SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2005

Yearly Examination

YEAR 11

Mathematics Extension

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 80 Marks

- Attempt Questions 1 - 4
- All questions are of equal value.

Examiner: *P. Bigelow*

Total marks – 80
Attempt Questions 1 - 4
All questions are of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A (Use a SEPARATE writing booklet)

Marks

Question 1 (20 marks)

(a) Find

(i) $\lim_{x \to 4} \frac{x^2 + x - 20}{x - 4}$ **1**

(ii) $\lim_{x \to \infty} \frac{4 - x + 2x^2}{x^2 + 7x - 9}$ **1**

(b) Find x in the following

(i) $8^x = 2^{x-4}$ **1**

(ii) $\log_x 36 = 2$ **1**

(iii) $\log_3 \frac{\sqrt{3}}{9} = x$ **1**

(c) Differentiate the following

(i) $y = (7 - 2x)^6$ **1**

(ii) $y = 7x^{-3} + 4x - 17$ **1**

(iii) $f(x) = x\sqrt{x - 2}$ **2**

(iv) $f(x) = \frac{x + 4}{x + 5}$ **2**

(d) Write down a quadratic equation with roots 4 and -7. **2**

(e) If $f(x) = \sqrt{x^2 + 16}$ find $f'(3)$. **2**

(f) Find the equation of the tangent to $y = x^2 + 3x + 4$ at the point where $x = 0$. **2**

(g) For the parabola $(x + 2)^2 = 4y - 8$, find the:

(i) coordinates of the vertex; **1**

(ii) coordinates of the focus; **1**

(iii) equation of the directrix. **1**

Section B (Use a SEPARATE writing booklet)

Marks

Question 2 (20 marks)

(a) α, β and γ are the roots of $x^3 + 4x^2 + 8x + 16 = 0$.
Write down the values of:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(iii) $\alpha\beta\gamma$ **1**

(iv) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **1**

(v) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(b) Derive the equation of the normal to $x^2 = 4ay$ at $P\left(2as, as^2\right)$ **2**

(c) Write $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x + \alpha)$ where $A > 0$ and $0 < \alpha < \dfrac{\pi}{2}$ **2**

(d) Solve:

(i) $\dfrac{4}{|x+1|} < 3$ **2**

(ii) $\dfrac{3}{x-1} \geq \dfrac{2}{x}$ **2**

(e) Find the coordinates of the point which divides the interval *AB*, where *A* is $(-4, 0)$ and *B* is $(3, -7)$, *externally* in the ratio 3 : 2 **2**

(f) (i) Show that -4 is a zero of $x^3 - 3x^2 - 18x + 40$. **1**

(ii) Hence solve $x^3 - 3x^2 - 18x + 40 = 0$ **2**

(g) Solve $\sin 2x = \cos x$ for $0 \leq x \leq 2\pi$ **2**

Section C (Use a SEPARATE writing booklet)

Marks

Question 3 (20 marks)

(a) Find the acute angle between the lines $2x - y - 7 = 0$ and $x - 3y + 3 = 0$. **2**

(b) Find the general solution of the equation $2\cos\theta - \sqrt{3} = 0$ **2**

(c) (i) Write down the expansion of $\sin(A - B)$. **1**

(ii) Find the exact value of $\sin 15°$. **2**

(d) The polynomial $x^3 + 2x^2 + ax + b$ has a factor of $x + 2$ and when divided by $x - 2$ the remainder is 12.
Find the values of *a* and *b*. **3**

(e) (i) If $0 \le \theta \le \dfrac{\pi}{2}$ prove that $\tan\theta = \sqrt{\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}}$ **2**

(ii) Hence show that the exact value of $\tan\dfrac{\pi}{8}$ is $\sqrt{2} - 1$ **2**

(f) (i) Show that $\cot x + \tan x = 2\operatorname{cosec} 2x$. **2**

(ii) Hence, or otherwise, prove that $\cot 15° = 2 + \sqrt{3}$ **1**

(g) Show that if $2\cos\theta = x + \dfrac{1}{x}$ then $2\cos 3\theta = x^3 + \dfrac{1}{x^3}$. **3**

You may use the identity $\cos 3A = 4\cos^3 A - 3\cos A$

Section D (Use a SEPARATE writing booklet)

Marks

Question 4 (20 marks)

(a) Differentiate $f(x) = 3 - x + x^2$ using first principles. **2**

(b) Sketch the following without calculus:

(i) $y = (x-1)(x+3)(x-7)$ **2**

(ii) $y = (x-2)^2(x+3)$ **2**

(c) $P(2p, p^2)$ and $Q(2q, q^2)$ are points on the parabola $x^2 = 4y$.

(i) Show that the equation of the chord joining P and Q is given by $y = \left(\frac{p+q}{2}\right)x - pq$ **2**

(ii) Find the coordinates of the midpoint, Z, of PQ. **2**

(iii) If the chord, joining P and Q, passes through $(0,2)$ find the locus of Z. **3**

(d) (i) θ and ϕ are acute angles such that $\cos\theta = \frac{3}{5}$ and $\sin\phi = \frac{1}{\sqrt{5}}$. **2**
Without finding the size of either angle, show that $\theta = 2\phi$.

(ii) Using the result above, find the exact value of $\sin 3\phi$ **3**

(e) Given that $\log_b a = 2$ and $\log_c b = 3$, find the value of $\log_a c$. **2**

End of paper

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Sample Solutions

Section	Marker
A	DM
B	FN
C	EC
D	AF

Q1

a i) $\lim_{x\to 4} \frac{x^2+x-20}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+5)}{x-4}$ (1)

$= \lim_{x\to 4} x+5$

$= 9.$

ii) $\lim_{x\to\infty} \frac{4-x+2x^2}{x^2+7x-9} = \lim_{x\to\infty} \frac{4-x+2x^2}{x^2+7x-9} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$ (1)

$= \lim_{x\to\infty} \frac{\frac{4}{x^2}-\frac{1}{x}+2}{1+\frac{7}{x}-\frac{9}{x^2}}$

$= \frac{0-0+2}{1+0-0}$

$= 2.$

bi) $8^x = 2^{x-4}$

$\log_2 8^x = \log_2 2^{x-4}$ (1)

$x\log_2 2^3 = (x-4)\log_2 2$

$3x\log_2 2 = x-4$

$3x = x-4$

$2x = -4$

$x = -2.$

ii) $\log_x 36 = 2$

$36 = x^2$ 1

$x = 6.$

iii) $\log_3 \frac{\sqrt{3}}{9} = x$

$\frac{\sqrt{3}}{9} = 3^x$ 1

$3^{\frac{1}{2}} \div 3^2 = 3^x$

$3^{-\frac{3}{2}} = 3^x$

$x = -\frac{3}{2}.$

c) i) $y = (7-2x)^6$ Chain rule

$\frac{dy}{dx} = 6(7-2x)^5(-2)$ 1

$= -12(7-2x)^5$

ii) $y = 7x^{-3} + 4x - 17$

$\frac{dy}{dx} = -21x^{-4} + 4$ 1

iii) $f(x) = x\sqrt{x-2}$ Chain and product rule.

$f'(x) = \sqrt{x-2} + \frac{x}{2}(x-2)^{-\frac{1}{2}}$ 2

$= \sqrt{x-2} + \frac{x}{2\sqrt{x-2}}$

$= \sqrt{x-2}\left(\frac{3x-4}{2x-4}\right)$

iv) $f(x) = \frac{x+4}{x+5}$ Quotient rule.

$$f'(x) = \frac{(x+5)-(x+4)}{(x+5)^2}$$ 2

$$= \frac{1}{(x+5)^2}$$

d) $(x-4)(x+7) = 0.$ 2

$x^2 - 4x + 7x - 28 = 0$

$x^2 + 3x - 28 = 0.$

e) $f(x) = \sqrt{x^2+16}$ Chain rule

$$f'(x) = \frac{1}{2}(x^2+16)^{-\frac{1}{2}}(2x)$$ 2

$$= \frac{x}{\sqrt{x^2+16}}$$

$$f'(3) = \frac{3}{\sqrt{9+16}}$$

$$= \frac{3}{5}$$

f) $y = x^2 + 3x + 4$

At $x=0$

$y = 0^2 + 3(0) + 4$

$y = 4$

$(0,4)$, is point of contact.

$\frac{dy}{dx} = 2x + 3$

At $(0,4)$

$\frac{dy}{dx} = 2(0) + 3$

$= 3$ 2

eqn of tangent $m=3$, $(0,4)$.

$y - 4 = 3(x-0)$

$y = 3x + 4.$

9 i) $(x+2)^2 = 4y - 8$

$(x+2)^2 = 4(y-2)$ 1

Vertex $(-2, 2)$

ii) focal length $= 1$ 1

So focus $(-2, 3)$

iii) Directrix $y = 1$ 1

QUESTION 2

(a) $x^3+4x^2+8x+16=0$

(i) $\alpha+\beta+\gamma=-\frac{b}{a}=-4$

(ii) $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}=8$

(iii) $\alpha\beta\gamma=-\frac{d}{a}=-16$

(iv) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}=\frac{8}{-16}=-\frac{1}{2}$

(v) $(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)=16-2\times8=0$

(b) $x^2=4ay \quad (2as, as^2)$

$y=\frac{x^2}{4a}$

$y'=\frac{2x}{4a}=\frac{x}{2a}$ (grad.of tangent)

$y'=\frac{2as}{2a}=s \,(x=2as)$

grad.of normal $=-\frac{1}{s}$

eqn.of normal is $y-as^2=-\frac{1}{s}(x-2as)$

$x+sy=as^3+2as$

c) $\cos x-\sqrt{3}\sin x=A\cos(x+\alpha)$

$\cos x-\sqrt{3}\sin x=A\cos x\cos\alpha-A\sin x\sin\alpha$

equating coefficients $A\cos\alpha=1$, $A\sin\alpha=\sqrt{3}$

$\tan\alpha=\sqrt{3} \quad \alpha=\frac{\pi}{3} \quad A=2$

$\cos x-\sqrt{3}\sin x=2\cos\left(x+\frac{\pi}{3}\right)$

d) (i) $\frac{4}{|x+1|}<3$

$4<3|x+1| \quad (|x+1|>0)$

$4<3(x+1)$ or $4<3(-x-1)$

$3x+3>4$ or $-3x-3>4$

$3x>1$ or $-3x>7$

$x>\frac{1}{3}$ or $x<-\frac{7}{3}$

(d) (ii) $\frac{3}{x-1}>\frac{2}{x} \quad x\neq1, x\neq0$

$\left[\text{multiply by } (x-1)^2x^2\right]$

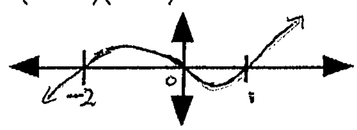
$3x^2(x-1)\geq2x(x-1)^2$

$3x^3-3x^2\geq2x^3-4x^2+2x$

$x^3+x^2-2x\geq0$

$x(x^2+x-2)\geq0$

$x(x+2)(x-1)\geq0$



$-2\leq x<0$ or $x>1$

(e) $m:n=3:-2 \quad (-4,0) \; (3,-7)$

$x=\frac{mx_2+nx_1}{m+n} \quad y=\frac{my_2+ny_1}{m+n}$

$x=\frac{3\times3-2\times-4}{3-2} \quad y=\frac{3\times-7-2\times0}{3-2}$

$(17,-21)$

(f) (i) $P(-4)=(-4)^3-3(-4)^2-18(-4)+40=0$

or

$$\begin{array}{r} x^2-7x+10 \\ x+4\,\overline{)\,x^3-3x^2-18x+40} \\ \underline{x^3+4x^2} \qquad\qquad \\ -7x^2-18x \qquad \\ \underline{-7x^2-28x} \qquad \\ 10x+40 \\ \underline{10x+40} \\ 0\;+\;0 \end{array}$$

remainder = 0 $\therefore -4$ is a zero of P(x)

(ii) $x^3-3x^2-18x+40=(x+4)(x-5)(x-2)=0$

$x=-4, 2, 5$

(g) $\sin2x-\cos x=0$

$2\sin x\cos x-\cos x=0$

$\cos x(2\sin x-1)=0$

$\cos x=0$ or $\sin x=\frac{1}{2}$

$x=\frac{\pi}{2},\frac{3\pi}{2}$ or $x=\frac{\pi}{6},\frac{5\pi}{6}$

Section (c) — Solutions

(a) $2x - y - 7 = 0$ —— (1)

$x - 3y + 3 = 0$ —— (2)

$y = 2x + 7, \quad m_1 = 2$

$y = \frac{x}{3} + 1, \quad m_2 = \frac{1}{3}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2 - 1/3}{1 + 2/3}\right| = 1$$

$\therefore \theta = 45$

(b) $2\cos\theta = \sqrt{3}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{6}$

(c) (i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\therefore \sin 15 = \sin(45 - 30)$

(ii) $= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}(\sqrt{3} - 1)}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}.$

(d) $x^3 + 2x^2 + ax + b = P(x)$

$P(-2) = 0$

$-8 + 8 - 2a + b = 0$

$\therefore b = 2a$ —— (1)

$P(2) = 12$

$8 + 8 + 2a + b = 12$

$2a + b = -4$ —— (2)

Subst. (1) into (2)

$4a = -4 \quad \therefore a = -1, \quad b = -2$

(e) (i) $\sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$

$= \sqrt{\frac{(\sin^2\theta + \cos^2\theta) - (\cos^2\theta - \sin^2\theta)}{2\cos^2\theta}}$

$= \sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}} = \tan\theta.$

$\therefore \tan\frac{\pi}{8} = \sqrt{\frac{1 - \cos\frac{\pi}{4}}{1 + \cos\frac{\pi}{4}}}$

$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}$

$= \sqrt{\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}}$

$= \sqrt{(\sqrt{2} - 1)^2} = \sqrt{2} - 1.$

(f)

(i) $\cot x + \frac{1}{\cot x}$

$= \frac{\cot^2 x + 1}{\cot x}$

$= \frac{\operatorname{cosec}^2 x}{\cot x}$

$= \frac{1}{\sin^2 x} \times \frac{\cancel{\sin x}}{\cos x}$

$= \frac{2}{2\sin x \cos x}$

$= \frac{2}{\sin 2x}$

$= 2\operatorname{cosec} 2x.$

(ii) $\cot 15 + \frac{1}{\cot 15} = \frac{2}{\sin 30^\circ}$

$\therefore \cot 15 + \frac{1}{\cot 15} = 4$

$\cot^2 15 - 4\cot 15 + 1 = 0$

Let $m = \cot 15$

$m^2 - 4m + 1 = 0$

$m = \frac{4 \pm \sqrt{16 - 4}}{2}$

$= 2 \pm \sqrt{3}.$

$\tan 15 = 2 + \sqrt{3}$

(g) If $2\cos\theta = x + \frac{1}{x}$.

$$8\cos^3\theta = \left(x + \frac{1}{x}\right)^3$$

$$= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$$

$$= \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) \quad \text{——— (1)}$$

but $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore$ $2\cos 3A = 8\cos^3 A - 6\cos A$

From (1)

$$x^3 + \frac{1}{x^3} = 8\cos^3\theta - 6\cos\theta$$

$$\therefore x^3 + \frac{1}{x^3} = 2\cos 3\theta \quad \text{——— (2)}$$

4.a. $f(x) = 3 - x + x^2$

$f(x+h) = 3 - (x+h) + (x+h)^2$

$= 3 - x - h + x^2 + 2xh + h^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - \cancel{x} - h + \cancel{x^2} + 2xh + h^2 - (\cancel{3} - \cancel{x} + \cancel{x^2})}{h}$$

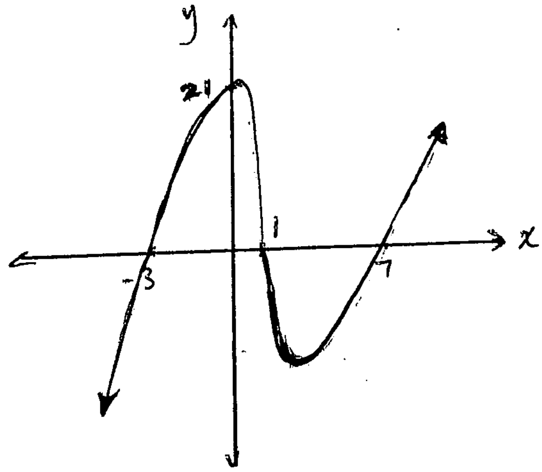
$$= \lim_{h \to 0} \frac{\cancel{h}(-1 + 2x + h)}{\cancel{h}}$$

$$= -1 + 2x$$

b. i. $y = (x-1)(x+3)(x-7)$

when $x = 0$, $y = 21$

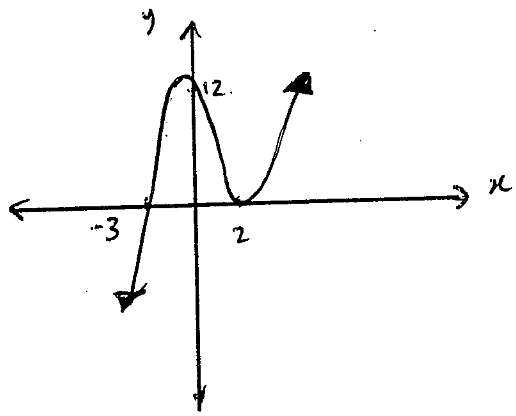
$y = 0$, $x = 1, -3, 7$



ii. $y = (x-2)^2(x+3)$

when $x = 0$, $y = 12$

$y = 0$, $x = 2, -3$



c. i. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{p^2 - q^2}{2p - 2q}$$

$$= \frac{\cancel{(p-q)}(p+q)}{2\cancel{(p-q)}}$$

$$= \frac{p+q}{2}$$

$y - y_1 = m(x - x_1)$

$y - p^2 = \frac{p+q}{2}(x - 2p)$

$y = \left(\frac{p+q}{2}\right)x - \cancel{p^2} - pq + \cancel{p^2}$

$y = \left(\frac{p+q}{2}\right)x - pq$

ii. $Z = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{2p + 2q}{2}, \frac{p^2 + q^2}{2}\right)$

$= \left(p + q, \frac{p^2 + q^2}{2}\right)$

iii. $(0, 2)$ satisfies $y = \left(\frac{p+q}{2}\right)x - pq$

$2 = \left(\frac{p+q}{2}\right).0 - pq$

$-pq = 2$

$pq = -2$ ——— 1

from(ii) $x = p + q$ ——— 2

$y = \frac{p^2 + q^2}{2}$ ——— 3

rearrange 3

$y = \frac{(p+q)^2 - 2pq}{2}$ ——— 3a

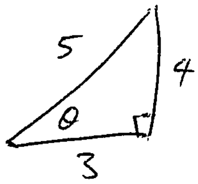
substitute 1 & 2 into 3a

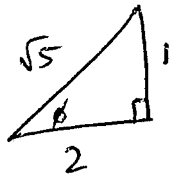
$y = \frac{x^2 - 2(-2)}{2}$

$y = \frac{x^2 + 4}{2}$

d. i. θ & ϕ are acute angles

$\cos\theta = \frac{3}{5}$, $\sin\phi = \frac{1}{\sqrt{5}}$





Show that $\theta = 2\phi$

$\sin\theta = \sin 2\phi$

$LHS = \frac{4}{5}$

$$RHS = 2\sin\phi\cos\phi$$
$$= 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}}$$
$$= \frac{4}{5}$$

$LHS = RHS.$

$\sin\theta = \sin 2\phi$

$\therefore\ \theta = 2\phi$

ii.
$$\sin 3\phi = \sin(2\phi + \phi)$$
$$= \sin(\theta + \phi)$$
$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$= \frac{4}{5} \cdot \frac{2}{\sqrt{5}} + \frac{3}{5} \cdot \frac{1}{\sqrt{5}}$$
$$= \frac{8}{5\sqrt{5}} + \frac{3}{5\sqrt{5}}$$
$$= \frac{11}{5\sqrt{5}} \quad \text{or} \quad \frac{11\sqrt{5}}{25}$$

e.
$\log_b a = 2 \implies a = b^2$ ——— ①

$\log_c b = 3 \implies b = c^3$ ——— ②

sub ② into ①

$$a = (c^3)^2$$
$$a = c^6$$

substitute.

let $\log_a c = x \Rightarrow c = a^x$

$$c = (c^6)^x$$
$$c^1 = c^{6x}$$
$$\therefore\ 6x = 1$$
$$x = \frac{1}{6}$$

$\therefore\ \log_a c = \frac{1}{6}$