SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013

Year 11 Mathematics
Yearly

Mathematics Extension Continuers

General Instruction

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 67

SECTION I – **7 Marks**

- Attempt Questions 1 – 7

SECTION II – **60 Marks**

- Attempt Questions 8 – 12

Examiner: *J. Chen*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SECTION I – 7 Marks
Attempt Questions 1 – 7

Answer this section on the Multiple Choice Answer Sheet

1)
$$\int \left(x^{-1/2} + 4\right).dx =$$

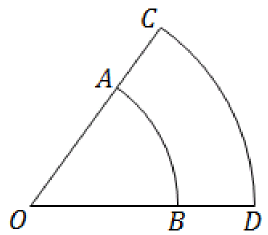
(A) $-\frac{3}{2}x^{-3/2} + 4x + C$

(B) $\frac{1}{2}\sqrt{x} + 4x + C$

(C) $2\sqrt{x} + 4x + C$

(D) $\frac{2}{3}x^{-3/2} + 4x + C$

2) In the figure, OAB and OCD are sectors with centre O. If the area of sector $OAB = 6\pi\ cm^2$, the area of $ABCD = \frac{14}{3}\pi\ cm^2$ and $OA = 6\ cm$, then $AC =$



(A) $2\ cm$

(B) $4\ cm$

(C) $6\ cm$

(D) $8\ cm$

3) $\frac{\sin 120°}{1+\sin(90°+\theta)} - \frac{\sin 240°}{1-\cos(-\theta)} =$

(A) $\frac{1}{\sin^2\theta}$

(B) $\frac{\sqrt{3}}{\sin^2\theta}$

(C) $\frac{\sqrt{3}}{1+\cos\theta}$

(D) $\frac{\sqrt{3}}{2(1+\sin\theta)(1+\cos\theta)}$

4) If $A = \frac{16}{r^2} - \pi\sqrt{r}$, then $\frac{dA}{dr} =$

(A) $\frac{8}{r} - \frac{\pi}{2\sqrt{r}}$

(B) $-\frac{8}{r} - \frac{\pi}{2\sqrt{r}}$

(C) $-\frac{32}{r^3} - \frac{\pi}{2\sqrt{r}}$

(D) $\frac{32}{r^3} + \frac{\pi}{2\sqrt{r}}$

5) If $2 \ln x = 6$, then x equals

(A) e^3

(B) $\sqrt{e^3}$

(C) e^{36}

(D) $2e^6$

6)
$$\int \frac{2}{3x+1}.dx =$$

(A) $2\log_e(3x+1) + C$

(B) $\frac{2}{3}\log_e(3x+1) + C$

(C) $6\log_e(3x+1) + C$

(D) $\frac{3}{2}\log_e(3x+1) + C$

7) $\frac{d}{dx}[\cos(3x+1)] =$

(A) $\sin(3x+1)$

(B) $-\sin(3x+1)$

(C) $3\sin(3x+1)$

(D) $-3\sin(3x+1)$

End of SECTION I

SECTION II – 60 Marks
Attempt Questions 8 – 12

Answer each question in a SEPARATE writing booklet.

QUESTION 8 (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\tan 75°$. **2**

(b) Solve the equation $2\cos\theta = \sqrt{3}$, where $0 \leq \theta \leq 2\pi$. **2**

(c) Let $y = x + \sin kx$. It is given that $\frac{dy}{dx} = 3$, when $x = 0$.

(i) Find the value of k. **2**

(ii) Find $\frac{d^2y}{dx^2}$ **2**

(d) Solve **2**

$$\frac{1}{x+1} \geq 1$$

(e) Evaluate **2**

$$\lim_{h\to 2} \frac{2h-4}{h^2-4}$$

End of Question 8

QUESTION 9 (12 marks) Use a SEPARATE writing booklet.

(a) Prove that **2**

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

(b) The region bounded by the curve $y = 2 + e^{-x}$, the x-axis, the y-axis and the line $x = 1$ is rotated about the x-axis. Find the volume of the solid generated. **2**

(c) Evaluate the following integrals.

(i)

$$\int_{-1}^{0} \frac{1}{\sqrt{1-2x}}.\,dx$$ **2**

(ii)

$$\int_{2}^{4} \frac{x}{x-1}\,dx$$ **2**

(d) The value of a car $\$C$ can be represented by the formula

$$C = 1000 + Ae^{-kt}$$

where t is the age in years from new.

(i) The price of a brand new car is \$18000 and after 3 years, the same car has depreciated to \$7500. Find the value of A and k. **2**

(ii) Find the age of the car when its value first falls below \$3000. **2**

End of Question 9

QUESTION 10 (12 marks) Use a SEPARATE writing booklet.

(a) Let $y = x^2 e^{-x}$, where $x > 0$. Show that $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - (x-3)y = 0$. **3**

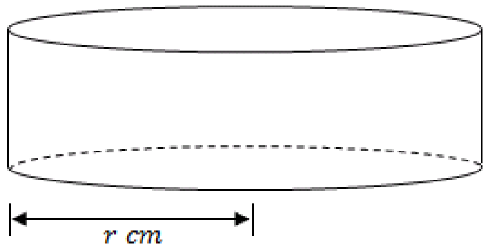
(b)

(i) Show that $\sin^2 x \cos^2 x = \frac{1-\cos 4x}{8}$. **3**

(ii) Hence find **2**

$$\int \sin^2 x \cos^2 x \,.\, dx$$

(c) The figure below shows a right circular cylindrical can without cover and its volume is $27\pi\ cm^3$. Let $A\ cm^2$ be the outer surface area of the circular cylindrical can and $r\ cm$ be the radius of the can.



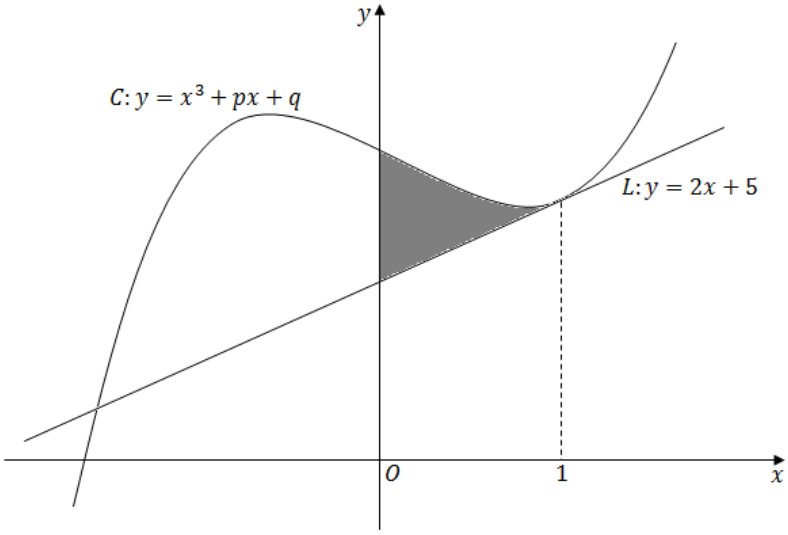
(i) Show that $A = \pi\left(\frac{54}{r} + r^2\right)$. **2**

(ii) Find r if A is minimum. **2**

End of Question 10

QUESTION 11 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graphs of the curve $C: y = x^3 + px + q$ and the line $L: y = 2x + 5$. L touches C at $x = 1$.



(i) Find the values of p and q. **4**

(ii) Find the area of the shaded region. **2**

(b) Find $\frac{dy}{dx}$ of the following functions.

(i) $y = \frac{x-2}{\sqrt{x+1}}$, where $x > -1$. **3**

(ii) $y = \ln[\ln(2x + 1)]$, where $x > 0$. **3**

End of Question 11

QUESTION 12 (12 marks) Use a SEPARATE writing booklet.

(a) Let **3**

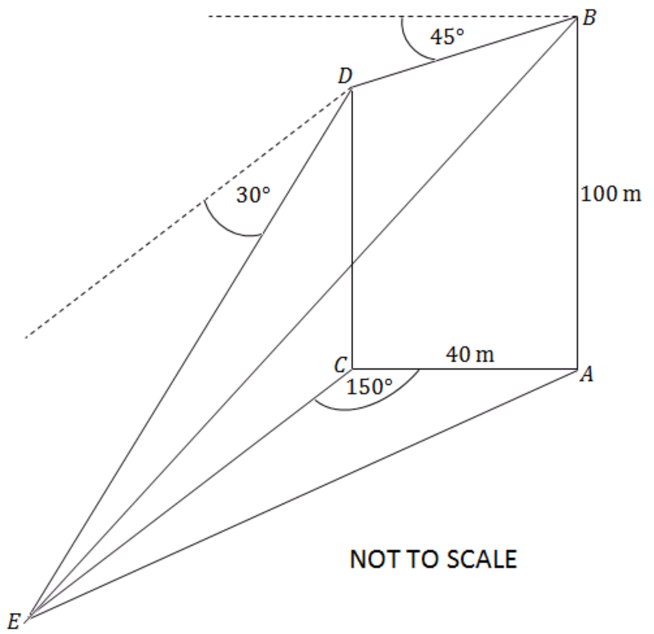
$$\int_a^b \frac{f(x)}{f(x)-g(x)}\,dx = 5.$$

Find the value of

$$\int_a^b \frac{g(x)}{f(x)-g(x)}\,dx$$

in terms of a and b.

(b) The figure below shows two vertical towers AB and CD. The height of tower AB is 100m. The tower CD is 40m away from AB. The angle of depression of D from B is 45°. A canopy BDE is constructed to cover the two towers. A, C and E are on the horizontal ground such that $\angle$ACE $= 150°$. The angle of depression of E from D is 30°.



(i) Find the angle of elevation of B from E, correct your answer to nearest degree. **5**

(ii) Find the area of canopy BDE, correct your answer to 3 significant figures. **4**

End of Question 12
End of Exam

Ext Continuers 11 Yearly 13

1) C

2) D

3) B

4) C

5) A

6) B

7) D

Q11 Actually Q8 solutions

(a) $\tan 75^\circ = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

$= \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

$= \dfrac{(\sqrt{3}+1)^2}{2}$

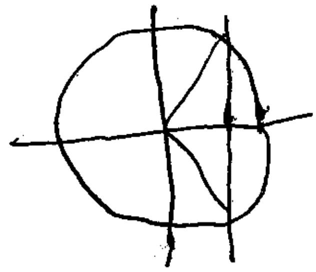
$= \dfrac{4+2\sqrt{3}}{2}$

$= 2+\sqrt{3}$

[2]

(b) $2\cos\theta = \sqrt{3}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$



$\theta = \pi/6,\ \dfrac{11\pi}{6}$ [2]

(c) $y = x + \sin kx$

$\dfrac{dy}{dx} = 1 + k\cos kx$

(i) When $x=0$

$\dfrac{dy}{dx} = 3 = 1 + k\cdot 1$

$\therefore k = 2$ [2]

(ii) $\dfrac{dy}{dx} = 1 + 2\cos 2x$

$\dfrac{d^2y}{dx^2} = -4\sin 2x$

[2]

(d) $\dfrac{1}{x+1} \geqslant 1,\ x \neq -1$

Multiply by $(x+1)^2$

$x+1 \geqslant (x+1)^2$

$x+1 \geqslant x^2+2x+1$

$0 \geqslant x^2 + x$

$0 \geqslant x(x+1)$

$\therefore -1 < x \leqslant 0$ [2]

Q11 (Cont'd)

(a) $\lim\limits_{h\to 2} \dfrac{2h-4}{h^2-4} = \lim\limits_{h\to 2} \dfrac{2(h-2)}{(h+2)(h-2)}$

$= \lim\limits_{h\to 2} \dfrac{2}{h+2}$

$= \dfrac{1}{2}$ [2]

Q12 Actually Q9 solutions

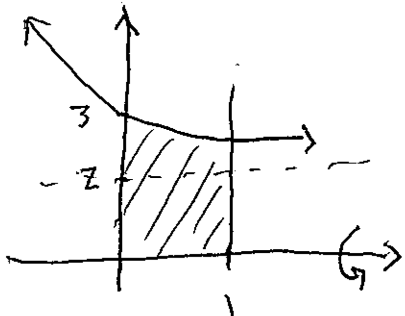
(a) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore \cos(A-B) - \cos(A+B) = 2\sin A\sin B$

[2]

(b)



$V = \pi\int_0^1 (2+e^{-x})^2\,dx$

$= \pi\int_0^1 (4 + 4e^{-x} + e^{-2x})\,dx$

$= \pi\left[4x - 4e^{-x} - \frac{1}{2}e^{-2x}\right]_0^1$

$= \pi\left[\left(4 - \frac{4}{e} - \frac{1}{2e^2}\right) - \left(0 - 4 - \frac{1}{2}\right)\right]$

$= \pi\left(8\frac{1}{2} - \frac{4}{e} - \frac{1}{2e^2}\right)$

$\doteqdot 21.868$ [2]

(c)

(i) $\int_{-1}^{0} \dfrac{1}{\sqrt{1-2x}}\,dx$

$= \int_{-1}^{0} (1-2x)^{-1/2}\,dx$

$= \left[\dfrac{-\frac{1}{2}(1-2x)^{1/2}}{\frac{1}{2}}\right]_{-1}^{0}$

$= \left[-\sqrt{1-2x}\right]_{-1}^{0}$

$= \left(-\sqrt{1-0}\right) - \left(-\sqrt{1+2}\right)$

$= \sqrt{3} - 1$ [2]

(ii) $\int_2^4 \dfrac{x}{x-1}\,dx = \int_2^4 \dfrac{x-1+1}{x-1}\,dx$

$= \int_2^4 \left(1 + \dfrac{1}{x-1}\right)dx$

$= \left[x + \ln(x-1)\right]_2^4$

$= (4 + \ln 3) - (2 + \ln 1)$

$= 2 + \ln 3$ [2]

(d) $C = 1000 + Ae^{-kt}$

(i) When $t=0$, $C = 18000$

$\therefore 18000 = 1000 + A.1$

$A = 17000$

Q12 (Cont'd)

When $t=3, C=7500$

$\therefore 7500 = 1000 + 17000e^{-3k}$

$$e^{-3k} = \frac{6500}{17000}$$

$$-3k = \ln\left(\frac{6500}{17000}\right)$$

$$k = \frac{\ln\left(\frac{65}{170}\right)}{-3}$$

$\therefore k = 0.32047\ldots$ [2]

(ii) $3000 > 1000 + 17000e^{-kt}$

$$\frac{2000}{17000} > e^{-kt}$$

$$-kt < \ln\frac{2}{17}$$

$$t > \frac{\ln\frac{2}{17}}{-k}$$

$$> 6.677\ldots$$

$\therefore$ 6yrs 9 months

or 7 yrs to nearest year.

[2]

Q13

(a) $y = x^2 e^{-x}$, $x > 0$.

$$\frac{dy}{dx} = 2x e^{-x} - x^2 e^{-x}$$

$$\frac{d^2y}{dx^2} = -2x e^{-x} + 2e^{-x} + x^2 e^{-x} - 2x e^{-x}$$

$$= e^{-x}(x^2 - 4x + 2)$$

$$LHS = x\frac{d^2y}{dx^2} - \frac{dy}{dx} - (x-3)y$$

$$= xe^{-x}(x^2-4x+2) - xe^{-x}(2-x) - (x-3)x^2e^{-x}$$

$$= xe^{-x}\left[x^2 - 4x + 2 - 2 + x - x^2 + 3x\right]$$

$$= xe^{-x}\left[x^2 - x^2 - 4x + 4x + 2 - 2\right]$$

$$= xe^{-x} \times 0$$

$$= 0$$

$$= RHS.$$

Q13 (cont'd)

(b) (i). $RHS = \dfrac{1-\cos 4x}{8}$

$= \dfrac{1-(1-2\sin^2 2x)}{8}$

$= \dfrac{2\sin^2 2x}{8}$

$= \dfrac{\sin^2 2x}{4}$

$= \dfrac{(2\sin x\cos x)^2}{4}$

$= \sin^2 x\cos^2 x.$

$= LHS.$

(ii) $\int \sin^2 x\cos^2 x\,dx = \int \dfrac{(1-\cos 4x)}{8}\cdot dx.$

$= \frac{1}{8}\int (1-\cos 4x)\,dx$

$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$

$\boxed{= \frac{x}{8} - \frac{1}{32}\sin 4x + C.}$

(c) (i) $V = \pi r^2 h$

$\therefore 27\pi = \pi r^2 h$

$r^2 h = 27$

$h = \dfrac{27}{r^2}$

$\downarrow$

Q13 (CONTD)

now $A = 2\pi r h + \pi r^2$

$$= 2\pi r \cdot \frac{27}{r^2} + \pi r^2$$

$$= \frac{54\pi}{r} + \pi r^2$$

$$= \pi\left(\frac{54}{r} + r^2\right)$$

(ii) $\frac{dA}{dr} = -54\pi r^{-2} + 2\pi r$

& $\frac{d^2A}{dr^2} = 108\pi r^{-3} + 2\pi, > 0$ for all r.

For max/min. let $\frac{dA}{dr} = 0$

$$-\frac{54\pi}{r^2} + 2\pi r = 0$$

$$\frac{54\pi}{r^2} = 2\pi r$$

$$r^3 = 27$$

$$r = 3$$

since $\frac{d^2A}{dr^2} > 0$ MIN at $\boxed{r = 3}$

Q14.

$$y = x^3 + px + q$$

$$\frac{dy}{dx} = 3x^2 + p$$

now slope of L is 2

$\therefore$ at $(1, 7)$ $m = 2$

$\therefore$ $3 + p = 2$ 4

$$\boxed{p = -1}$$

and $7 = 1^3 + -1 \times 1 + q$

$$\boxed{q = 7}$$

$$A = \int_0^1 \left[(x^3 - x + 7) - (2x + 5)\right] dx$$

$$= \int_0^1 (x^3 - 3x + 2)\,dx$$ 2.

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1$$

$$= \frac{1}{4} - \frac{3}{2} + 2$$

$$= \left|\frac{3}{4} \text{ u}^2\right|$$

Q14 (CONTD)

(i) $$y = \frac{x-2}{\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x+1} \cdot 1 - (x-2) \times \frac{1}{2} (x+1)^{-1/2}}{x+1}$$

$$= \frac{\sqrt{x+1} - \frac{(x-2)}{2\sqrt{x+1}}}{x+1}$$

$$= \frac{2(x+1) - (x-2)}{2\sqrt{x+1}\,(x+1)}$$ 3

$$y' = \frac{x+4}{2(x+1)\sqrt{x+1}}$$

(ii) $$y = \ln(\ln(2x+1))$$ 3

$$y' = \frac{\frac{2}{2x+1}}{\ln(2x+1)}$$

$$\boxed{y' = \frac{2}{(2x+1)\ln(2x+1)}}$$

Q12 (a) now $\displaystyle\int_a^b \frac{f(x)-g(x)}{f(x)-g(x)}\,dx$

$$= \int_a^b 1\,.\,dx$$

$$= \Big[x\Big]_a^b$$

$$= b - a.$$

Also $\displaystyle\int_a^b \frac{f(x)-g(x)}{f(x)-g(x)}\,dx = \int_a^b \frac{f(x)}{f(x)-g(x)}\,.dx - \int_a^b \frac{g(x)}{f(x)-g(x)}\,.dx$

$$= 5 - \int_a^b \frac{g(x)}{f(x)-g(x)}\,.dx$$

$\therefore \quad \cancel{b - a = \int_a^b g(x)}$

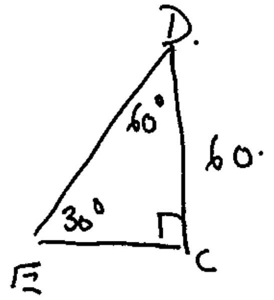
$$\therefore \quad b - a = 5 - \int_a^b \frac{g(x)}{f(x)-g(x)}\,.dx$$

$$\int_a^b \frac{f(x)}{f(x)-g(x)}\,dx = 5 - (b-a)$$

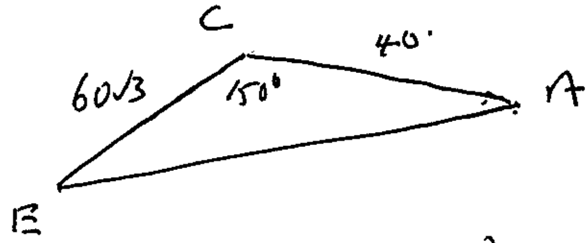
$$\boxed{= 5 + a - b}$$

Q15 (CONTD)

(b) (i)



$\therefore EC = 60 \tan 60^\circ$
$= 60\sqrt{3}$



$$EA^2 = (60\sqrt{3})^2 + 40^2 - 2 \times 60\sqrt{3} \times 40 \times \cos 150^\circ$$

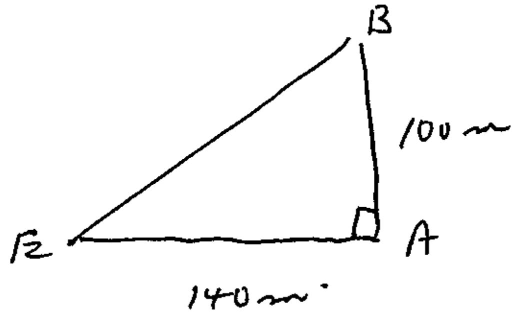
$$= 10800 + 1600 - 2 \times 60\sqrt{3} \times 40 \times -\frac{\sqrt{3}}{2}$$

$$= 12400 + 7200$$

$$= 19600$$

$\therefore EA = 140$ m.

(ii)



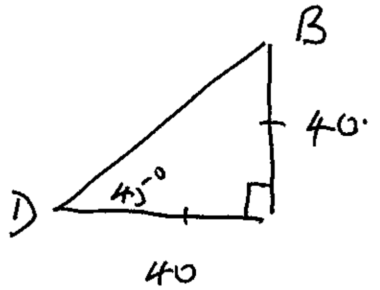
The angle of elevation of B from E

is $\angle BEA = \tan^{-1} \frac{100}{140}$

$= \tan^{-1} \frac{5}{7}$

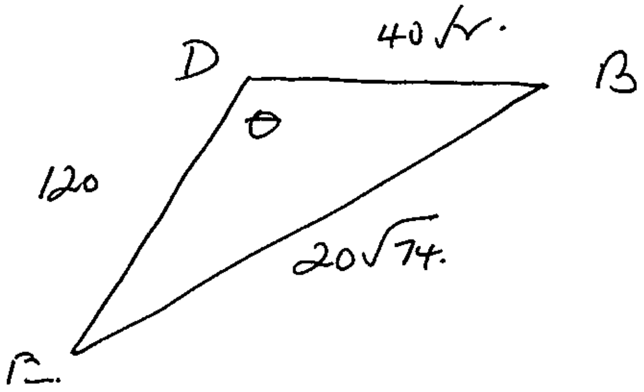
$\doteqdot 35^\circ 32'$

Q15 (CONTD)



$$DB^2 = 40^2 + 40^2$$

$$\therefore DB = 40\sqrt{2}$$



$$EB^2 = 140^2 + 100^2$$

$$= 29600$$

$$\therefore EB = 20\sqrt{74}$$

$$\cos \angle EDB = \frac{120^2 + (40\sqrt{2})^2 - (20\sqrt{74})^2}{2 \times 120 \times 40\sqrt{2}}$$

$$= \frac{14400 + 3200 - 29600}{9600\sqrt{2}}$$

$$\cos\theta = -\frac{5\sqrt{2}}{2}$$

now $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \frac{50}{64} = 1$$

$$\sin^2\theta = 1 - \frac{25}{32}$$

$$= \frac{7}{32}$$

$$\sin\theta = \frac{\sqrt{7}}{4\sqrt{2}}$$

$$= \frac{\sqrt{14}}{8}$$

Q15 (CONTD)

(ii) $\therefore$ Area of $\triangle EDB$

$$= \frac{1}{2} \times 120 \times 40\sqrt{2} \times \frac{\sqrt{14}}{8}$$

$$= \underline{600\sqrt{7}} \text{ m}^2 \quad (\text{APPROX. } 1587.5 \text{ m}^2)$$