SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2009

Yearly Examination

YEAR 11

Mathematics (2 unit) & Extension Continuers

Instructions:

- Each Question is to be returned in a separate booklet.
- **Question 1 & 2 are to be collected after 60 minutes** at which time the 2 unit Mathematics students will be dismissed.
- **Question 3 & 4 are to be collected after 105 minutes.**
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise stated.
- Full marks may not be awarded for careless or badly arranged work.

Examiner: *A Fuller*

(Use a SEPARATE writing booklet)

Question 1 (28 marks)

(a) Find the value of $\log_3 9$. 1

(b) Solve the following for x: 6

(i) $\log_6 x = 3$

(ii) $\log_x 3 = -1$

(iii) $2^{2x+1} = \frac{1}{16}$

(iv) $x^2 + 3x - 18 = 0$

(c) Differentiate the following with respect to x: 7

(i) $2x + 5$

(ii) $\dfrac{1}{2x+5}$

(iii) $\dfrac{2x+5}{x}$

(iv) $\dfrac{x}{2x+5}$

(d) Sketch the following on separate axes showing any intercepts with the co-ordinate axes and any asymptotes: 4

(i) $y = \frac{1}{x} + 1$

(ii) $y = \log_2(x + 1)$

(e) Write $2x^2 - 7x - 4$ in the form $a(x + 2)^2 + b(x + 2) + c$. 3

(f) Consider the arithmetic series: $-1 + 3 + 7 + 11 + 15 + ...$ 4

(i) Which term of the series is 391?

(ii) Hence, find the sum up to the term which is 391.

(g) If $f(x) = x^3 - 3x^2 - 6x$. Evaluate: 3

(i) $f(-2)$

(ii) $f'(2)$

(Use a SEPARATE writing booklet)

Question 2 (28 marks)

(a) Consider the geometric series: $27 + 18 + 12 + 8 + ...$ [2]

(i) Explain why the series has a limiting sum.

(ii) Find the limiting sum of the series.

(b) Find the co-ordinates of the focus and the equation of the directrix of the parabola $y = \dfrac{x^2}{4} - 1$. [3]

(c) State whether the following functions are odd, even, or neither: [3]

(i) $f(x) = x^6 + 10$

(ii) $f(x) = \dfrac{x^2}{2 - x}$

(iii) $f(x) = \log_{10} 2^x$

(d) Find using first principles the derivative $f'(x)$ given that $f(x) = x^3$. [2]

(e) Let $\log_5 3 = a$ and $\log_5 2 = b$. [4]

(i) Find the following in terms of a and b:

(α) $\log_5 6$

(β) $\log_5 \left(\frac{1}{4}\right)$

(ii) Evaluate 5^{2a}.

(f) Find the value(s) of k for which $x^2 - kx + 4 = 0$ has: [5]

(i) one root equal to -1

(ii) real roots

(iii) one root double the other.

(g) Find the domain and the range of the following: [4]

(i) $y = \sqrt{1 - x}$

(ii) $y = \sqrt{1 - x^2}$

(h) Caleb plans to deposit an amount of money into an account which will pay him 1% interest each month on the balance of his account at the time. 5

Immediately after each interest payment is made, Caleb plans to withdraw \$1000.

Let his deposit be $\$D$.

(i) Show that when he has made his second withdrawal, the balance of his account will be $\$[D(1.01)^2 - 1000(1 + 1.01)]$.

(ii) Caleb wants his deposit to be sufficient to be able to make withdrawals for 10 years. Find, to the nearest \$100, what his deposit must be.

(Use a SEPARATE writing booklet)

Question 3 (18 marks)

(a) Find the point dividing the interval from $(-3, 4)$ to $(5, -2)$ in the ratio $1 : 3$. [2]

(b) Find the acute angle between the lines $y = 3x - 2$ and $x + 2y - 3 = 0$. [3]

Give the answer to the nearest degree.

(c) (i) Show that $(x + 2)$ is a factor of $6x^3 + 7x^2 - 9x + 2$. [3]

(ii) Hence, or otherwise find all of the factors of $6x^3 + 7x^2 - 9x + 2$.

(d) Find the general solution for $\tan\theta + 1 = 0$ (in radians). [3]

(e) (i) Write $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$. [4]

(ii) Hence, or otherwise, solve $\cos\theta - \sqrt{3}\sin\theta = 1$ for $0 \leq \theta \leq 2\pi$.

(f) Given that $\sin\theta = \frac{1}{\sqrt{3}}$ and $\frac{\pi}{2} < \theta < \pi$. Find the exact value of the following: [3]

(i) $\tan\theta$

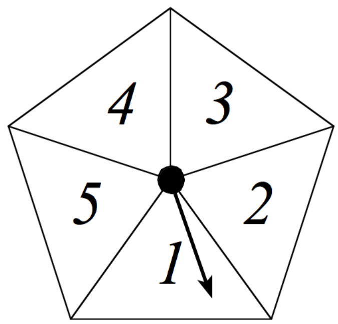
(ii) $\cos 2\theta$

(Use a SEPARATE writing booklet)

Question 4 (17 marks)

(a) Solve $\dfrac{x+4}{x-2} \geq 3.$ 3

(b) 6



(i) The arrow on the regular pentagon is spun twice and the sum of the two numbers is recorded. Find the probability of getting:

(α) an odd result

(β) a result of at least 7

(ii) How many times must the arrow on the regular pentagon be spun to be 99.9% sure of getting at least one 5?

(c) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.The tangents at P and Q intersect at the point T. Let $S(0, a)$ be the focus of the parabola. 8

(i) Show that the equation of the tangent to the parabola at P is given by

$y = px - ap^2$.

(ii) Find the co-ordinates of T.

(iii) Show that $SP = a(p^2 + 1)$.

(iv) Suppose P and Q move on the parabola so that $SP + SQ = 4a$.

Show that the locus of the point T is a parabola.

End of paper

1 (a) $\log_3 9 = 2$ [1]

(b) (i) $\log_6 x = 3$

$x = 6^3$

$x = 216$ [1]

(ii) $\log_x 3 = -1$

$3 = x^{-1}$

$x = \frac{1}{3}$ [1]

(iii) $2^{2x+1} = \frac{1}{16}$

$2^{2x+1} = 2^{-4}$

$2x+1 = -4$

$x = -2\frac{1}{2}$ [2]

(iv) $x^2 + 3x - 18 = 0$

$(x+6)(x-3) = 0$

$x = -6$ or 3 [2]

(c) (i) $\frac{d}{dx}(2x+5) = 2$ [1]

(ii) $\frac{d}{dx}\left(\frac{1}{2x+5}\right) = \frac{d}{dx}\left((2x+5)^{-1}\right)$

$= -1 \times (2x+5)^{-2} \times 2$

$= \frac{-2}{(2x+5)^2}$ [2]

(iii) $\frac{d}{dx}\left(\frac{2x+5}{x}\right) = \frac{d}{dx}\left(2 + \frac{5}{x}\right)$

$= 5 \times -1 x^{-2}$

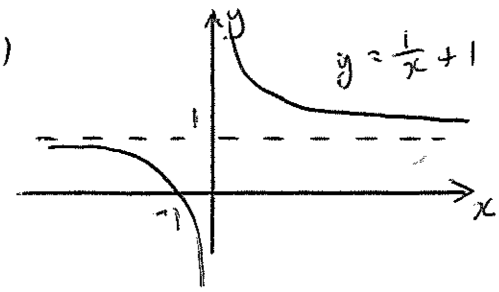
$= \frac{-5}{x^2}$ [2]

(iv) $\frac{d}{dx}\left(\frac{x}{2x+5}\right) = \frac{(2x+5).1 - x.2}{(2x+5)^2}$

$= \frac{2x+5-2x}{(2x+5)^2}$

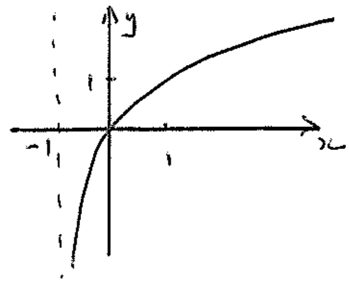
$= \frac{5}{(2x+5)^2}$ [2]

(d) (i)



[2]

(ii)



[2]

(e) $2x^2 - 7x - 4 \equiv a(x+2)^2 + b(x+2) + c$

Equating leading coefficients $a = 2$

Let $x = -2$: $8 + 14 - 4 = c$

$\therefore c = 18$

Let $x = 0$: $-4 = 2 \times 4 + b \times 2 + 18$

$-4 = 2b + 26$

$2b = -30$

$b = -15$ [3]

$2x^2 - 7x - 4 = 2(x+2)^2 - 15(x+2) + 18$

(f) $a = -1$

$d = 4$

(i) $T_n = -1 + (n-1).4 = 391$

$4(n-1) = 392$

$n - 1 = 98$

$n = 99$ [2]

(ii) $S_{99} = \frac{99}{2}(-1 + 391)$

$= \frac{99}{2} \times 390$

$= 19305$ [2]

(g) (i) $f(-2) = -8 - 12 + 12$

$= -8$ [1]

(ii) $f'(x) = 3x^2 - 6x - 6$

$f'(2) = 12 - 12 - 6$

$= -6$ [2]

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Question 2.

a) if $|r| < 1$ the sum of the series is limited to some finite number (1)

(1)

ii) $a = 27 \quad r = 2/3$

$S_\infty = \dfrac{27}{1 - 2/3}$

$= 81$ (1)

b) $y = \dfrac{x^2}{4} - 1$

$4y = x^2 - 4$

$x^2 = 4y + 4$

$x^2 = 4(y+1)$

$\therefore h = 0, \ k = -1, \ a = 1$ (1)

focus: $(0, 0)$ (1)

directrix: $y = -1 - 1$

$y = -2$ (1)

c) i) $f(x) = x^6 + 10$

$f(-x) = (-x)^6 + 10$

$= x^6 + 10$

$\therefore f(x)$ is even (1)

ii) $f(x) = \dfrac{x^2}{2 - x}$

$f(-x) = \dfrac{(-x)^2}{2 - (-x)}$

$= \dfrac{x^2}{2 + x}$

$\therefore f(x)$ is neither (1)

iii) $f(x) = \log_{10} 2^x = x \log_{10} 2$

$f(-x) = \log_{10} 2^{-x}$

$= -x \log_{10} 2$

$= -f(x)$

$\therefore f(x)$ is odd. (1)

d) $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

$= \dfrac{(x+h)^3 - x^3}{h}$

$(x+h)^3 = (x^2 + 2xh + h^2)(x + h)$

$= x^3 + x^2 h + 2x^2 h + 2xh^2 + h^2 x + h^3$

$= x^3 + 3x^2 h + 3xh^2 + h^3$

$= \dfrac{x^3 + 3x^2 h + 3xh^2 + h^3 - x^3}{h}$

$= \dfrac{3x^2 h + 3xh^2 + h^3}{h}$

$= \dfrac{h(3x^2 + 3xh + h^2)}{h}$

$\lim_{h \to 0} = 3x^2$ (2)

$\therefore f'(x) = 3x^2$

e) $\log_5 3 = a$
$\log_5 2 = b$

i) α) $\log_5 6 = \log_5 3 + \log_5 2$

$= a + b$ (1)

β) $\log_5 (1/4) = \log_5 1 - \log_5 4$
$= 0 - \log_5 2^2$
$= 0 - 2\log_5 2$
$= -2b$ (1)

ii) $5^{2a} = (5^a)^2$
$= 3^2$
$= 9$ (2)

f) $x^2 - kx + 4 = 0$

i) $(-1)^2 + k + 4 = 0$
$k = -5$ (1)

ii) $\Delta > 0$
$\therefore \sqrt{b^2 - 4ac} \geq 0$
$b^2 - 4ac \geq 0$
$(-k)^2 - 4 \times 1 \times 4 \geq 0$
$k^2 - 16 \geq 0$
$k^2 \geq 16$
$k \geq 4,\ k \leq -4$ (2)

iii) $\alpha + \beta = -b/a$
$\alpha + 2\alpha = k$
$3\alpha = k$

$\alpha \times 2\alpha = 4$
$2\alpha^2 = 4$
$\alpha^2 = 2$
$\alpha = \pm\sqrt{2}$

$\therefore k = 3\sqrt{2},\ -3\sqrt{2}$ (2)

g) i) D: $x \leq 1$ (1)
R: $y \geq 0$ (1)

ii) D: $-1 \leq x \leq 1$ (1)
R: $0 \leq y \leq 1$ (1)

h) i) After 1 month
$A_1 = (\$D \times 1.01) - 1000$

After 2nd month
$A_2 = (A_1 \times 1.01) - 1000$
$= [(\$D \times 1.01) - 1000] \times 1.01 - 1000$
$= \$[D \times (1.01)^2 - 1000 \times 1.01 - 1000]$

$= \$[D(1.01)^2 - 1000(1.01 + 1)]$

$= \$[D(1.01)^2 - 1000(1 + 1.01)]$ (2)

$D(1.01)^{120} - 1000\,[1.01^{119} + 1.01^{118} + 1.01^{117} + \ldots + 1.01 + 1] \geq 0$

this part is a series
$a = 1,\ r = 1.01 \quad n = 120$

$S_n = \dfrac{1(1.01^{120} - 1)}{1.01 - 1}$

$D(1.01)^{120} - 1000 \times \dfrac{1(1.01^{120} - 1)}{1.01 - 1} \geq 0$

$D(1.01)^{120} \geq 1000 \times \dfrac{1(1.01^{120} - 1)}{1.01 - 1}$

$D \geq \dfrac{1000 \times \dfrac{(1.01^{120} - 1)}{1.01 - 1}}{(1.01)^{120}}$

$D \geq \$69,700$. (nearest \$100)

Extension 1 Year 11 Yearly 2009. QUESTION 3.

a) $(-3, 4)$. $(5, -2)$. $1:3$.
$x_1 \; y_1 \quad x_2 \; y_2 \quad m:n$.

Co-ords of point dividing

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

$$= \left(\frac{5 + 3(-3)}{4}, \frac{-2 + 3(4)}{4} \right)$$

$$= (-1, 2.5)$$

b)
$y = 3x - 2 \qquad m_1 = 3$.
$x + 2y - 3 \qquad m_2 = -\frac{1}{2}$.

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{3 - (-\frac{1}{2})}{1 + (3)(-\frac{1}{2})} \right|$$

$$\tan\theta = |-7| = 7.$$

$\theta = 81.87$
$\theta = 82^\circ$ (nearest degree)

c)
$f(-2) = 6(-2)^3 + 7(-2)^2 - 9(-2) + 2$
$= 0$
$\therefore (x+2)$ is a factor

ii) Divide to find remaining factors.

$$\begin{array}{r} 6x^2 - 5x + 1 \\ x+2 \,\big|\, 6x^3 + 7x^2 - 9x + 2 \\ 6x^3 + 12x^2 \\ \hline -5x^2 - 9x \\ -5x^2 - 10x \\ \hline x + 2 \end{array}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 - 4(6)(1)}}{12}$$

$$= \frac{1}{3}, \frac{1}{2}$$

- but we need $6x^2$ so.
Factors are

$6x^3 + 7x^2 - 9x + 2$
$= (x+2)(2x-1)(3x-1)$

d.) $\tan\theta = -1$.
$\tan\theta = \tan(-\frac{\pi}{4})$

$\therefore \theta = -\frac{\pi}{4} + n\pi \qquad n \in \mathbb{Z}$

e) i) $\cos\theta - \sqrt{3}\sin\theta = R\cos(\theta + \alpha)$

$R = \sqrt{1^2 + \sqrt{3}^2} \qquad \tan\alpha = \frac{\sqrt{3}}{1}$
$= 2$

$R > 0 \qquad 0 \le \alpha \le \frac{\pi}{2} \qquad \alpha = \frac{\pi}{3}$.

$\therefore \cos\theta - \sqrt{3}\sin\theta = 2\cos(\theta + \frac{\pi}{3})$

ii) $\cos\theta - \sqrt{3}\sin\theta = 1$.

$2\cos(\theta + \pi/3) = 1$.

$\cos(\theta + \pi/3) = 1/2$

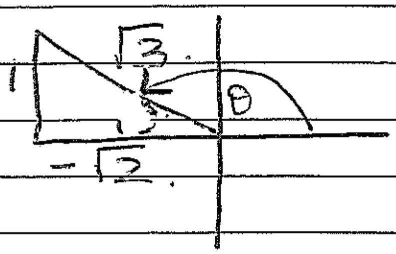
as $0 \le \theta \le 2\pi$.

$\pi/3 \le \theta + \pi/3 \le 2\pi + \pi/3$.

$\theta + \pi/3 = \pi/3, \frac{5\pi}{3}, \frac{7\pi}{3}$.

$\underline{\theta = 0, \frac{4\pi}{3}, 2\pi}$ #

f) $\sin\theta = \frac{1}{\sqrt{3}}, \frac{\pi}{2} < \theta < \pi$



i) $\tan\theta = -\frac{1}{\sqrt{2}}$.

ii) $\cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2(1/\sqrt{3})^2$

$= +1/3$ #

QUESTION FOUR

a) $\dfrac{x+4}{x-2} \geq 3 \qquad x \neq 2$

$\dfrac{x+4}{x-2} \cdot (x-2)^2 \geq 3(x-2)^2$

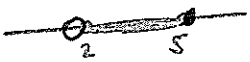
$(x+4)(x-2) \geq 3(x-2)^2$

$x^2 + 2x - 8 \geq 3x^2 - 12x + 12$

$2x^2 - 14x + 20 \leq 0$

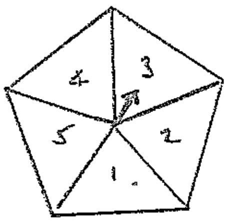
$x^2 - 7x + 10 \leq 0$

$(x-5)(x-2) \leq 0$



$2 < x \leq 5$

b)



Events are equiprobable

~~P(odd) relies only on~~

(1,1) 2	(1,2) 3	(1,3) 4	(1,4) 5	(1,5) 6
(2,1) 3	(2,2) 4	(2,3) 5	(2,4) 6	(2,5) 7
(3,1) 4	(3,2) 5	(3,3) 6	(3,4) 7	(3,5) 8
(4,1) 5	(4,2) 6	(4,3) 7	(4,4) 8	(4,5) 9
(5,1) 6	(5,2) 7	(5,3) 8	(5,4) 9	(5,5) 10

(i) α $P(\text{Odd Sum}) = \frac{12}{25}$, β $P(\text{at least } 7) = \frac{10}{25} = \frac{2}{5}$

ii) $P(\text{No 5 with 1 spin}) = 4/5$

$P(\text{No 5 with 2 spins}) = (4/5)^2$

$\vdots$

$P(\text{No 5 with } n \text{ spins}) = (4/5)^n$

$P(\text{at least 1 Five}) = 1 - P(\text{no FIVES})$

So $1 - \left(\frac{4}{5}\right)^n \geq 99.9\%$

$1 - \left(\frac{4}{5}\right)^n \geq .999$

$-(.8)^n \geq -.001$

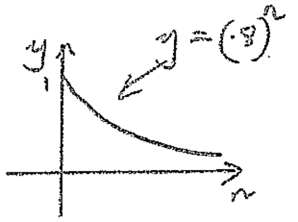
$(.8)^n \leq .001$

$n \ln(.8) \leq \ln .001$

$n \geq \dfrac{\ln .001}{\ln .8}$

$n \geq 30.9$

$n = 31$



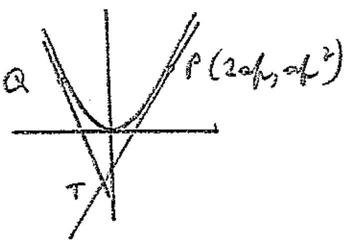
NOTE
ln means $\log_e$
$\log_{10}$ could also be used.

CHECK $n = 30 \Rightarrow .8^{30} = .00123\ldots$

$n = 31 \Rightarrow .8^{31} = .00099\ldots$

$\boxed{n = 31}$

c)



i) $x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$y' = \dfrac{2x}{4a}$

$= \dfrac{x}{2a}$

$= \dfrac{2ap}{2a}$

$= p$ (when $x = 2ap$)

OR
$x = 2ap$
$y = ap^2$

$\dfrac{dy}{dx} = \dfrac{dy}{dp} \times \dfrac{dp}{dx}$

$= 2ap \times \dfrac{1}{2a}$

$= p$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ — ①

Tangent at Q is

$y = qx - aq^2$ — ②

ii) Subtracting ① − ②

$0 = x(p-q) - a(p^2 - q^2)$

$0 = x(p-q) - a(p-q)(p+q)$

$0 = x - a(p+q) \qquad p \neq q$

$x = a(p+q)$

THEN

$$y = px - ap^2$$
$$= pa(p+q) - ap^2$$
$$= ap^2 + apq - ap^2$$
$$= apq$$

So $x = a(p+q)$
$y = apq$

iii) NOW

$$SP = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$$
$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$$
$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$
$$= \sqrt{(ap^2+a)^2}$$
$$= ap^2 + a$$
$$= a(p^2+1)$$

iv) $SQ = a(q^2+1)$

AND

$$SP + SQ = a(p^2+1) + a(q^2+1)$$
$$a(p^2+1) + a(q^2+1) = 4a$$
$$\Rightarrow p^2 + q^2 = 2$$

NOW

$$\frac{x}{a} = p + q \quad \& \quad \frac{y}{a} = pq$$

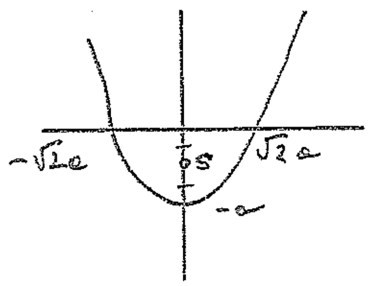
$$\frac{x^2}{a^2} = p^2 + 2pq + q^2$$
$$= (p^2+q^2) + 2pq$$

$$\frac{x^2}{a^2} = 2 + \frac{2y}{a}$$

$$\frac{x^2}{a^2} = 2 + \frac{2y}{a}$$

$$x^2 = 2ay + 2a^2$$
$$x^2 = 2a(y+a)$$
$$x^2 = 4 \cdot \frac{a}{2}(y+a)$$
$$X^2 = 4AY$$

Parabola
Vertex $(0, -a)$
Focal length $\frac{a}{2}$



Focus is point S