometric series $a + \frac{a}{2} + \frac{a}{4} + \cdots$ is 6. Find the value of *a*. **3**

(d) Consider the function $f(x) = x^2 + 3x$.

Show that $f(x+h) - f(x) = 2xh + h^2 + 3h$ **2**

(e) (i) Show that there are 21 terms in the arithmetic series **2**

$$-2 + 1 + 4 + \cdots + 58.$$

(ii) Hence, or otherwise, find the sum of the 21 terms. **1**

(f) Find an equation in terms of *x* and *y* that is independent of θ **2**

$$\left.\begin{aligned} x &= 2\cos\theta \\ y &= \sin\theta \end{aligned}\right\}$$

(g) (i) Show that $\frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$ **3**

(ii) Hence find the exact value of $\tan 15°$. **2**

(h) *A* and *B* are points $(-5, 1)$ and $(2, 2)$ respectively. Find the coordinates of the point which divides *A* and *B* **externally** in the ratio 3 : 2. **3**

(i) If $a + b = 1$, show that $(a^2 - b^2)^2 + ab = a^3 + b^3$ **3**

Question 3 (20 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$. **2**

(ii) Hence, solve the equation $\sqrt{3}\sin x - \cos x = 1$, for $0 \le x \le 2\pi$. **2**

(b) Find the general solutions of $\sin x + \cos 2x = 1$. **2**

(c) (i) Show that the equation of the normal to the curve $x^2 = 4y$ at the point $(2p, p^2)$ is $x + py = 2p + p^3$. **2**

(ii) If the normal passes through the point $(-2, 5)$ find the values of *p*. **2**

(d) When the polynomial $P(x)$ is divided by $A(x) = (2x+1)(x-3)$, it gives a quotient $Q(x)$ and a remainder $R(x)$.
Write the general form of $R(x)$. Justify your answer. **1**

(e) (i) Show that the point $P(2,7)$ lies on the line $2x - y + 3 = 0$ **1**

(ii) Hence find the distance between the parallel lines $2x - y + 3 = 0$ and $2x - y - 11 = 0$ **2**

(f) $A(-2,-5)$ and $B(1,4)$ are 2 points. Find the acute angle θ between the line joining *A* and *B* and the line $x + 2y + 1 = 0$, giving the answer correct to the nearest minute. **2**

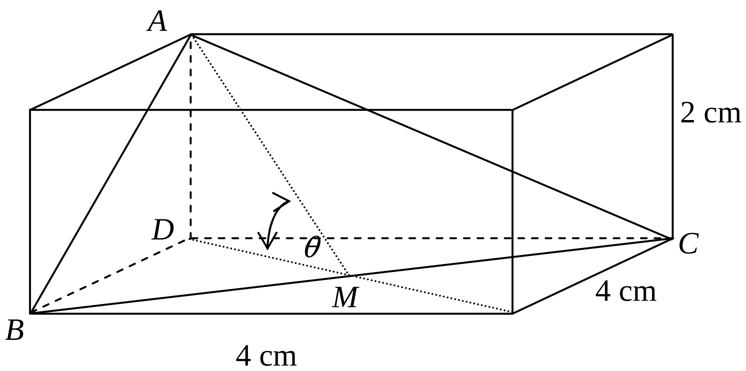
Question 3 is continued on the next page

 Marks

(g) The prism in the diagram below has a square base of side 4 cm and its height is 2 cm. *ABC* is a diagonal plane of the prism. Let θ be the acute angle between the diagonal plane and the base of the prism.

(i) Show that $MD = 2\sqrt{2}$ cm. **2**

(ii) Hence find θ, correct to the nearest minute. **2**



NOT TO SCALE

Question 4 (33 marks) Use a SEPARATE writing booklet **Marks**

(a) The quartic equation $x^4 - 4x^3 + 2x^2 - 3x + 2 = 0$ has roots α, β, γ and δ. Find the value of:

(i) $\alpha + \beta + \gamma + \delta$ **1**

(ii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **1**

(iii) $\alpha\beta\gamma\delta$ **1**

(iv) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **2**

(b) Given that $x = 1$ is a double root of the equation

$$6x^4 - 7x^3 + cx^2 + 13x - 4 = 0$$

(i) Show that $c = -8$ **2**

(ii) Hence find the other roots. **2**

(c) If $\log_5 8 = a$, prove that $\log_{10} 2 = \dfrac{a}{a+3}$ **3**

(d) (i) By expanding $\cos(2A + A)$, show that **3**

$$\cos 3A = 4\cos^3 A - 3\cos A$$

(ii) Hence show that if $2\cos A = x + \dfrac{1}{x}$, then $2\cos 3A = x^3 + \dfrac{1}{x^3}$ **2**

Question 4 is continued on the next page.

Question 4 continued **Marks**

(e) (i) Show that the equation of the tangent at the point $P(2ap, ap^2)$ to $x^2 = 4ay$ is given by $y = px - ap^2$. **2**

(ii) Write down the equation of the tangent at the point $Q(2aq, aq^2)$. **1**

(iii) Find the coordinates of *M* the midpoint of chord *PQ*. **1**

(iv) The tangents at *P* and *Q* meet at *T*. Find the coordinates of *T*. **2**

(v) Show that *TM* is parallel to the axis of the parabola. **2**

(vi) *K* is the midpoint of *TM*. Find the locus of *K*.

(f) Three tangents to the parabola $x^2 = 4ay$ form a triangle *PQR* and the lines *QR*, *RP* and *PQ* make acute angles $\alpha_1, \alpha_2, \alpha_3$ respectively with the tangent at the vertex.
If d_1, d_2 and d_3 are the respective distances of the focus from these tangents and if r_1, r_2 and r_3 are the respective distances of the focus from the vertices *P*, *Q* and *R* of ΔPQR, show that:

(i) $d_1 \cos\alpha_1 = d_2 \cos\alpha_2 = d_3 \cos\alpha_3 = a$ **2**

(ii) $d_1 r_1 = d_2 r_2 = d_3 r_3$ **2**

(iii) $r_1 r_2 r_3 = \dfrac{d_1^{\,2} d_2^{\,2} d_3^{\,2}}{a^3}$ **2**

THIS IS THE END OF THE PAPER

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Sample Solutions

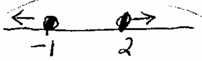
QUESTION 1.

(a) $x^3 = 3^6$

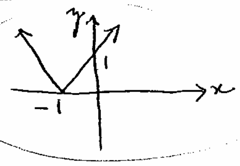
$x = (3^6)^{\frac{1}{3}}$

$= 3^2$

$\boxed{x = 9}$ ✓✓

(b) 

$\boxed{x \le -1,\ x \ge 2}$ ✓✓

(c) 

✓✓

(d) $\sec\theta = 2$

now $\sec^2\theta = 1 + \tan^2\theta$.

$4 = 1 + \tan^2\theta$

$\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$. ✓✓

(e) $m = \tan 150^\circ$

$= -\frac{1}{\sqrt{3}}$

$\therefore\ y - 3 = -\frac{1}{\sqrt{3}}(x+1)$

$\sqrt{3}y - 3\sqrt{3} = -x - 1$

$x + \sqrt{3}y + 1 - 3\sqrt{3} = 0$. ✓✓✓

(f) $f(x) = \frac{x}{1-3x}$

$f(\frac{1}{x}) = \frac{\frac{1}{x}}{1 - \frac{3}{x}}$

$= \frac{1}{x-3}$ ✓✓

(g) $|x+1| < 2$

$-2 < x+1 < 2$

$-3 < x < 1$ ✓✓

(h) $y = 8\sqrt{x}$

$= 8x^{\frac{1}{2}}$

$\frac{dy}{dx} = 4x^{-\frac{1}{2}}$

$= \frac{4}{\sqrt{x}}$ ✓✓

(i) $y = x^2 + 4x - 6$

$= x^2 + 4x + 4 - 6 - 4$

$= (x+2)^2 - 10$.

$\therefore$ MIN value is -10 ✓✓

(j) $\frac{-4}{5}$ ✓✓ (1 for $-\frac{3}{5}$)

(k) $2\sin 2\theta = -1$ $\quad 0 \le \theta \le 2\pi$

$\sin 2\theta = -\frac{1}{2}$. $\quad \therefore 0 \le 2\theta \le 4\pi$

$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\therefore \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$. ✓✓

(l) $P(1) = 0$

$\therefore\ 1 + a - 2 - 4 = 0$

$a - 5 = 0$

$a = 5$. ✓✓

(m) $\frac{\log_2 32}{\log_2 16} = \frac{\log_2 2^5}{\log_2 2^4} = \frac{5\log_2 2}{4\log_2 2} = \frac{5}{4}$ ✓

(n) $x^3 - 27 = (x-3)(x^2+3x+9)$

$\therefore\ x^2 + 3x + 9 \equiv x^2 + Ax + B$ $\quad \therefore A = 3,\ B = 9$ ✓

(o) $f(g(x)) = (x-2)^2 - 4 = x^2 - 4x$

Q2. a) $p+q = 2\sqrt{2}$ while ii) $pq = -1$

b) i) $\frac{dy}{dx} = 20x^4 + 14x$ ii) $-7/x^2$ iii) $12(8x-3)(4x^2-3x)^{?}$

c) $6 = \frac{a}{1-\frac{1}{2}}$

$a = 3.$

d) $f(x+h) - f(x) = (x+h)^2 + 3(x+h) - x^2 - 3x$

$= 2xh + h^2 + 3h$

e) i) $58 = -2 + (n-1)\times 3$ so $n = 21$

ii) $S_{21} = \frac{n}{2}(-2+58) = \frac{21}{2} \times 56 = 588$

f) $x = 2\cos\theta$ so $x^2 = 4\cos^2\theta$ ie $x^2 = 4\cos^2\theta$

$y = \sin\theta$ $y^2 = \sin^2\theta$ $4y^2 = 4\sin^2\theta$

$4y^2 + x^2 = 4\cos^2\theta + 4\sin^2\theta$

$x^2 + 4y^2 = 4$

g) i) $\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2} \Rightarrow \frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$ where $t = \tan\frac{\theta}{2}$

$= \frac{2t^2}{2t} = t = \tan\frac{\theta}{2}$

ii) $\tan 15 = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$

h) ratio 3 : -2 $\left(\frac{3\times 2 - 2\times -5}{1}, \frac{3\times 2 - 2\times 1}{1}\right) = (16, 4)$

i) $(a^2-b^2)^2 + ab$ a^3+b^3

$((a+b)(a-b))^2 + ab$ while $= (a+b)(a^2-ab+b^2)$

$(a-b)^2 + ab$ $= a^2 - ab + b^2$

$a^2 - ab + b^2$ $\therefore (a^2-b^2)^2 + ab = a^3+b^3.$

Question 3

(a) (i) $\sqrt{3}\sin x - \cos x \equiv R\sin(x-\alpha)$

$R\sin(x-\alpha) = R(\sin x\cos\alpha - \cos x\sin\alpha)$

$= R\sin x\cos\alpha - R\cos x\sin\alpha$

$= (R\cos\alpha)\sin x - (R\sin\alpha)\cos x$

So $R\cos\alpha = \sqrt{3}$ $\quad -(1)$

$R\sin\alpha = 1$ $\quad -(2)$

$(2)\div(1) \Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = 30^\circ \text{ or } \dfrac{\pi}{6}^c$

$(2)^2 + (1)^2 \Rightarrow \quad R^2\sin^2\alpha + R^2\cos^2\alpha = 4$

$\therefore R^2(\sin^2\alpha + \cos^2\alpha) = 4 \Rightarrow R^2 = 4$

$\therefore R = 2$

(ii) $\sqrt{3}\sin x - \cos x = 1 \Rightarrow 2\sin\left(x - \dfrac{\pi}{6}\right) = 1$

$\sin\left(x-\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$0 \le x \le 2\pi \Rightarrow 0 - \dfrac{\pi}{6} \le x - \dfrac{\pi}{6} \le 2\pi - \dfrac{\pi}{6}$ $-\dfrac{\pi}{6} \le x - \dfrac{\pi}{6} \le \dfrac{11\pi}{6}$

$x - \dfrac{\pi}{6} = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6} = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore x = \dfrac{2\pi}{6}, \dfrac{6\pi}{6} = \dfrac{\pi}{3}, \pi$

(b) $\sin x + \cos 2x = 1$

$\sin x + 1 - 2\sin^2 x = 1 \Rightarrow \sin x - 2\sin^2 x = 0 \Rightarrow \sin x(1 - 2\sin x) = 0$

$\therefore \sin x = 0, \dfrac{1}{2}$

$\sin x = c, \qquad -1 \le c \le 1$ $x = n\pi + (-1)^n \sin^{-1}(c)$

$$\left.\begin{aligned} x &= n\pi + (-1)^n\sin^{-1}(0) \\ x &= n\pi + (-1)^n \sin^{-1}(\tfrac{1}{2}) \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= n\pi \\ x &= n\pi + (-1)^n\frac{\pi}{6} \end{aligned}\right\}$$

Question 3

(c) (i) $x^2 = 4y \Rightarrow y = \frac{1}{4}x^2$

$\frac{dy}{dx} = \frac{x}{2} \Rightarrow \frac{dy}{dx}_{x=2p} = \frac{2p}{2} = p$

$\therefore m_\perp = -\frac{1}{p}$ | $m_\perp$ is the gradient of the normal |

$\therefore y - p^2 = -\frac{1}{p}(x-2p) \Rightarrow py - p^3 = -x + 2p \Rightarrow x + py = 2p + p^3$

(ii) $(-2,5)$ lies on the normal.

$(-2) + p(5) = 2p + p^3 \Rightarrow p^3 - 3p + 2 = 0$

Let $P(x) = x^3 - 3x + 2$

$P(1) = 0 \Rightarrow (x-1)$ is a factor of $P(x)$.

$x^2 + x - 2 = (x-1)(x+2)$

$$
\begin{array}{r}
x^2 + x - 2 \\
x-1 \overline{)\, x^3 \quad -3x + 2} \\
x^3 - x^2 \\
x-1 \overline{)\, x^2 - 3x + 2} \\
x^2 - x \\
x-1 \overline{)\, -2x + 2} \\
-2x + 2 \\
\hline
0
\end{array}
$$

$\therefore P(x) = (x-1)^2(x+2)$

$\therefore p^3 - 3p + 2 = 0 \Rightarrow (p-1)^2(p+2) = 0$

$\therefore p = 1, -2$

(d) $\deg(A(x)) = 2$ and $\deg(R(x)) < \deg(A(x)) = 2$

(The degree of the remainder is always less than the degree of the divisor).

So the degree of $R(x)$ is at most 1 ie $R(x) = mx + b$ is the most general form.

(e) (i) $\text{LHS} = 2\times2 - 7 + 3 = 4 - 7 + 3 = 0 = \text{RHS}$

So $(2,7)$ lies on $2x - y + 3 = 0$

(ii) $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$, $Ax + By - C = 0 \Leftrightarrow 2x - y - 11 = 0$

$(x_1, y_1) = (2,7)$

$d = \frac{|2\times2 - 7 - 11|}{\sqrt{2^2 + (-1)^2}} = \frac{|4-7-11|}{\sqrt{5}} = \frac{14}{\sqrt{5}} = \frac{14\sqrt{5}}{5} \approx 6\cdot261$

Question 3

(f) $A(-2,-5), B(1,4)$

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4+5}{1+2} = 3 = m_1$$

$$x + 2y + 1 = 0 \Rightarrow m_2 = -\frac{1}{2}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3-(-\frac{1}{2})}{1+3(-\frac{1}{2})}\right| = \left|\frac{\frac{7}{2}}{-\frac{1}{2}}\right| = 7$$

$$\therefore \theta = 81°52'$$

(g) (i) $MD = \frac{1}{2}BC = \frac{1}{2} \times \sqrt{4^2 + 4^2} = \frac{1}{2} \times \sqrt{32} = \frac{1}{2} \times 4\sqrt{2} = 2\sqrt{2}$

(ii) $\tan\theta = \frac{AD}{MD} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 35°16'$

QUESTION 4

(a)

(i) $\frac{-(-4)}{1} = 4$ 1

(ii) $\frac{-(-3)}{1} = 3$ 1

(iii) 2 1

(iv) $= \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$ 2

$= \frac{3}{2}$

b)

(i) $6(1)^4 - 7(1)^3 + c(1)^2 + 13(1) - 4 = 0$

$\Rightarrow c = -8$ 2

(ii) $P(x)$ $6x^4 - 7x^3 - 8x^2 + 13x - 4 = 0$

$x^2 - 2x + 1 \overline{)\, P(x)}$ quotient $6x^2 + 5x - 4$ 2

Other roots are solutions of $6x^2 + 5x - 4 = 0$ ie $x = -\frac{4}{3}$ or $x = \frac{1}{2}$

(c) $\log_{10} 2 = \frac{\log_5 2}{\log_5 10}$ (Change of Base)

$= \frac{\log_5 2}{\log_5 2 + \log_5 5}$ 3

let $u = \frac{\log_5 2}{\log_5 2 + 1}$

(Since $\log_5 8 = a \Rightarrow 3\log_5 2 = a$)

(cont)

$\therefore u = \frac{\frac{a}{3}}{\frac{a}{3} + 1} = \frac{a}{a+3}$

(d)

(i) $\cos 3A = \cos(2A + A)$ 3

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \sin A$

$= 2\cos^3 A - \cos A - 2\cos A[1 - \cos^2 A]$

$= 4\cos^3 A - 3\cos A$

(ii)

$2\cos 3A = 8\cos^3 A - 6\cos A$

$= (2\cos A)^3 - 3(2\cos A)$

$= (x + \frac{1}{x})^3 - 3(x + \frac{1}{x})$

$=$ etc.

$= x^3 + \frac{1}{x^3}$ 2

e) Bookwork

(i) 2

(ii) $y = qx - aq^2$ 1

(iii) $M\left[a(p+q), \frac{a}{2}(p^2+q^2)\right]$ 1

(iv) $T\left[a(p+q), apq\right]$ 2

(v) gradient of TM 2

$$\frac{\frac{a}{2}(p^2+q^2) - apq}{0}$$

$\Rightarrow$ undefined gradient.

and gradient of axis of parabola is undefined

$\therefore$ TM // axis

(vi) $K\left[a(p+q), \dfrac{\frac{a}{2}(p^2+q^2) + apq}{2}\right]$ 2

let $x = a(p+q)$, $y = \frac{a}{4}(p^2+q^2) + \frac{apq}{2}$

$$= \frac{a}{4}\left[p^2+q^2 + 2pq\right]$$

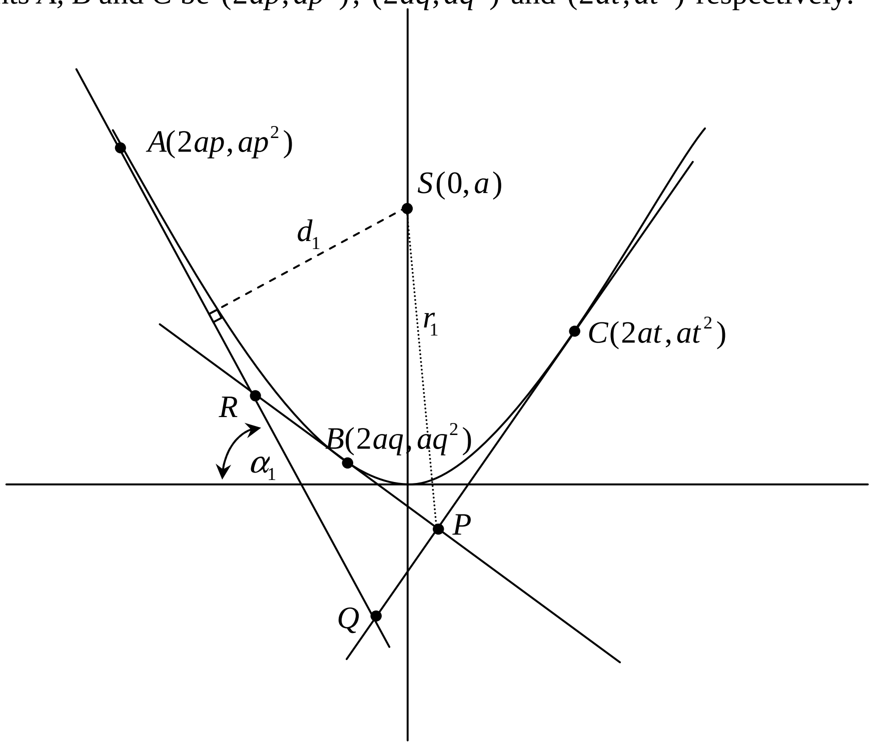
$$= \frac{a}{4}\left[p+q\right]^2$$

$$= \frac{a}{4}\left[\frac{x}{a}\right]^2$$

$y = \frac{x^2}{4a}$ locus

Question 4 (f)

Let the points *A*, *B* and *C* be $(2ap, ap^2)$, $(2aq, aq^2)$ and $(2at, at^2)$ respectively.



The gradient of *AR* is p, so that $\tan\alpha_1 = |p|$ $[\because \alpha_1 < 90°]$

So the equation of *RQ* is $y = px - ap^2 \Leftrightarrow px - y - ap^2 = 0$.

Similarly, the equations of *PQ* and *RP* are respectively $y = tx - at^2$ & $y = qx - aq^2$

By solving simultaneously ie the intersection of lines *RBP* and *CPQ*, *P* has coordinates $\left(a(t+q), atq\right)$. This was proved in 4(e).

(Similarly *Q* and *R* have coordinates $\left(a(t+p), atp\right)$ & $\left(a(t+q), atq\right)$ respectively)

If $\tan\alpha_1 = |p| \Rightarrow \cos\alpha_1 = \dfrac{1}{\sqrt{1+p^2}}$ [Pythagoras' Theorem]

(Similarly $\cos\alpha_2 = \dfrac{1}{\sqrt{1+q^2}}$ & $\cos\alpha_3 = \dfrac{1}{\sqrt{1+t^2}}$)

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \Rightarrow d_1 = \frac{a(1+p^2)}{\sqrt{1+p^2}} = a\sqrt{1+p^2}$$

(Similarly $d_2 = a\sqrt{1+q^2}$ & $d_3 = a\sqrt{1+t^2}$)

$$r_1^2 = PS^2 = \left(a^2(t+q)^2 + a^2(tq-1)^2\right) = a^2\left(t^2 + q^2 + 1 + t^2q^2\right) = a^2(1+t^2)(1+q^2)$$

(Similarly $r_2^2 = a^2(1+p^2)(1+t^2)$ & $r_3^2 = a^2(1+p^2)(1+q^2)$)

(i) $d_1 \cos\alpha_1 = a\sqrt{1+p^2} \times \frac{1}{\sqrt{1+p^2}} = a$.

Similarly for $d_2 \cos\alpha_2$ & $d_3 \cos\alpha_3$ ie $d_2 \cos\alpha_2 = d_3 \cos\alpha_3 = a$

QED

(ii) $d_1^{\,2} r_1^{\,2} = a^2(1+p^2) \times a^2(1+q^2)(1+t^2) = a^4(1+p^2)(1+q^2)(1+t^2)$

Similarly $d_2^{\,2} r_2^{\,2} = d_3^{\,2} r_3^{\,2} = a^4(1+p^2)(1+q^2)(1+t^2)$

Thus $d_1 r_1 = d_2 r_2 = d_3 r_3$

QED

(iii) $r_1 r_2 r_3 = \frac{d_1^{\,2} d_2^{\,2} d_3^{\,2}}{a^3}$

$\Leftrightarrow a^3 r_1 r_2 r_3 = d_1^{\,2} d_2^{\,2} d_3^{\,2}$

$\Leftrightarrow (ar_1)(ar_2)(ar_3) = d_1^{\,2} d_2^{\,2} d_3^{\,2}$

$\Leftrightarrow (d_1 r_1 \cos\alpha_1)(d_2 r_2 \cos\alpha_2)(d_3 r_3 \cos\alpha_3) = d_1^{\,2} d_2^{\,2} d_3^{\,2}$ (from(i))

$\Leftrightarrow (r_1 \cos\alpha_1)(r_2 \cos\alpha_2)(r_3 \cos\alpha_3) = d_1 d_2 d_3$

$(r_1^{\,2} \cos^2\alpha_1)(r_2^{\,2} \cos^2\alpha_2)(r_3^{\,2} \cos^2\alpha_3)$

$= \frac{a^2(1+q^2)(1+t^2)}{(1+p^2)} \times \frac{a^2(1+p^2)(1+t^2)}{(1+q^2)} \times \frac{a^2(1+q^2)(1+p^2)}{(1+t^2)}$

$= a^6(1+q^2)(1+t^2)(1+p^2)$

$= a^2(1+p^2) \times a^2(1+q^2) \times a^2(1+t^2)$

$= d_1^{\,2} d_2^{\,2} d_3^{\,2}$

$\therefore (r_1 \cos\alpha_1)(r_2 \cos^2\alpha_2)(r_3 \cos\alpha_3) = d_1 d_2 d_3$

QED