SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

SEPTEMBER 2011
YEARLY
YEAR 11 Continuers

Mathematics Ext 1

General Instructions:

- Reading time—5 minutes.
- Working time—90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—68 Marks

- Attempt questions 1–4.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 2 sections:
  Section A (Questions 1 and 2),
  Section B (Questions 3 and 4).

Examiner: Mr P. Bigelow

Section A

Marks

Question 1 (17 marks)

(a) Differentiate the following: **5**

(i) $y = \dfrac{4}{3\sqrt{x}}$,

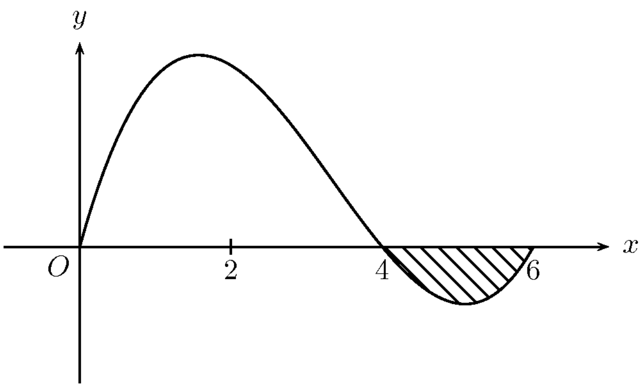
(ii) $y = \dfrac{x+1}{x-2}$,

(iii) $y = (2 - 3x^2)^5$.

(b) Find the equation of the normal to $y = \dfrac{3}{x}$ at the point where $x = -3$. **2**

(c) Differentiate $y = x^2 + bx + c$, and hence find b and c if the parabola has a y-intercept of -3 and its gradient at that point is -2. **3**

(d) **2**



You are given that $\displaystyle\int_0^4 f(x)\,dx = 22$ and $\displaystyle\int_0^6 f(x)\,dx = 15$.

(i) What is the area of the shaded region?

(ii) What is the value of $\displaystyle\int_4^6 f(x)\,dx$?

(e) Find $\displaystyle\int_1^2 \left(2x - 3x^2\right) dx$. **2**

(f) θ and ϕ are acute angles and $\cos\phi = \frac{1}{3}$ and $\sin\theta = \frac{2}{3}$, find the exact value of $\cos(\theta - \phi)$. **3**

Marks

Question 2 (17 marks)

(a) Consider the function $f(x) = 2 + 3x - x^3$. **7**

(i) Find $f'(x)$.

(ii) Find $f''(x)$.

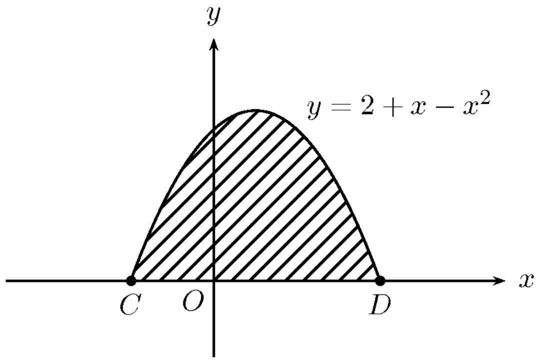
(iii) Locate any stationary points and determine their nature.

(iv) For what values of x is the curve increasing?

(v) For what values of x is the curve concave up?

(vi) Sketch the curve.

(b) **4**



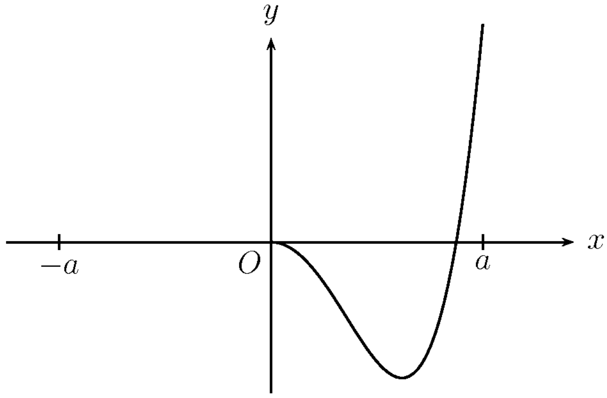
(i) Find the co-ordinates of C and D.

(ii) Find the ratio in which the shaded area is divided by the y-axis.

(c) (i) Expand $\sin(A + B)$. **3**

(ii) Hence show that $\sin 75° = \dfrac{\sqrt{6} + \sqrt{2}}{4}$.

(d) **3**



The diagram shows the graph of the function for $0 \leqslant x \leqslant a$.
It is known that $f(x)$ is an odd function and stationary at $(0, 0)$.

(i) Copy the diagram and continue the graph for $-a \leqslant x < 0$.

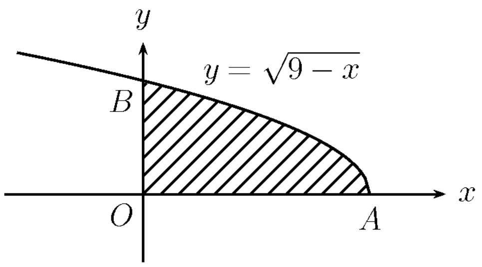
(ii) On another diagram, sketch $y = f'(x)$.

Section B

(Use a separate writing booklet.)

Marks

Question 3 (20 marks)

(a) 

The shaded area is bounded by the curve $y = \sqrt{9-x}$ and the co-ordinate axes. 7

(i) Write down the values of A and B.

(ii) Find the area AOB.

(iii) Find the volume when the shaded area is rotated about

(α) the x-axis,

(β) the y-axis.

(b) Solve the following inequalities and graph the solutions on separate number lines. 4

(i) $\dfrac{2}{|x-1|} > 1$,

(ii) $\dfrac{3}{x-2} \leqslant \dfrac{2}{x+1}$.

(c) Given $\triangle ABC$ with vertices $A(3, 2)$, $B(5, 6)$ and $C(7, 4)$: 5

(i) Find the co-ordinates of E and F, the midpoints of AB and BC respectively.

(ii) Show that EF is

(α) half of

(β) and parallel to AC.

(d) Prove the identity 2

$$\frac{\sin 2A}{1+\cos 2A} = \tan A.$$

(e) Solve $4\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 1$ for $0 \leqslant \theta \leqslant 2\pi$. 2

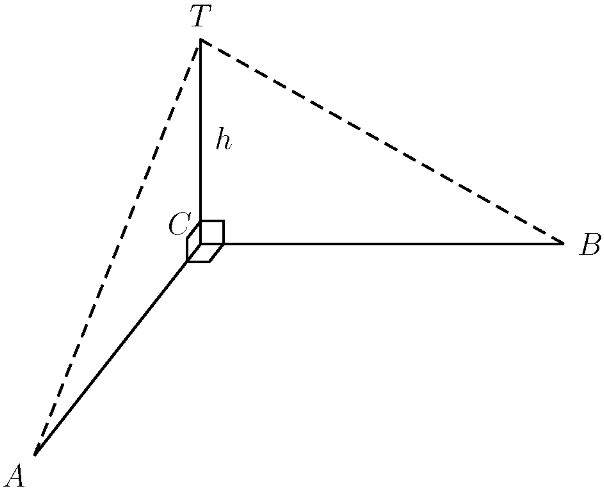
[Hint: use a double angle result.]

Marks

Question 4 (14 marks)

(a) The graph of $y = f(x)$ passes through (2, 1) and $f'(x) = \dfrac{x^3 - 4}{x^2}$. Find $f(x)$. 2

(b) 4



CT is a tower of height h. From A, due south of the base of the tower, the angle of elevation of T, the top of the tower, is 25°. From B, due east, the angle of elevation of T is 30°. Given that AB is 200 m, find h to the nearest metre.

(c) Let $f(x) = \dfrac{2x}{(1 + x^2)^2}$. 8

(i) Copy this table and complete it in decimal form.

x	1	2	3
f(x)			

(ii) Use the table to find values for $\int_1^3 f(x)\,dx$ using

(α) the Trapezoidal rule,

(β) Simpson's rule.

(iii) Find $\dfrac{d}{dx}\left(\dfrac{1}{x^2+1}\right)$.

(iv) Using (iii), find the exact value of the integral in (ii).

End of Paper

STANDARD INTEGRALS

$$
\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \\
\int \frac{1}{x} dx &= \ln x, \quad x > 0 \\
\int e^{ax} dx &= \frac{1}{a}e^{ax}, \quad a \neq 0 \\
\int \cos ax dx &= \frac{1}{a}\sin ax, \quad a \neq 0 \\
\int \sin ax dx &= -\frac{1}{a}\cos ax, \quad a \neq 0 \\
\int \sec^2 ax dx &= \frac{1}{a}\tan ax, \quad a \neq 0 \\
\int \sec ax \tan ax dx &= \frac{1}{a}\sec ax, \quad a \neq 0 \\
\int \frac{1}{a^2+x^2} dx &= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0 \\
\int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a \\
\int \frac{1}{\sqrt{x^2-a^2}} dx &= \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0 \\
\int \frac{1}{\sqrt{x^2+a^2}} dx &= \ln\left(x + \sqrt{x^2+a^2}\right)
\end{aligned}
$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Mathematics Ext 1
Solutions

Section A

Marks

Question 1 (17 marks)

(a) Differentiate the following: [5]

(i) $y = \dfrac{4}{3\sqrt{x}}$,

> **Solution:**
> $$\begin{aligned} y &= \frac{4x^{-\frac{1}{2}}}{3}. \\ \frac{dy}{dx} &= -\frac{1}{2}.\frac{4x^{-\frac{3}{2}}}{3}, \\ &= -\frac{2}{3x\sqrt{x}}. \end{aligned}$$

(ii) $y = \dfrac{x+1}{x-2}$,

> **Solution:**
> $$\begin{aligned} \frac{dy}{dx} &= \frac{(x-2)\times 1 - (x+1)\times 1}{(x-2)^2}, \\ &= \frac{-3}{(x-2)^2}. \end{aligned}$$

(iii) $y = (2-3x^2)^5$.

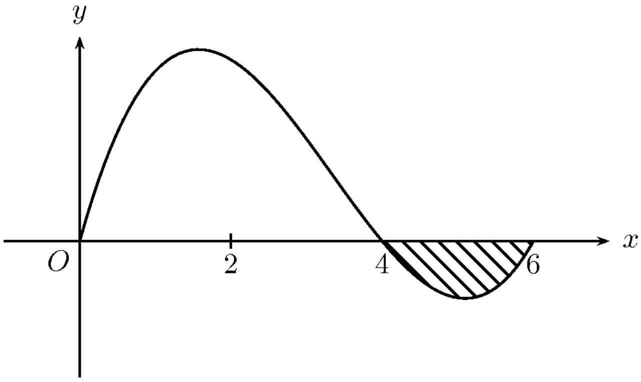
> **Solution:**
> $$\begin{aligned} \frac{dy}{dx} &= 5\times(-3\times 2x)\times(2-3x^2)^4, \\ &= -30x(2-3x^2)^4. \end{aligned}$$

(b) Find the equation of the normal to $y = \dfrac{3}{x}$ at the point where $x = -3$. [2]

> **Solution:**
> $$\begin{aligned} \frac{dy}{dx} &= -\frac{3}{x^2}, \\ &= -\tfrac{1}{3} \text{ at the point } (-3,\ -1). \end{aligned}$$
> Normal equation:
> $$\begin{aligned} y+1 &= 3(x+3), \\ \textit{i.e., } 3x - y + 8 &= 0. \end{aligned}$$

(c) Differentiate $y = x^2 + bx + c$, and hence find b and c if the parabola has a y-intercept of -3 and its gradient at that point is -2. [3]

> **Solution:** $y' = 2x + b$.
> Now, at $(0,\ -3)$:
> $$\begin{aligned} -2 &= 2\times 0 + b, \\ \textit{i.e., } b &= -2, \\ \text{and } -3 &= 0^2 - 2\times 0 + c, \\ \textit{i.e., } c &= -3. \end{aligned}$$

(d) 

[2]

You are given that $\int_0^4 f(x)\,dx = 22$ and $\int_0^6 f(x)\,dx = 15$.

(i) What is the area of the shaded region?

> **Solution:** $$\begin{aligned}\text{Shaded area} &= |15 - 22|,\\ &= 7.\end{aligned}$$

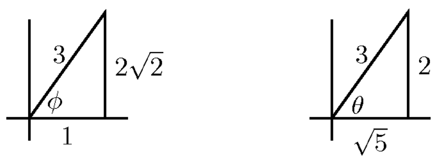
(ii) What is the value of $\int_4^6 f(x)\,dx$?

> **Solution:** $$\begin{aligned}\int_4^6 f(x)\,dx &= \int_0^6 f(x)\,dx - \int_0^4 f(x)\,dx,\\ &= -7.\end{aligned}$$

(e) Find $\int_1^2 \left(2x - 3x^2\right) dx$. **[2]**

> **Solution:** $$\begin{aligned}\int_1^2 \left(2x - 3x^2\right) dx &= \Big[x^2 - x^3\Big]_1^2,\\ &= 4 - 8 - (1 - 1),\\ &= -4.\end{aligned}$$

(f) θ and ϕ are acute angles and $\cos\phi = \frac{1}{3}$ and $\sin\theta = \frac{2}{3}$, find the exact value of $\cos(\theta - \phi)$. **[3]**

> **Solution:** 

>
> $$\begin{aligned}\cos(\theta - \phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi,\\ &= \frac{\sqrt{5}}{3}.\frac{1}{3} + \frac{2}{3}.\frac{2\sqrt{2}}{3},\\ &= \frac{\sqrt{5} + 4\sqrt{2}}{9}.\end{aligned}$$

Marks

Question 2 (17 marks)

(a) Consider the function $f(x) = 2 + 3x - x^3$. **7**

(i) Find $f'(x)$.

Solution: $f'(x) = 3 - 3x^2$.

(ii) Find $f''(x)$.

Solution: $f''(x) = -6x$.

(iii) Locate any stationary points and determine their nature.

Solution: When $f'(x) = 0$, $\quad 3(1 - x^2) = 0$, $\quad x = \pm 1$, $\quad y = 4, 0$. and when $f''(x) = 0$, $\quad x = 0$, $y = 2$, $\quad f''(1) = -6$, $\quad f''(-1) = 6$.

$\therefore$ A maximum at (1, 4) and a minimum at (−1, 0).

(iv) For what values of x is the curve increasing?

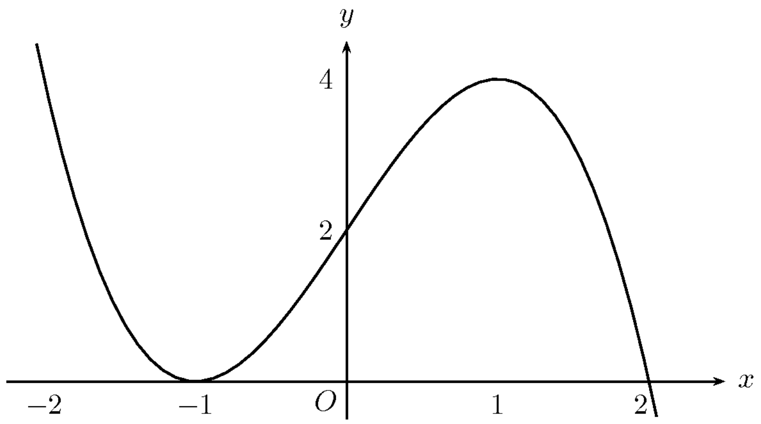
Solution: $-1 < x < 1$.

(v) For what values of x is the curve concave up?

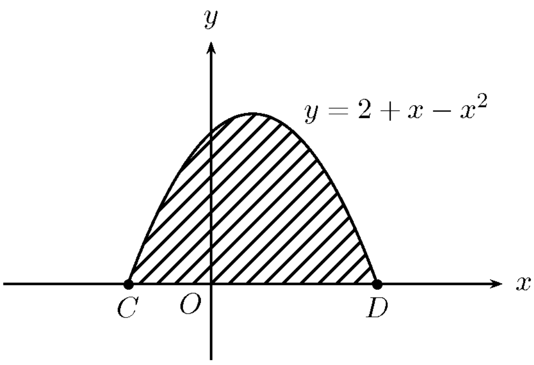
Solution: $x < 0$.

(vi) Sketch the curve.

Solution:



(b) $\boxed{4}$



(i) Find the co-ordinates of C and D.

Solution:
$$\begin{aligned} x^2 - x - 2 &= 0, \\ (x-2)(x+1) &= 0, \\ \therefore x &= -1,\, 2. \end{aligned}$$
i.e., $C(-1,\,0)$ and $D(2,\,0)$.

(ii) Find the ratio in which the shaded area is divided by the y-axis.

Solution:
$$\begin{aligned} \int_{-1}^{0} \left(2 + x - x^2\right) dx &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0}, \\ &= 0 - \left\{-2 + \tfrac{1}{2} - \tfrac{1}{3}\right\}, \\ &= 1\tfrac{5}{6}. \\ \int_{0}^{2} \left(2 + x - x^2\right) dx &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{2}, \\ &= \left\{4 + \frac{4}{2} - \frac{8}{3}\right\} - 0, \\ &= 5. \end{aligned}$$
i.e., the ratio is $11 : 30$.

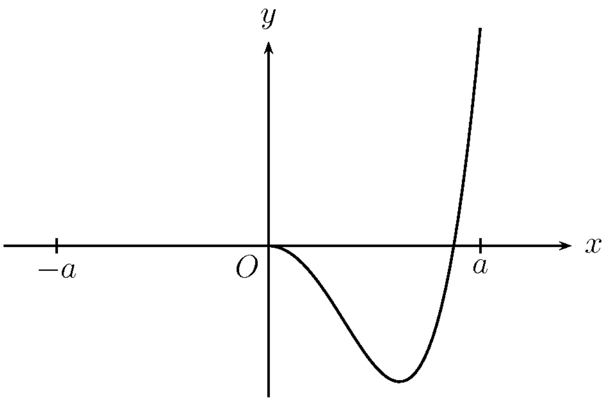
(c) (i) Expand $\sin(A+B)$. $\boxed{3}$

Solution: $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(ii) Hence show that $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Solution:
$$\begin{aligned} \sin(45^\circ + 30^\circ) &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ, \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}, \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

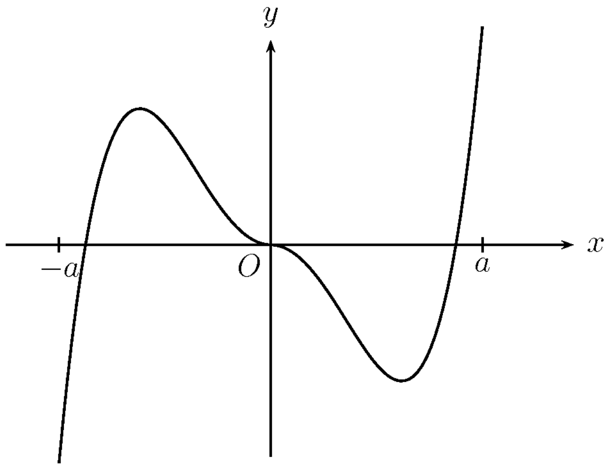
(d) $\boxed{3}$



The diagram shows the graph of the function for $0 \leqslant x \leqslant a$.
It is known that $f(x)$ is an odd function and stationary at (0, 0).

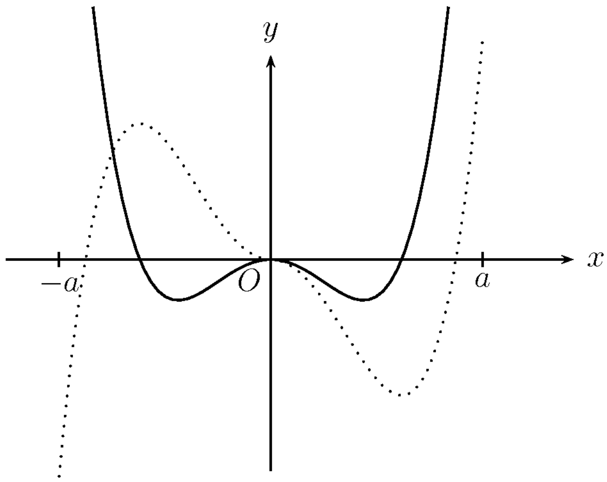
(i) Copy the diagram and continue the graph for $-a \leqslant x < 0$.

Solution:



(ii) On another diagram, sketch $y = f'(x)$.

Solution:

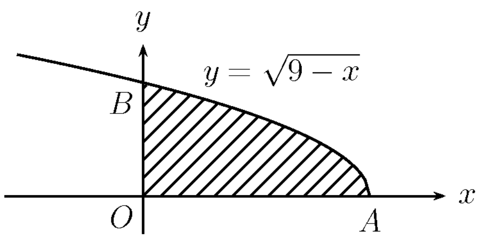


Section B

(Use a separate writing booklet.)

Marks

Question 3 (20 marks)

(a) 

The shaded area is bounded by the curve $y = \sqrt{9-x}$ and the co-ordinate axes. **7**

(i) Write down the values of A and B.

> **Solution:** $A(9,\,0)$, $B(0,\,3)$

(ii) Find the area AOB.

> **Solution:**
> $$\begin{aligned}\int_0^9 (9-x)^{\frac{1}{2}}\,dx &= \left[-1\times\tfrac{2}{3}\times(9-x)^{\frac{3}{2}}\right]_0^9,\\ &= -\tfrac{2}{3}\{0-27\},\\ &= 18.\end{aligned}$$

(iii) Find the volume when the shaded area is rotated about

(α) the x-axis,

> **Solution:** Note that $y^2 = 9-x$.
> $$\begin{aligned}\text{Volume} &= \pi\int_0^9 (9-x)\,dx,\\ &= \pi\left[9x-\frac{x^2}{2}\right]_0^9,\\ &= \pi\{81-\tfrac{81}{2}-0\},\\ &= \frac{81\pi}{2}.\end{aligned}$$

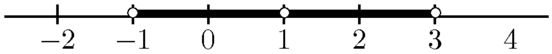
(β) the y-axis.

> **Solution:**
> $$\begin{aligned}\text{Note: } y^2 &= 9-x,\\ x &= 9-y^2,\\ x^2 &= 81-18y^2+y^4.\\ \text{So Volume} &= \pi\int_0^3 (81-18y^2+y^4)\,dy,\\ &= \pi\left[81y-6y^3+\frac{y^5}{5}\right]_0^3,\\ &= \pi\left\{243-162+\tfrac{243}{5}\right\},\\ &= \frac{648\pi}{5}.\end{aligned}$$

(b) Solve the following inequalities and graph the solutions on separate number lines. 4

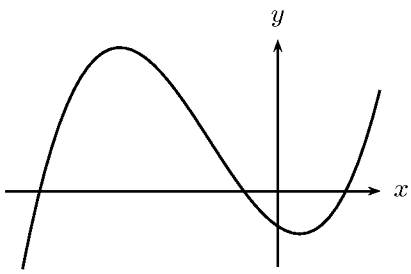
(i) $\dfrac{2}{|x-1|} > 1$,

Solution:
$$\begin{aligned} 2 &> |x-1|, \\ -2 &< x-1 < 2, \\ -1 &< x < 3,\ x \neq 1. \end{aligned}$$



(ii) $\dfrac{3}{x-2} \leqslant \dfrac{2}{x+1}$.

Solution:
$$\begin{aligned} \frac{3}{x-2} - \frac{2}{x+1} &\leqslant 0, \\ \frac{3(x+1)-2(x-2)}{(x-2)(x+1)} &\leqslant 0, \\ \frac{x+7}{(x-2)(x+1)} &\leqslant 0, \\ (x+7)(x-2)(x+1) &\leqslant 0. \end{aligned}$$



We note that $x \neq -1$ and $x \neq 2$,
$\therefore x \leqslant -7, \quad -1 < x < 2$.

(c) Given $\triangle ABC$ with vertices $A(3,\,2)$, $B(5,\,6)$ and $C(7,\,4)$: 5

(i) Find the co-ordinates of E and F, the midpoints of AB and BC respectively.

Solution: E is $\left(\dfrac{3+5}{2}, \dfrac{2+6}{2}\right) = (4,\,4)$,
F is $\left(\dfrac{5+7}{2}, \dfrac{6+4}{2}\right) = (6,\,5)$.

(ii) Show that EF is

(α) half of

Solution:
$$\begin{aligned} AC^2 &= (3-5)^2 + (2-6)^2, \\ &= 20, \\ AC &= 2\sqrt{5}. \\ EF^2 &= (4-6)^2 + (4-5)^2, \\ &= 5, \\ EF &= \sqrt{5} = \tfrac{1}{2}AC. \end{aligned}$$

(β) and parallel to AC.

Solution: Slope, $m_{AC} = \dfrac{2-4}{3-7} = \dfrac{1}{2}$.
$m_{EF} = \dfrac{4-5}{4-6} = \dfrac{1}{2}$. *i.e.*, $EF \parallel AC$.

(d) Prove the identity **2**

$$\frac{\sin 2A}{1+\cos 2A} = \tan A.$$

Solution:
$$\begin{aligned}\text{L.H.S.} &= \frac{2\sin A\cos A}{1+2\cos^2 A-1},\\ &= \frac{\cancel{2}\sin A\cancel{\cos A}}{\cancel{2}\cos^{\cancel{2}} A},\\ &= \frac{\sin A}{\cos A},\\ &= \tan A,\\ &= \text{R.H.S.}\end{aligned}$$

(e) Solve $4\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 1$ for $0 \leqslant \theta \leqslant 2\pi$. **2**
[Hint: use a double angle result.]

Solution:
$$\begin{aligned}4\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} &= 2\sin\theta.\\ \text{So }\ \sin\theta &= \tfrac{1}{2},\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}.\end{aligned}$$

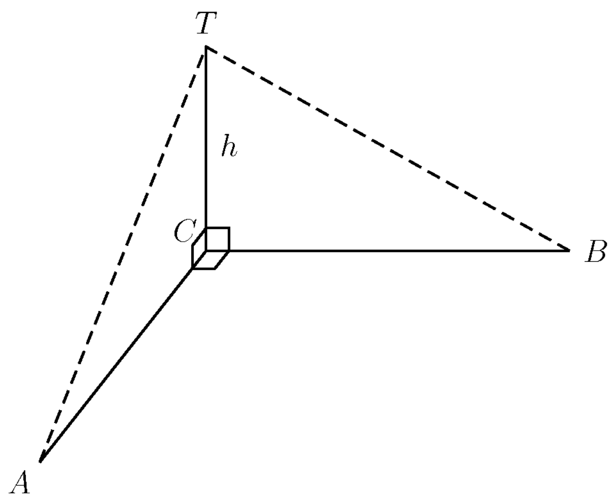
Marks

Question 4 (14 marks)

(a) The graph of $y = f(x)$ passes through $(2,\ 1)$ and $f'(x) = \dfrac{x^3 - 4}{x^2}$. Find $f(x)$. 2

Solution:
$$\begin{aligned}
\int \frac{x^3-4}{x^2}\,dx &= \int \left(x - 4x^{-2}\right) dx,\\
&= \frac{x^2}{2} - \frac{4x^{-1}}{-1} + c,\\
&= \frac{x^2}{2} + \frac{4}{x} + c.
\end{aligned}$$
So at $(2,\ 1)$ we have
$$\begin{aligned}
1 &= \frac{2^2}{2} + \frac{4}{2} + c,\\
\textit{i.e.},\ c &= -3.\\
\text{Hence } f(x) &= \frac{x^2}{2} + \frac{4}{x} - 3.
\end{aligned}$$

(b) 4



CT is a tower of height h. From A, due south of the base of the tower, the angle of elevation of T, the top of the tower, is $25°$. From B, due east, the angle of elevation of T is $30°$. Given that AB is 200 m, find h to the nearest metre.

Solution:
$$\begin{aligned}
\frac{h}{AC} &= \tan 25° \implies AC = h\cot 25°,\\
\frac{h}{BC} &= \tan 30° \implies BC = h\cot 30°.\\
AC^2 + BC^2 &= 200^2,\\
h^2\cot^2 25° + h^2\cot^2 30° &= 40\,000,\\
h^2 &= \frac{40\,000}{\cot^2 25° + \cot^2 30°},\\
&\approx 5263.912898.\\
\therefore h &\approx 73\,\text{m (nearest metre)}.
\end{aligned}$$

(c) Let $f(x) = \dfrac{2x}{(1+x^2)^2}$. **8**

(i) Copy this table and complete it in decimal form.

x	1	2	3
$f(x)$			

Solution:

x	1	2	3
$f(x)$	0.5	0.16	0.06

(ii) Use the table to find values for $\displaystyle\int_1^3 f(x)\,dx$ using

(α) the Trapezoidal rule,

Solution: $$\int_1^3 f(x)\,dx \approx \tfrac{1}{2}\{0.5 + 2\times 0.16 + 0.06\},$$
$$\approx 0.44.$$

(β) Simpson's rule.

Solution: $$\int_1^3 f(x)\,dx \approx \tfrac{1}{3}\{0.5 + 4\times 0.16 + 0.06\},$$
$$\approx 0.4.$$

(iii) Find $\dfrac{d}{dx}\left(\dfrac{1}{x^2+1}\right)$.

Solution: $$\frac{d}{dx}\big((x^2+1)^{-1}\big) = -1\times 2x(x^2+1)^{-2},$$
$$= \frac{-2x}{(1+x^2)^2}.$$

(iv) Using (iii), find the exact value of the integral in (ii).

Solution: $$\int_1^3 f(x)\,dx = -\int_1^3 \frac{-2x}{(1+x^2)^2}\,dx,$$
$$= -\left[\frac{1}{x^2+1}\right]_1^3,$$
$$= -\left\{\frac{1}{10} - \frac{1}{2}\right\},$$
$$= 0.4.$$

End of Paper