SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2004

PHSC Examination

YEAR 11

Mathematics Extension

Sample Solutions

Question	Marker
1	PSP
2	Mr Choy
3	Mr Hespe
4	Mr Bigelow

Question 1

(a) $$\frac{1}{3\quad\sqrt{2}}+\frac{1}{3+\sqrt{2}}=\frac{3+\sqrt{2}+3\quad\sqrt{2}}{\left(3\quad\sqrt{2}\right)\left(3+\sqrt{2}\right)}$$

$$=\frac{6}{9\quad 2}$$

$$=\frac{6}{7}$$

$\frac{6}{7}\quad\mathbb{Q}\qquad\frac{1}{3\quad\sqrt{2}}+\frac{1}{3+\sqrt{2}}$ is a rational number

QED

(b) (i) $\left|2x\quad 1\right|=5$

$2x\quad 1=5$ or $2x\quad 1=\quad 5$

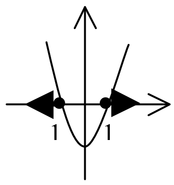
$2x=6,\quad 4$

$x=3,\quad 2$

(ii) $x^2\quad 1\qquad x^2\quad 1\quad 0$

$\left(x\quad 1\right)\left(x+1\right)\quad 0$

$x\qquad 1,x\quad 1$



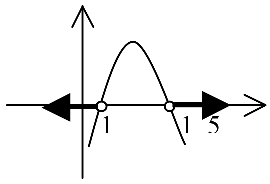
(iii) $\frac{1}{x\quad 1}<2\qquad\frac{1}{x\quad 1}\quad 2<0$

$\frac{1\quad 2\left(x\quad 1\right)}{x\quad 1}<0\qquad\frac{1\quad 2x+2}{x\quad 1}<0$

$\frac{3\quad 2x}{x\quad 1}<0\qquad\left(x\quad 1\right)^2\binom{(}{)}$

$\left(x\quad 1\right)\left(3\quad 2x\right)<0$

$x<1,x>\frac{3}{2}$



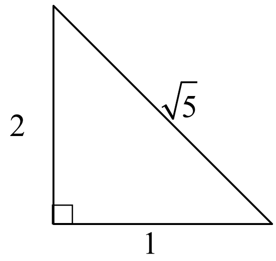
(c) $P\left(x\right)=2x^3\quad 3x^2+x\quad 4$

By the Remainder Theorem:

Remainder = $P\left(2\right)=16\quad 12+2\quad 4=2$

(d) $$\frac{x^3\quad 1}{x^2\quad 2x+1}=\frac{\left(x\quad 1\right)\left(x^2+x+1\right)}{\left(x\quad 1\right)^2}=\frac{x^2+x+1}{x\quad 1}$$

(e) $\sin \quad + \frac{\ }{4}(= \sin \quad \cos\frac{\ }{4} + \cos \quad \sin\frac{\ }{4}$



$= \frac{1}{\sqrt{2}}(\sin \quad + \cos \quad)$

$= \frac{1}{\sqrt{2}} \quad \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}($

$= \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

(f) $y = \frac{1}{4}(x^2 \quad 2x + 9)$

$4y = x^2 \quad 2x + 9 \qquad x^2 \quad 2x + 1 + 8$

$(x \quad 1)^2 = 4y \quad 8 = 4(y \quad 2)$

$a = 1$

Vertex $(1,2)$, Focus $(1,3)$

(g) $\text{LHS} = \cos 3$

$= \cos(2 \quad + \quad)$

$= \cos 2 \quad \cos \qquad \sin 2 \quad \sin$

$= (2\cos^2 \qquad 1)\cos \qquad (2\sin \quad \cos \quad)\sin$

$= 2\cos^3 \qquad \cos \qquad 2\cos \quad \sin^2$

$= 2\cos^3 \qquad \cos \qquad 2\cos \quad (1 \quad \cos^2 \quad)$

$= 2\cos^3 \qquad \cos \qquad 2\cos \quad + 2\cos^3$

$= 4\cos^3 \quad 3\cos$

$= \text{RHS}$

QED

Question 2

Question (2)

(a) (i) $\frac{d}{dx}(1+2x-4x^2-x^3)$

$= \boxed{2-8x-3x^2}$ 2

(ii) $\frac{d}{dx}(1-x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$= \boxed{-x(1-x^2)^{-1/2}}$ 2

(iii) $\frac{d}{dx}(x-1)^4(3x+1)$

$= (3x+1)4(x-1)^3 + (x-1)^4 \cdot 3$

$= (x-1)^3(12x+4+3x-3)$

$= \boxed{(x-1)^3(15x+1)}$ 2

(iv) $\frac{d}{dx}\left(\frac{2}{x^3-1}\right) = 2\frac{d}{dx}(x^3-1)^{-1}$

$= -2(x^3-1)^{-2}(3x^2)$

$= \boxed{-6x^2(x^3-1)^{-2}}$ 2

(i)

$\sin x - \sqrt{3}\cos x = A\sin(x-\alpha)$

$A(\sin x\cos\alpha - \cos x\sin\alpha)$

$\therefore A\cos\alpha = 1$

$A\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}, \quad \alpha = \frac{\pi}{3}$

$A^2 = 1+3 = 4, \quad A = 2$

$\therefore \boxed{\sin x - \sqrt{3}\cos x = 2\sin\left(x-\frac{\pi}{3}\right)}$ 2

(ii) $2\sin\left(x-\frac{\pi}{3}\right) = \frac{2}{\sqrt{2}}$

$\therefore \sin\left(x-\frac{\pi}{3}\right) = \sin\frac{\pi}{4}$

$x-\frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$

$\therefore \boxed{x = \left(n\pi+\frac{\pi}{3}\right) + (-1)^n\frac{\pi}{4}}$

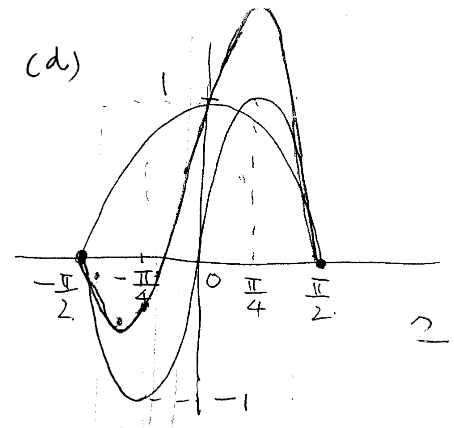
$n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3}$ 2

(c) $(x-1)[(x-1)-4] < 0$

$(x-1)(x-5) < 0$ 2

$\therefore$ $\boxed{1 < x < 5}$

(d)



2

(e) $P(x) = x^3 - 19x - 30$

(i) $P(-2) = -8 + 38 - 30 = 0$

$\therefore (x+2)$ is a factor. 1

$$\begin{array}{r} x^2-2x-15 \\ x+2\,\overline{)\,x^3-19x-30} \\ -(x^3+2x^2) \\ \hline -2x^2-19x-30 \\ -(-2x^2-4x) \\ \hline -15x-30 \\ -(-15x-30) \\ \hline 0 \end{array}$$

1

$\therefore$ (ii) $\boxed{P(x) = (x+2)(x+3)(x-5)}$

Question 3

3. (a) (i) Express the decimal $0{\cdot}1\dot{5}\dot{4}$ as a common fraction in lowest terms.

Solution:
$$\begin{aligned} x &= 0{\cdot}1\dot{5}\dot{4}, \\ 100x &= 15{\cdot}4\dot{5}\dot{4}, \\ 99x &= 15{\cdot}3, \\ x &= \frac{153}{990}, \\ &= \frac{17}{110}. \end{aligned}$$

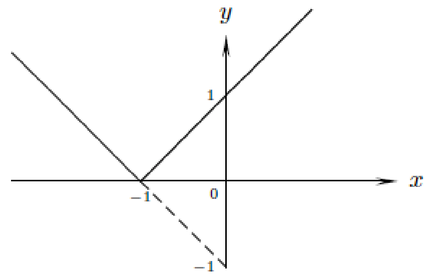
(ii) Find $\log_2 74$ correct to three decimal places.

Solution:
$$\begin{aligned} \log_2 74 &= \frac{\log 74}{\log 2}, \\ &\approx 6{\cdot}209 \quad [6{\cdot}20945336562 \text{ on calculator}]. \end{aligned}$$

(b) Draw neat sketches of the following functions, showing their principal features:

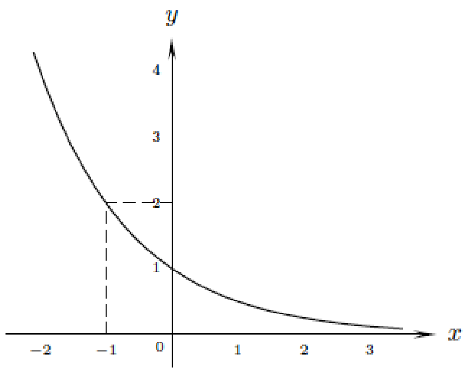
(i) $y = |x + 1|$

Solution: When $x < -1$, $y = -x - 1$; when $y \geq -1$, $y = x + 1$.



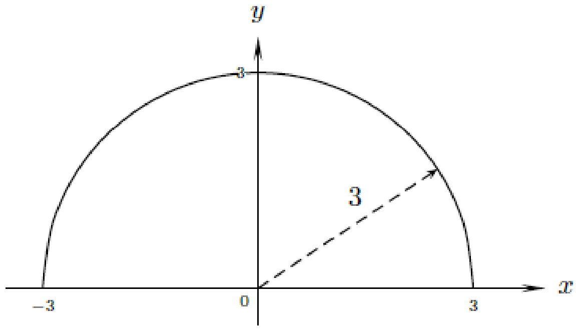
(ii) $y = 2^{-x}$

Solution:



(iii) $y = \sqrt{9 - x^2}$

Solution:



(c) Given the function $f(x) = \dfrac{x}{x^2+1}$

(i) Find $f(-1)$

Solution: $$\begin{aligned} f(-1) &= \frac{-1}{(-1)^2+1}, \\ &= -\tfrac{1}{2}. \end{aligned}$$

(ii) Show that $f(x)$ is odd.

Solution: $$\begin{aligned} f(-x) &= \frac{-x}{(-x)^2+1}, \\ &= -\frac{x}{x^2+1}, \\ &= -f(x). \end{aligned}$$
$\therefore\ f(x)$ is odd.

(iii) Find x such that $f(x) = 0$.

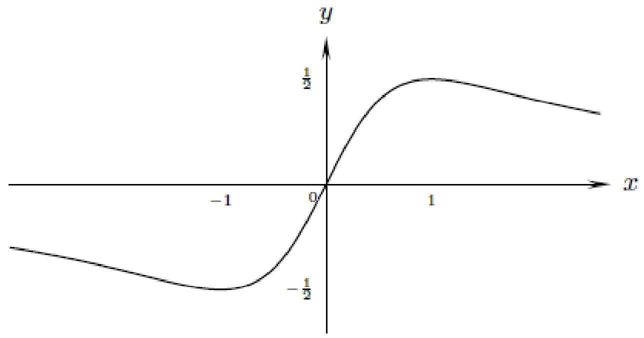
Solution: $$\begin{aligned} \frac{x}{x^2+1} &= 0, \\ \therefore\ x &= 0. \end{aligned}$$

(iv) State the domain and range of $f(x)$.

Solution: Domain: $x \in \mathbb{R}$, or all real x.
Now, putting $y = f(x)$ and rearranging,
$$\begin{aligned} yx^2 + y &= x, \\ yx^2 - x + y &= 0, \\ \Delta &= 1 - 4y^2 \ \geq 0 \text{ for real values of x}, \\ i.e.\ \tfrac{1}{4} &= y^2. \end{aligned}$$
And thus the range is $-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$.

(v) Sketch the function.

Solution:



(d) Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals.

(i) Find the other root.

Solution: Let the roots be α, $\frac{1}{\alpha}$, β, then

$$\alpha \times \frac{1}{\alpha} \times \beta = -4 \text{ (product of roots)},$$
$$i.e.\ \beta = -4.$$

(ii) Find the reciprocal roots.

Solution:

$$\begin{aligned} \alpha + \frac{1}{\alpha} - 4 &= 0 \text{ (sum of roots)}, \\ \alpha^2 - 4\alpha + 1 &= 0, \\ \alpha &= \frac{4 \pm \sqrt{16-4}}{2}, \\ &= 2 \pm \sqrt{3}. \end{aligned}$$

$i.e.$ the reciprocal roots are $2 \pm \sqrt{3}$.

(e) Find the distance between the parallel lines $4x + 3y = 12$ and $4x + 3y = 5$.

Solution: One point on $4x + 3y = 12$ is $(0,\ 4)$.

$$\begin{aligned} \therefore\ \text{Distance} &= \frac{|0 \times 4 + 4 \times 3 - 5|}{\sqrt{16+9}}, \\ &= \frac{7}{5}. \end{aligned}$$

Question 4

(a) (i) $\frac{AM}{c} = \cos A$

$\therefore \underline{AM = c\cos A}$.

$\frac{AN}{b} = \cos A$.

$\therefore \underline{AN = b\cos A}$.

(ii) $MN^2 = AN^2 + AM^2 - 2AN.AM.\cos A$.

$= b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^2 A.\cos A$.

$= \cos^2 A\,(b^2 + c^2 - 2bc\cos A)$

$= \cos^2 A.\, a^2$

(NB. $a^2 = b^2 + c^2 - 2bc\cos A$ from Cosine Rule.)

$\therefore \underline{MN = a\cos A}$.

(b) (i) $y - px + ap^2 = 0$.

(ii) $m_{SP} = \frac{ap^2 - a}{2ap}$

$= \frac{a(p^2-1)}{2ap}$

$= \frac{p^2-1}{2p}$.

$\tan\theta = \left|\frac{\frac{p^2-1}{2p} - p}{1 + \frac{p^2-1}{2p}\times p}\right|$

$= \left|\frac{\frac{p^2-1-2p^2}{2p}}{\frac{2+p^2-1}{2}}\right|$

$= \left|\frac{-(1+p^2)}{p(1+p^2)}\right|$

$= \left|\frac{-1}{p}\right|$

$= \frac{1}{|p|}$.

(iii) $\tan\theta$ is undefined when $p=0$. At this point $(0,0)$ the angle is 90°.

(c) LHS $= \cot\theta + \tan\theta$

$= \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{1}{\sin\theta\cos\theta}$

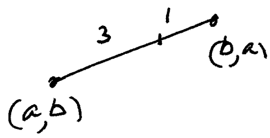
$= \dfrac{2}{2\sin\theta\cos\theta}$

$= \dfrac{2}{\sin 2\theta}$

$= 2\,\mathrm{cosec}\,2\theta$

$=$ RHS.

(d)



$\dfrac{3b+a}{4} = 0 \Rightarrow 3b+a=0$ —(1).

$\dfrac{3a+b}{4} = 4 \Rightarrow 3a+b=16$ —(2)

Solving (1) & (2) simultaneously.

From (1) $a = -3b$.

Substitute in (2)

$-9b+b = 16$

$-8b = 16$

$b = -2$.

Sub in (1)

$-6+a = 0$

$a = 6$.

$\therefore \boxed{a=6,\ b=-2}$

(e) (i) $\dfrac{a}{AP} = \tan\theta \ \therefore\ AP = \dfrac{a}{\tan\theta}$ Also $\dfrac{b}{BP} = \tan\theta \ \therefore\ BP = \dfrac{b}{\tan\theta}$.

By Pythagoras $d^2 = AP^2 + BP^2$

$d^2 = \dfrac{a^2}{\tan^2\theta} + \dfrac{b^2}{\tan^2\theta}$ — (A)

(ii) LHS $= \tan^2\alpha + \tan^2\beta = \dfrac{a^2}{d^2} + \dfrac{b^2}{d^2}$

$= \dfrac{a^2+b^2}{d^2}$

$= \dfrac{d^2\tan^2\theta}{d^2}$

$= \tan^2\theta$

$=$ RHS.

(From (A) $a^2+b^2 = d^2\tan^2\theta$.)

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2004

YEAR 11

PRELIMINARY HIGH SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – One and a half hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt all questions.
- All questions are of equal value.
- Each question is to be answered in a separate booklet.

Examiner: *A.M.Gainford*

Question 1. (18 Marks)

(a) Show that $\dfrac{1}{3-\sqrt{2}}+\dfrac{1}{3+\sqrt{2}}$ is a rational number. **2**

(b) Solve for x: **6**

(i) $|2x-1|=5$

(ii) $x^2 \geq 1$

(iii) $\dfrac{1}{x-1}<2$

(c) Find the remainder when the polynomial $P(x)=2x^3-3x^2+x-4$ is divided by $x-2$. **1**

(d) Simplify $\dfrac{x^3-1}{x^2-2x+1}$. **2**

(e) If $\tan\theta=2$, and $0<\theta<\dfrac{\pi}{2}$, find the exact value of $\sin\left(\theta+\dfrac{\pi}{4}\right)$. **2**

(f) Find the vertex and focus of the parabola $y=\dfrac{1}{4}\left(x^2-2x+9\right)$. **2**

(g) Show that for all θ: **3**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta .$$

Question 2. (18 Marks)

(a) Differentiate: **8**

(i) $1+2x-4x^2-x^3$

(ii) $\sqrt{1-x^2}$

(iii) $(x-1)^4(3x+1)$

(iv) $\dfrac{2}{x^3-1}$

(b) (i) Express $\sin x-\sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$, where $A>0$ and $0<\alpha<\dfrac{\pi}{2}$. **2**

(ii) Find the general solution to the equation $\sin x-\sqrt{3}\cos x=\dfrac{2}{\sqrt{2}}$. **2**

(c) Solve $(x-1)^2 < 4(x-1)$, and graph the solution on the number line. **2**

(d) Sketch the graph of $y = \cos x + \sin 2x$ in the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **2**

(e) Given the polynomial $P(x) = x^3 - 19x - 30$. **2**

(i) Use the factor theorem to find a zero of the polynomial.

(ii) Express $P(x)$ as a product of three linear factors.

Question 3. (18 Marks)

(a) (i) Express the decimal $0\cdot1\dot{5}\dot{4}$ as a common fraction in lowest terms. **2**

(ii) Find $\log_2 74$ correct to three decimal places.

(b) Draw neat sketches of the following functions, showing their principle features: **6**

(i) $y = |x+1|$ (ii) $y = 2^{-x}$ (iii) $y = \sqrt{9 - x^2}$

(c) Given the function $f(x) = \frac{x}{x^2+1}$ **6**

(i) Find $f(-1)$.

(ii) Show that $f(x)$ is odd.

(iii) Find x such that $f(x) = 0$.

(iv) State the domain and range of $f(x)$.

(v) Sketch the function.

(d) Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals. **2**

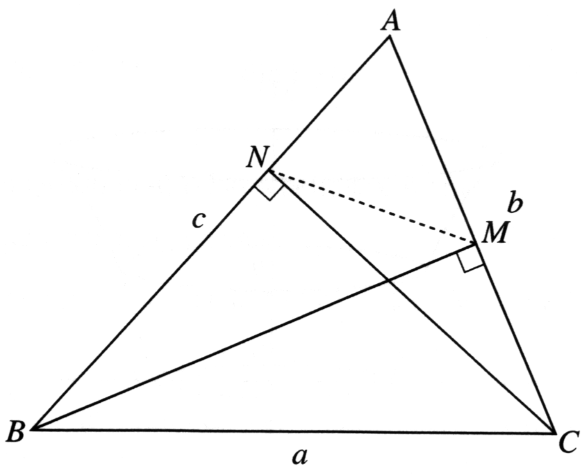
(i) Find the other root.

(ii) Find the reciprocal roots.

(e) Find the distance between the parallel lines $4x + 3y = 12$ and $4x + 3y = 5$. **2**

Question 4. (18 Marks)

(a) **4**



Triangle *ABC* has sides of length *a*, *b*, *c* as shown.
BM is perpendicular to *AC* and *CN* is perpendicular to *AB*.

(i) Show that $AM = c\cos A$ and $AN = b\cos A$.

(ii) Hence, using the cosine rule, prove that $MN = a\cos A$.

(b) Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$. **4**

(i) Write down the equation of the tangent at *P*.

(ii) Let θ be the acute angle between the tangent at *P* and the line *SP*, which joins *P* with the focus *S*.

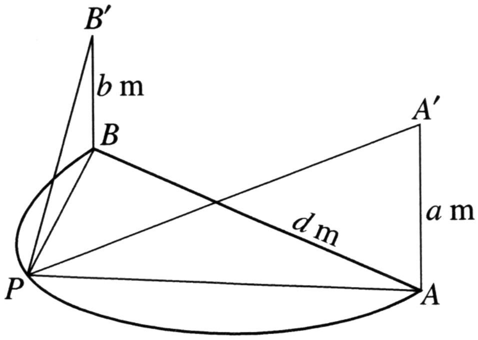
Show that $\tan\theta = \dfrac{1}{|p|}$.

(iii) Explain the situation at the one point where this angle is not acute.

(c) Show that $\cot\theta + \tan\theta = 2\operatorname{cosec}2\theta$. **2**

(d) The point $P(0,4)$ divides the interval from (*a*, *b*) to (*b*, *a*) in ratio 3 : 1. **2**
Find the values of *a* and *b*.

(e) *APB* is a horizontal semicircle, diameter *d* m. **6**
At *A* and *B* are vertical posts of height *a* m and *b* m respectively.
From *P*, the angle of elevation of the tops of both posts is θ.
The angle *APB* is a right angle.



(i) Prove that $d^2 = \dfrac{a^2}{\tan^2\theta} + \dfrac{b^2}{\tan^2\theta}$.

(ii) From *B*, the angle of elevation of A' is α, and from *A*, the angle of elevation of B' is β.

Prove that $\tan^2\alpha + \tan^2\beta = \tan^2\theta$.

End of the paper.