SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
YEAR 11
YEARLY EXAM

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 60

- Attempt questions 1-3
- Hand up in 3 sections clearly marked A,B & C

Examiner: *A.M.Gainford*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SECTION A

Question 1 (20 marks) **Marks**

a) Find the remainder when the polynomial $P(x) = x^3 + 3x - 2$ is divided by $x - 3$. **1**

b) If $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, find the possible values of $\tan\theta$. **2**

c) The equation $x^2 - (1-2k)x + k + 3 = 0$ has consecutive integral roots. Find the possible values of k. **3**

d) i) In how many ways can 6 committee members be selected from 10 people? **4**

ii) In how many ways can this be done if two particular people will only serve together?

e) Express $\sin 3\theta$ as an expression in powers of $\sin\theta$ only. **2**

f) i) Write the equation of the tangent to the curve $y = x^3$ at the point on the curve where $x = 1$. **3**

ii) Find the co-ordinates of the point where this tangent crosses the curve.

g) Solve $|2x + 6| < 4$ **2**

h) Solve for x: $\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) - 12 = 0$ **3**

SECTION B

Question 2 (20 marks) **Marks**

Start a New Booklet

a) Find the coordinates of the point which divides AB with $A(1,4)$ and B $(5,2)$ externally in the ratio 1:3. **2**

b) Given the curve with equation $y = \dfrac{x}{x^2+1}$: **7**

i) Find the first and second derivatives.

ii) Identify and determine the nature of any turning points and points of inflexion.

iii) Make a neat sketch of the curve.

3

c) Solve the inequality $\dfrac{4}{5-x} \geq 1$.

4

d) Find the general solution to the equation $\cos(\theta + \dfrac{\pi}{6}) = \dfrac{\sqrt{3}}{2}$.

4

e) i) Express $\sin\theta + \sqrt{3}\cos\theta$ in the form $R\sin(\theta + \alpha)$.

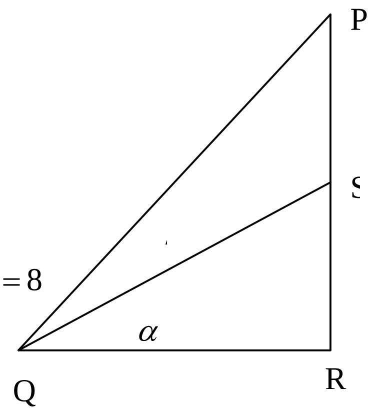
ii) Hence or otherwise sketch the graph of $y = \sin x + \sqrt{3}\cos x$ in the domain $0 \leq x \leq \pi$.

SECTION C

Question 3 (20 marks) **Marks**

Start a New Booklet

a) In the diagram $\angle\alpha > \angle\beta > 0$ and $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \dfrac{4}{5}$ **8**



i) Show that $\tan\alpha.\tan\beta = \dfrac{1}{9}$

ii) If PR = QR, show that $9(\tan\alpha + \tan\beta) = 8$

iii) Hence find α and β.

4

b) i) The polynomial equation $P(x) = 0$ has a double root at $x = 9$. By writing $P(x) = (x-a)^2 Q(x)$, where $Q(x)$ is a polynomial, show $P'(a) = 0$.

ii) Hence or otherwise find the values of a and b if $x = 1$ is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$

c) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. **8**

i) Derive the equation of the tangent to the parabola at P.

ii) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q.

iii) Given that the tangents at P and Q in (ii) intersect at and angle of 45°, show that $p - q = 1 + pq$.

iv) By Evaluating the expression $x^2 = 4ay$ at T, or otherwise, find the locus of T.

SECTION A. (Q1).

(a) $R = P(3) = 27 + 9 - 2$
$= \boxed{34}$ $\boxed{1}$

(b) $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$

$\therefore \sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60^\circ, 120^\circ$ etc..

$\therefore \boxed{\tan\theta = \pm\sqrt{3}}$ $\boxed{2}$

(c) Let the roots be α & $\alpha+1$.

now $\alpha + (\alpha+1) = \frac{-b}{a} = 1-2k$ ie $2\alpha+1 = 1-2k$.
ie $\alpha = -k$.

& $\alpha(\alpha+1) = \frac{c}{a} = k+3$ ie. $-k(-k+1) = k+3$

$k^2 - k = k+3$

$k^2 - 2k - 3 = 0$

$(k+1)(k-3) = 0$

$\boxed{k = 3, -1}$ $\boxed{3}$

(d) (i) $\binom{10}{6} = \boxed{210}$ $\boxed{2}$

(ii) If they (ie 2 particular people) will only serve together, they are either both on or both off.

$\therefore \binom{8}{4} + \binom{8}{6} = 70 + 28 = \boxed{98}$ $\boxed{2}$

(e) $\sin 3\theta = \sin(2\theta+\theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta.\cos\theta + (1-2\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\therefore \boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$. $\boxed{2}$

(f) (i) $y = x^3$

$y' = 3x^2$ $\therefore$ slope of tangent at $x=1$, is 3.

$\therefore$ eqn. of tangent at $(1,1)$

is $y-1 = 3(x-1)$

$y-1 = 3x-3$

$\boxed{y = 3x-2}$ $\boxed{1\frac{1}{2}}$

(ii) Solving $y = x^3$ and $y = 3x-2$.

ie $x^3 = 3x-2$

$x^3 - 3x + 2 = 0$ — (A)

now (A) will have roots $1, 1$ & α

now $1+1+\alpha = -\frac{b}{a} = 0$.

$\therefore \alpha = -2$.

$\therefore$ the other point is $\boxed{(-2,-8)}$ $\boxed{1\frac{1}{2}}$

(g) $|2x+6| < 4$

$\therefore -4 < 2x+6 < 4$

$-10 < 2x < -2$

$\boxed{-5 < x < -1}$

$\boxed{2}$

(h) let $x + \frac{1}{x} = u$.

$\therefore$ equation becomes

$u^2 - u - 12 = 0$.

$(u-4)(u+3) = 0$

$u = 4, -3$

If $u = 4$

$x + \frac{1}{x} = 4$

$x^2 + 1 = 4x$

$x^2 - 4x + 1 = 0$.

$x = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm \sqrt{12}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2}$

$\boxed{x = 2 \pm \sqrt{3}}$

If $u = -3$

$x + \frac{1}{x} = -3$.

$x^2 + 1 = -3x$.

$x^2 + 3x + 1 = 0$.

$x = \frac{-3 \pm \sqrt{9-4}}{2}$

$\boxed{x = \frac{-3 \pm \sqrt{5}}{2}}$

$\boxed{3}$

YR 11 Sept 2006 ext 1 paper

/20

Section B

Q 2. (a) $A(\overset{x_1}{1}, \overset{y_1}{4})$ $B(\overset{x_2}{5}, \overset{y_2}{2})$ $\overset{m}{-1} : \overset{n}{3}$

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$\Rightarrow \left(\frac{-1\times5+3\times1}{-1+3}, \frac{-1\times2+3\times4}{-1+3}\right) = (-1, 5)$$

2

(b) $y = \dfrac{x}{x^2+1}$

(i) $$y' = \frac{(x^2+1)\times1 - x\times2x}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$\frac{1}{2}$

$$y'' = \frac{(x^2+1)^2\times-2x - (1-x^2)\times2(x^2+1)^1\times2x}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1)^2 - 4x(1-x^2)(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$$

$$= \frac{-2x^3-2x-4x+4x^3}{(x^2+1)^3} = \frac{2x^3-6x}{(x^2+1)^3}$$

$$= \frac{2x(x^2-3)}{(x^2+1)^3}$$

$\frac{1}{2}$

(b) (ii) Stat pts exist when $y' = 0$

Since $(x^2+1)^2 \neq 0$, $1 - x^2 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = +1$, $y = \frac{1}{2}$ $(1, \frac{1}{2})$

when $x = -1$, $y = \frac{-1}{2}$ $(-1, -\frac{1}{2})$.

At $(1, \frac{1}{2})$ $y'' = \frac{\cancel{2}(1-3)}{\cancel{8}\,4} = -\frac{2}{4} < 0$ MAX. stat pt 1

At $(-1, -\frac{1}{2})$ $y'' = \frac{-\cancel{2}(1-3)}{\cancel{8}\,4} = \frac{2}{4} > 0$ min stat pt 1

Inflexions occur when $y'' = 0$ and $\exists$ a sign change

because $(x^2+1)^3 \neq 0$

$2x(x^2-3) = 0$

$x = 0$, $x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$.

At $x = 0$, $y = \frac{0}{0+1} = 0 \Rightarrow (0, 0)$ 1

At $x = +\sqrt{3}$, $y = \frac{\sqrt{3}}{4} \Rightarrow (\sqrt{3}, \frac{\sqrt{3}}{4})$ 1

At $x = -\sqrt{3}$, $y = -\frac{\sqrt{3}}{4} \Rightarrow (-\sqrt{3}, -\frac{\sqrt{3}}{4})$ 1

At $x = -\sqrt{3} - \varepsilon$ $y'' < 0$

$x = -\sqrt{3} + \varepsilon$ $y'' > 0$

sign change

At $x = 0 - \varepsilon$ $y'' > 0$

$x = 0 + \varepsilon$ $y'' < 0$

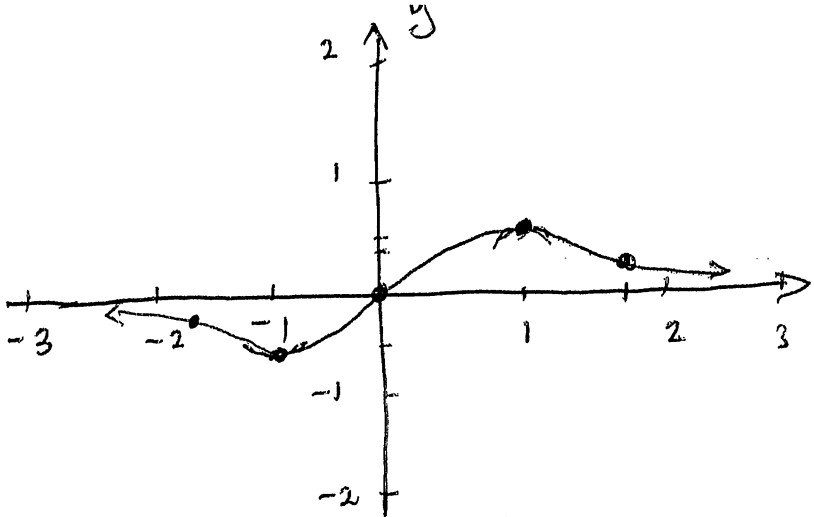
sign change

At $x = \sqrt{3} - \varepsilon$ $y'' < 0$

$x = \sqrt{3} + \varepsilon$ $y'' > 0$

sign change

b) (iii)



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$(1, \frac{1}{2})$ MAX $\qquad$ $(0,0)$ infl.

$(-1, -\frac{1}{2})$ min $\qquad$ $(\sqrt{3}, \frac{\sqrt{3}}{4}) \doteqdot (1.7, .4)$ infl

$(-\sqrt{3}, -\frac{\sqrt{3}}{4}) \doteqdot (-1.7, -.4)$ infl

(c) $\dfrac{4}{5-x} \geq 1$ now $x \neq 5$.

$$\frac{4(5-x)^2}{(5-x)} \geq 1.(5-x)^2$$

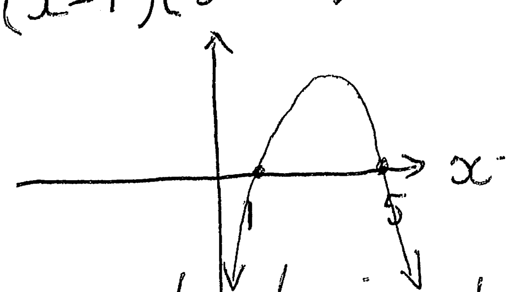
$$4(5-x) \geq (5-x)^2$$

$$4(5-x) - (5-x)^2 \geq 0$$

$$(5-x)[4-(5-x)] \geq 0$$

$$(5-x)[-1+x] \geq 0$$

So $(x-1)(5-x) \geq 0$



above or touching $\quad 1 \leq x \leq 5$

but $x \neq 5$, so $\{x : 1 \leq x < 5\}$ //

3

2 (d) in radians

If $\cos\theta = \cos\alpha$

$$\theta = 2\pi \times n \pm \alpha.$$

So $\cos(\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$$\theta + \frac{\pi}{6} = 2\pi \times n \pm \frac{\pi}{6}$$

$$\theta = 2\pi n \pm \frac{\pi}{6} - \frac{\pi}{6}$$

So $\theta = 2\pi n$, $\theta = 2\pi n - \frac{\pi}{3}$ // 4

2 (e) (i) $1\sin\theta + \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)$ $R = \sqrt{1^2 + \sqrt{3}^2} = 2.$

no $2\left(\frac{1}{2}\cancel{\sin\theta} + \frac{\sqrt{3}}{2}\cancel{\cos\theta}\right)$

$= R(\cancel{\sin}\theta\cos\alpha + \cancel{\cos}\theta\sin\alpha)$

now $R = 2$, $\left.\begin{array}{l}\cos\alpha = \frac{1}{2} \text{ quad } 1, 4 \\ \sin\alpha = \frac{\sqrt{3}}{2} \text{ quad } 1, 2.\end{array}\right\}$ quad 1.

So $\alpha = \frac{\pi}{3}$ (radian measure).

3

So $2\sin(\theta + \frac{\pi}{3}) = \sin\theta + \sqrt{3}\cos\theta$ //

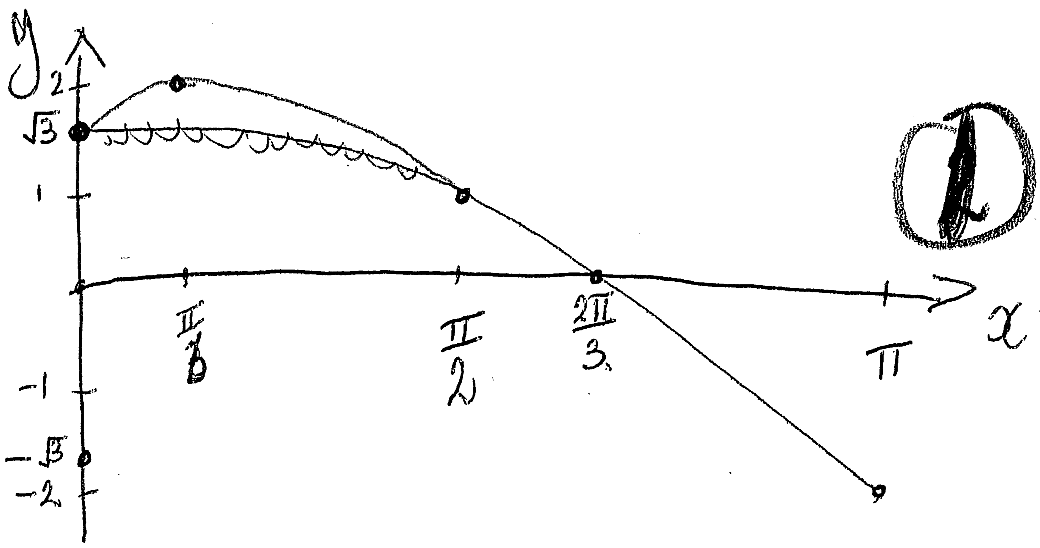
(ii)

$y = 2\sin(\theta + \frac{\pi}{3})$

$\theta = 0$, $y = 2\sin\frac{\pi}{3} = \sqrt{3}$

$\theta = \frac{\pi}{2}$ $y = 2\sin\frac{5\pi}{6} = 1$

$= \pi$ $y = 2\sin\frac{4\pi}{3} = -\sqrt{3}$



1

Question [3] Solution:

$$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{4}{5}$$

(8)

$$5[\cos\alpha\cos\beta - \sin\alpha\sin\beta]$$
$$= 4[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$$

$\cos\alpha\cos\beta = 9\sin\alpha\sin\beta.$ [2]

$\therefore \tan\alpha\tan\beta = \frac{1}{9}$ —— (1)

If $PR = QR \Rightarrow \alpha+\beta = 45^\circ$

$\therefore \tan(\alpha+\beta) = 1$ [1]

Now, $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = 1$

$\therefore \tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$
$= 1 - \frac{1}{9}$
$= 8/9.$ [2]

$\Rightarrow 9(\tan\alpha + \tan\beta) = 8$ —— (2)

Let $\tan\alpha, \tan\beta$ be the roots to $x^2 - \frac{8}{9}x + \frac{1}{9} = 0$ [1]

$\therefore 9x^2 - 8x + 1 = 0$

$$x = \frac{8 \pm \sqrt{64-36}}{18}$$ [2]

$\therefore \tan\alpha = \frac{8+2\sqrt{7}}{18}$, $\alpha = 36^\circ 27'$

or $\tan\beta = \frac{8-2\sqrt{7}}{18}$ $\beta = 8^\circ 33'$

(b) $P(x) = (x-9)^2 Q(x).$ (4)

(i) $P'(x) = 2Q(x)(x-9) + (x-9)^2 Q'(x)$

$\therefore P'(9) = 0$ [2]

(ii) $P(1) = 0 \Rightarrow 1 + a + b - 4 = 0$

$\therefore a + b = 3$ —— (1) [2].

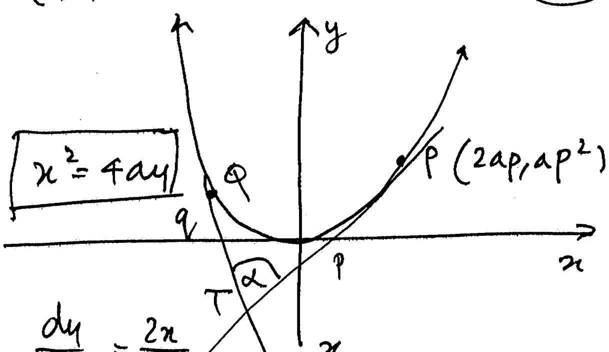
$P'(1) = 0$ $4 + 3a + 2b - 5 = 0$

i.e. $3a + 2b = 5$ —— (2)

(1) $\times [-2]$ $-2a + 2b = -6$ —— (3)

$$\boxed{a = -1} \quad \boxed{b = 4}$$

(c) (8)



$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

$$\left.\frac{dy}{dx}\right|_{x=2ap} = \frac{2ap}{2a} = p$$

(i) $y - ap^2 = p(x - 2ap)$ [1]

$y = px - ap^2$ —— (1)

Equation of QT is

(ii) $y = qx - aq^2$ —— (3)

Solve (1) & (2) [2]

$(p-q)x = a(p+q)(p-q)$

$\therefore \left.\begin{aligned} x &= a(p+q) \\ y &= apq \end{aligned}\right\}$ —— (4)

$\tan\alpha = 1$

$(= \tan\alpha = \frac{p-q}{1+pq}$ [2].

$\therefore p - q = 1 + pq$ —— (5)

$\frac{x}{a} = p+q$

$\frac{y}{a} = pq$ [3]

$(p-q)^2 = (p+q)^2 - 4pq.$

$\therefore (1+pq)^2 = (p+q)^2 - 4pq.$

$\therefore \left(1+\frac{y}{a}\right)^2 = \frac{x^2}{a^2} - \frac{4y}{a}.$

$1 + \frac{2y}{a} + \frac{y^2}{a^2} = \frac{x^2}{a^2} - \frac{4y}{a}.$

$a^2 + 2ay + y^2 = x^2 - 4ay.$ (6)

Locus of T

$$\boxed{x^2 - y^2 - 6ay - a^2 = 0}$$