SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014

Year 11 Yearly

Mathematics Continuers

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may NOT be awarded for messy or badly arranged work.
- **EACH <u>SECTION</u> IS TO BE RETURNED IN A SEPARATE BUNDLE**

  Section A: Q1 and Q2.

  Section B: Q3 and Q4.
- All necessary working should be shown in every question if full marks are to be awarded.
- Leave all answers in simplified exact form unless indicated otherwise.

Total Marks – 90

- Attempt all questions.
- All questions are not of equal value.

Examiner: *A Ward*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SECTION A

Start Question 1 in a new answer booklet.

Question 1 (25 Marks) **Marks**

a) What is the range of $y=\sqrt{4-x^2}$? 1

b) Graph the solution of $|2x+3|\geq 2$ on a number line. 2

c) If $f(x)=2ax+b$, find the values of a and b given that $f(-1)=5$ and $f(2)=-1$ 2

d) Differentiate with respect to x: 10

(i) $6x^5-3x^3-7x+1$

(ii) $\dfrac{7}{x}$

(iii) $4(1-x^3)^5$

(iv) $\ln(6x^2-3)$

(v) $\dfrac{x-1}{3x-4}$

(vi) xe^x

e) Find a primitive of: 6

(i) $(x^2-2)^2$

(ii) $\dfrac{x^2-2}{x}$

(iii) $\dfrac{x}{x^2-2}$

f) Sketch the curves in separate diagrams, showing intercepts and asymptotes: 4

(i) $y=\log(x+2)$

(ii) $y=\dfrac{1}{x+2}$

End of Question 1.

Start Question 2 in a new answer booklet.

Question 2 (20 Marks) **Marks**

a) Find the value(s) of x for which $\ln x = 2\ln x$ 1

b) If $f(x) = e^{x-2}$ find $f(-1)$ to 4 decimal places. 1

c) Find the value of the following: 6

(i) $\int_0^3 \sqrt{x}\,dx$

(ii) $\int_0^3 \frac{dx}{(2x-3)^2}$

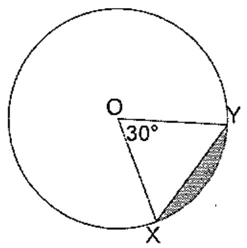
(iii) $\int_1^{e-1} \frac{1}{x+1}\,dx$

d) 3

(i) Show that $g(x) = \frac{2}{x^2-1}$ is an even function.

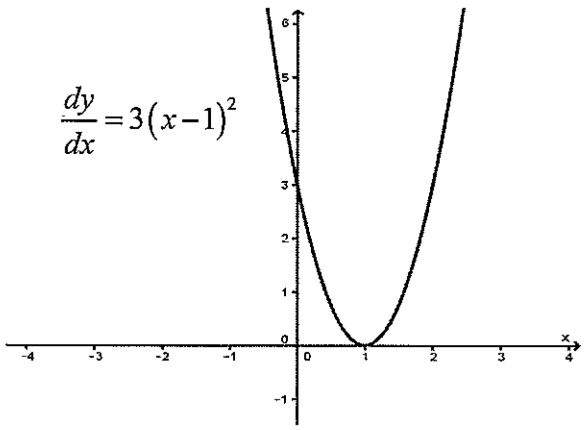
(ii) State the domain of $y = g(x)$.

e) Calculate the exact shaded area where arc XY, of length 10πcm subtends an angle of 30° at, O the centre of the circle. 2



Not to scale

f) The gradient function of a curve is illustrated by the graph below. 7



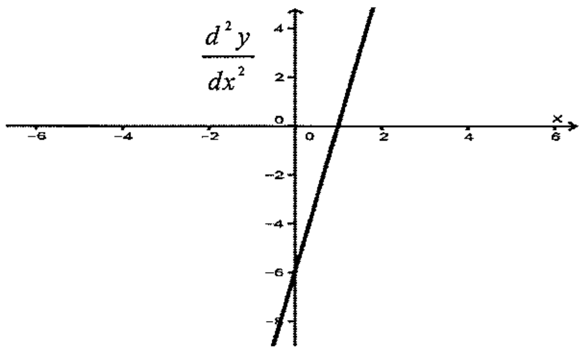
(i) A stationary point is located at $x = 1$.

$\mathcal{A}$) Justify this statement with reference to the graph.

$\mathcal{B}$) Comment on the sign of *dy/dx* for all *x*, $x \neq 1$.

$\mathcal{C}$) What does this imply about the curve $y = f(x)$?

(ii) The graph of $\frac{d^2y}{dx^2}$, related to the above, is given below:



Copy and complete the table below:

x	0	1	2
Sign of $\frac{d^2y}{dx^2}$			

(iii) What is the nature of the stationary point at $x = 1$?

(iv) If the curve $y = f(x)$ passes through $(0,0)$, find the equation of the curve.

End of Question 2

End of Section A

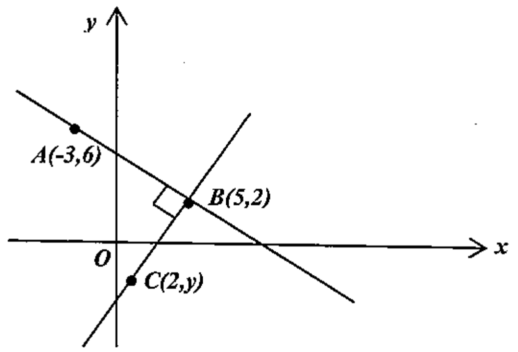
SECTION B

Start Question 3 in a new answer booklet

Question 3 (25 Marks) **Marks**

a) The diagram shows the origin O and the points $A(-3,6)$, $B(5,2)$ and $C(2,y)$. 11

The lines AB and BC are perpendicular.



NOT TO SCALE

(i) Copy this diagram into your writing booklet.

(ii) Show that A and B lie on the line $x+2y=9$.

(iii) Find the exact length AB.

(iv) Find the perpendicular distance from O to AB exactly.

(v) Find the area of triangle OAB.

(vi) Show that C has coordinates $(2,-4)$

(vii) Does the line AC passes through the origin? Explain.

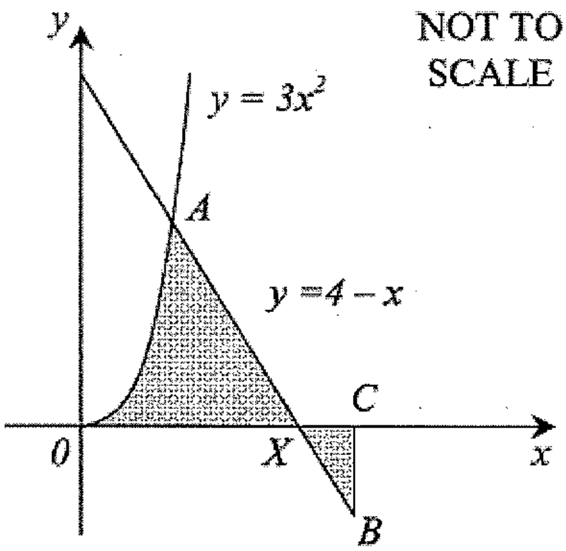
(viii) The point D is not shown on the diagram. The point D lies on the x-axis and $ABCD$ is a rectangle. Find the coordinates of D.

(ix) On your diagram, shade the region satisfying the inequality $x+2y\geq 9$.

b) Find the equation of the normal to the curve $y = x\log_e x$ at the point (e,e). Give your answer in general form. 3

c) For what value of x is the tangent to the curve $y=e^{3x}$ parallel to the line $y=6x$? 3

d) The shaded region $OABC$ is bounded by the lines $x = 0$, $x = 5$, the curve $y = 3x^2$, the line $y = 4 - x$ and the x-axis, as in the diagram below. 5



(i) Find the co-ordinates of X and A.

(ii) Find the area of the shaded region $OABC$.

e) Differentiate $y = (\ln x)^2$ and hence evaluate $\int_1^2 \frac{\ln x}{x}\,dx$. 3

End of Question 3

Start Question 4 in a new answer booklet.

Question 4 (20 Marks) **Marks**

a) Given $\frac{d}{dx}\left(b^x\right)=b^x\log_e b$, evaluate $\int_0^\pi \pi^x dx$ correct to one decimal place. 2

b) To calculate the area of the region bounded by the curve $y=x^2-3x$ and the x-axis between the ordinates of $x=0$ and $x=6$, a student used $y=\int_0^6\left(x^2-3x\right)dx$. 3

a) Explain why the student's method of calculating the area is incorrect.

b) Find the area of the required region.

c) Consider $f(x)=\frac{\log_e\sqrt{x}}{x}$. 5

What is the domain of $f(x)$?

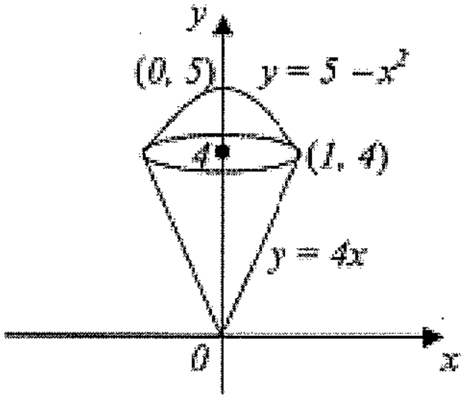
(i) Find $f'(x)$ and hence determine all the stationary points.

(ii) Sketch the curve of $y=f(x)$ clearly showing all its essential features.

d) Differentiate $y=\log_2 x$. 2

e) Determine the equation of the locus of the points $P(x,y)$ equidistant from the y-axis and the point $(1,0)$. 3

f) The diagram shows a cone and a paraboloid. It represents an ice-cream cone which is completely full of ice-cream and which has an additional scoop of ice-cream on top. 5



To calculate the volume of ice-cream, the area bounded by the section of the line $y = 4x$ between $(0,0)$ and $(1,4)$, the part of the parabola between $(1,4)$, $(0,5)$ and the y-axis, was rotated about the y-axis. Determine the total quantity of ice-cream contained in the cone and the scoop on top, in terms of π.

End of Question 4

End of Section B

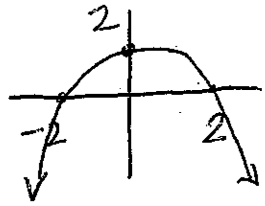
End of Examination

YR 11 2014 Yearly maths continuers exam

25

1 (a) $4-x^2 \geq 0$

$(2-x)(2+x) \geq 0$



D $-2 \leq x \leq 2$. 1

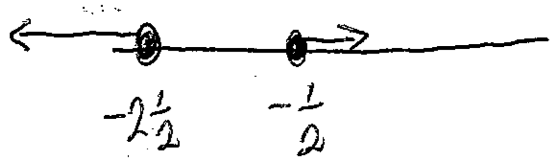
* Range $0 \leq y \leq 2$

b) $|2x+3| \geq 2$

$2x+3 \geq 2$ $\quad$ $2x+3 \leq -2$

$2x \geq -1$ $\quad$ $2x \leq -5$

$x \geq -\frac{1}{2}$ $\quad$ $x \leq -2\frac{1}{2}$



2

c) $f(x) = 2ax + b$

$f(-1) = -2a + b = 5$

$f(2) = 4a + b = -1$ $\quad -$

$-6a = 6$

$a = -1$ *

so $-4 + b = -1 \Rightarrow b = 3$ * 2

d) (i) $\frac{d}{dx}(6x^5 - 3x^3 - 7x + 1) = 30x^4 - 9x^2 - 7$ 1

(ii) $\frac{d}{dx}(7x^{-1}) = -7x^{-2} = \frac{-7}{x^2}$ 1

(iii) $\frac{d}{dx}\left(4(1-x^3)^5\right) = 20(1-x^3)^4 \times -3x^2$

$= -60x^2(1-x^3)^4$ 2

d) (iv) $\frac{d}{dx}\left(\ln(6x^2-3)\right) = \frac{1}{6x^2-3} \times 12x$

$= \frac{12x}{6x^2-3} = \frac{4x}{2x^2-1}$ 2

(v) $\frac{d}{dx}\left(\frac{x-1}{3x-4}\right) = \frac{(3x-4)\times 1-(x-1)\times 3}{(3x-4)^2}$

$= \frac{3x-4-3x+3}{(3x-4)^2} = \frac{-1}{(3x-4)^2}$ 2

(vi) $\frac{d}{dx}\left(xe^x\right) = xe^x + e^x \times 1$

$= e^x(x+1)$ 2

e) (i) $\int (x^2-2)^2\,dx = \int (x^4 - 4x^2 + 4)\,dx$

$= \frac{x^5}{5} - \frac{4x^3}{3} + 4x + c$ 2

(ii) $\int \frac{x^2-2}{x}\,dx = \int \frac{x^2}{x} - \frac{2}{x}\,dx$

$= \int x\,dx - 2\int \frac{1}{x}\,dx$ 2

$= \frac{x^2}{2} - 2\ln x + c$

(iii) $\frac{1}{2}\int \frac{2x}{x^2-2}\,dx = \frac{1}{2}\ln(x^2-2) + c$ 2

Q2 (a) $\ln x = 2\ln x$

$\frac{}{20}$

$2\ln x - 1\ln x = 0$

$\ln x = 0$

$\log_e x = 0$

$e^0 = 1 = x$ 1

$\ln x = \ln x^2$
$x = x^2$
$x^2 - x = 0$
$x(x-1) = 0$
$\cancel{x=0}$, $x = 1$ *
undef

(b) $f(x) = e^{x-2}$

$f(-1) = e^{-3}$

$\doteq 0.0498$ 1

(4 DP)

c) (i) $\int_0^3 x^{\frac{1}{2}}\,dx = \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

exact or approx $= \left.\frac{2}{3}x\sqrt{x}\right]_0^3$

$= \frac{2}{3} \times 3\sqrt{3}$

$= 2\sqrt{3}$ 2

$(3.4641\ldots)$

(ii) $\int_0^3 \frac{1\,dx}{(2x-3)^2}$

$= \int_0^3 (2x-3)^{-2}$

$= \left.\frac{(2x-3)^{-1}}{-1\times 2}\right]_0^3$

$= \left.-\frac{1}{2}\cdot\frac{1}{(2x-3)}\right]_0^3$

$= \left(-\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times -\frac{1}{3}\right)$

$= \left(-\frac{1}{6} - \frac{1}{6}\right) = -\frac{1}{3}$

2

(iii) $\int_1^{e-1} \frac{1}{x+1}\,dx$

$= \ln(x+1)\Big]_1^{e-1}$

$= \ln(e) - \ln(2)$

$= 1 - \ln 2$ 2

(d) (i) $g(x) = \frac{2}{x^2-1}$

$g(-x) = \frac{2}{(-x)^2-1} = \frac{2}{x^2-1}$

$g(x) = g(-x)$ even fn. 2

(ii) $x^2 - 1 \neq 0$

$x^2 \neq 1$

$x \neq \pm 1$ 1

12 (e) $L = r\theta$

$10\pi = r \times \frac{\pi}{6}$

$r = 10\cancel{\pi} \times \frac{6}{\cancel{\pi}}$

$= 60$ cm.

shaded area $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 3600\left(\frac{\pi}{6} - \sin 30^\circ\right)$

$= 1800\left(\frac{\pi}{6} - \frac{1}{2}\right)$

$= 1800\left(\frac{\pi - 3}{6}\right)$

exact $= 300(\pi - 3)$ cm^2 (2)

(f) (i) (A) when $\frac{dy}{dx} = 3(x-1)^2 = 0$

$x = 1$. (1)

LHS of $x = 1$, $\frac{dy}{dx} > 0$

RHS of $x = 1$, $\frac{dy}{dx} > 0$

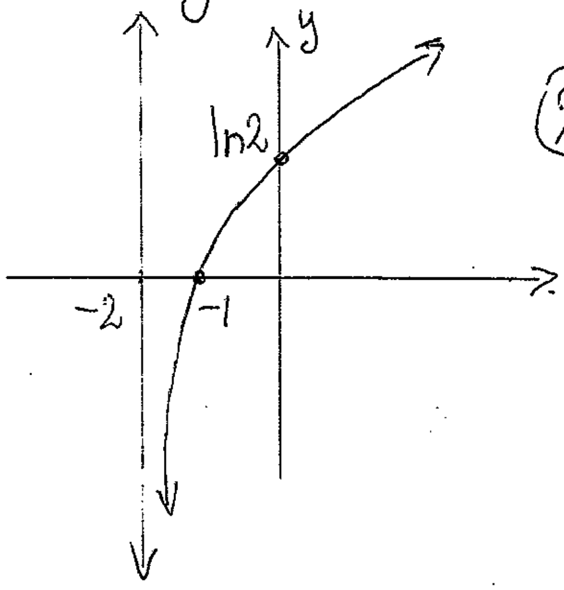
(B) except for $x = 1$, $\frac{dy}{dx} > 0$ for all x. (1)

(c) at $x = 1$, there is no max or min stat. pt but a stat pt of inflexion (1)

(f) (i) $y = \log(x+2)$

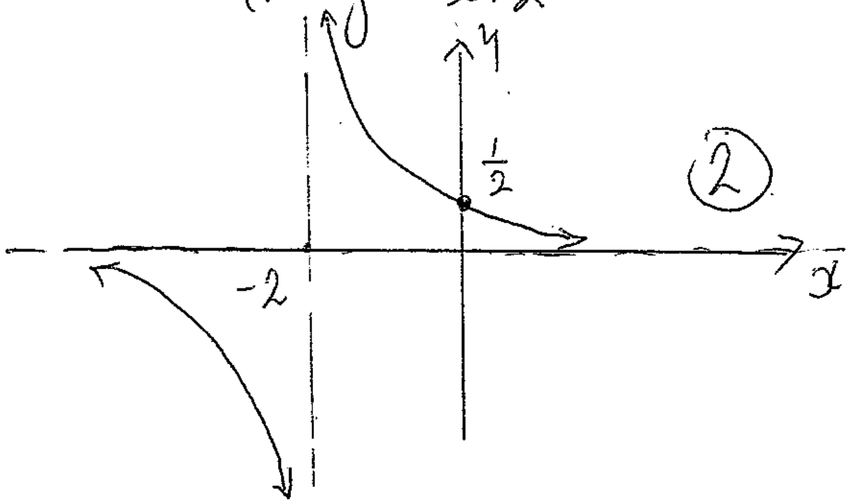
$x+2 > 0$

$x > -2$



(2)

(ii) $y = \dfrac{1}{x+2}$



(2)

Q2 (F) (ii)

x	0	1	2
sign y''	$-$	0	$+$

(1)

(iii) because of the y'' sign change to the left and right of $x=1$, we have a point of inflexion. (1)

(iv) $y = \int 3(x-1)^2\,dx$

$$y = \frac{\cancel{3}(x-1)^3}{\cancel{3}} + c$$

at $(0,0)$ $0 = -1^3 + c$

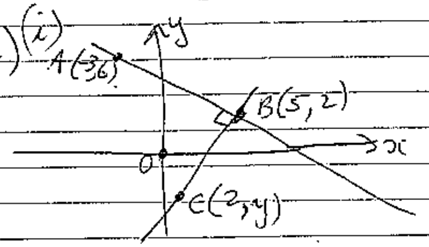
$c = 1$

$y = (x-1)^3 + 1$ (2)

or $c = 0$ and $y = x^3 - 3x^2 + 3x$

Section B Q3 (25)

Q3 (a) (i)



(ii) $x + 2y = 9$

$A(-3,6) \Rightarrow -3 + 12 = 9$ ✓

$B(5,2) \Rightarrow 5 + 4 = 9$ ✓ A, B satisfy eqn ∴ they lie on line.

1

(iii) $d_{AB} = \sqrt{(5+3)^2 + (2-6)^2}$

$= \sqrt{64 + 16}$

$= \sqrt{80} = 4\sqrt{5}$

1

(iv) Area ΔOAB

Now perp. distance from O to AB ie (0,0) to $x + 2y = 9$

is $d = \dfrac{|0 + 0 - 9|}{\sqrt{1+4}} = \dfrac{9}{\sqrt{5}}$

1

(v) Then Area $\Delta OAB = \frac{1}{2}bh$

$= \frac{1}{2} \times 4\sqrt{5} \times \frac{9}{\sqrt{5}}$

$= 18$ unit²

1

(vi) $C(2,y) = ?$ Now $m_{AB} = \dfrac{6-2}{-3-5} = \dfrac{4}{-8} = -\dfrac{1}{2}$

Then $m_{BC} = \dfrac{y-2}{2-5} = \dfrac{y-2}{-3} = 2$

$\Rightarrow y - 2 = -6$

$y = -4$

∴ $C = (2,-4)$

2

(2)

(vii) Line AC Passes through $A(-3,6)$ and $C(2,-4)$

Eqn of AC : $y - 6 = \dfrac{6+4}{-3-2}(x+3)$

$y - 6 = -2(x+3)$

$y = -2x$

2

$\Rightarrow (0,0) = 0 = -2 \times 0$

∴ (0,0) does lie on line AC as it satisfies eqn.

(viii) $D = (d, 0)$. ABCD is rect. and $C = (2,-4)$

$\Rightarrow AB \parallel CD$, $AB \perp BC$

$m_{AB} = -\frac{1}{2}$ and $m_{BC} = 2$

2

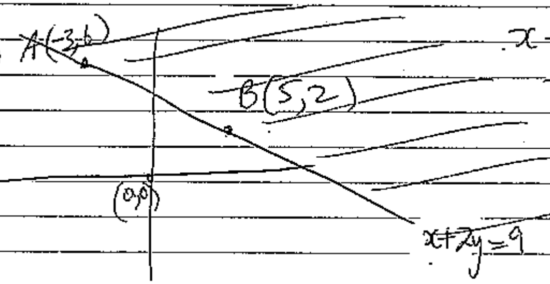
Then $m_{AB} = m_{CD} \Rightarrow -\dfrac{1}{2} = \dfrac{-4}{d-2}$

$-d + 2 = 8$

$d = -6$

∴ $D = (-6, 0)$

(ix)



$x + 2y \geq 9$

Test (0,0)

$\Rightarrow 0 + 0 < 9$

∴ need other side

1

9 (b) $y = x\ln x$

$y' = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$.

At (e, e) $y' = 1 + \ln e$

$y' = 2 = m$

Then gradient of normal $= -\frac{1}{2}$

At (e, e), eqn of normal is

$y - e = -\frac{1}{2}(x - e)$

$2y - 2e = -x + e$

$\Rightarrow x + 2y - 3e = 0$

3

c) $y = e^{3x}$

Parallel to line $y = 6x \Rightarrow m = 6$

$y' = 3e^{3x}$

Then $3e^{3x} = 6$

$e^{3x} = 2$

$3x = \ln 2$

$x = \frac{\ln 2}{3}$

3

(d) (i) $y = 4 - x$

Find X When $y = 0$, $0 = 4 - x$

$x = 4$

$\therefore X = (4, 0)$

1

Find A

$y = 4 - x$ (1)

$y = 3x^2$ (2)

Sub (1) into (2)

$\Rightarrow 3x^2 = 4 - x$

$3x^2 + x - 4 = 0$

$(3x + 4)(x - 1) = 0$

$x = -\frac{4}{3}$ or $x = 1$

For A, $x, y > 0 \Rightarrow x = 1$. Sub into (2)

$\Rightarrow y = 3$

$\therefore A = (1, 3)$

1

(ii) Area OABC $= \int_0^1 3x^2\,dx + \int_1^4 (4 - x)\,dx + \left|\int_4^5 (4-x)\,dx\right|$

$= \left[x^3\right]_0^1 + \left[4x - \frac{x^2}{2}\right]_1^4 + \left|\left[4x - \frac{x^2}{2}\right]_4^5\right|$

$= (1 - 0) + \left[(16 - 8) - (4 - \tfrac{1}{2})\right] + \left|(20 - \tfrac{25}{2}) - (16 - 8)\right|$

$= 1 + 4\frac{1}{2} + \left|-\frac{1}{2}\right|$

$= 6$ square units

3

(5)

(e) $y = (\ln x)^2$

$y' = 2\ln x \cdot \frac{1}{x} = \frac{2\ln x}{x}$

Then $\int_1^2 \frac{\ln x}{x}\,dx = \frac{1}{2}\int_1^2 \frac{2\ln x}{x}\,dx$

$= \frac{1}{2}\left[(\ln x)^2\right]_1^2$

$= \frac{1}{2}\left((\ln 2)^2 - (\underbrace{\ln 1}_{0})^2\right)$

$= \frac{1}{2}(\ln 2)^2$

3

Section B - Q4 (20) (1)

(a) $\frac{d}{dx}(b^x) = b^x \ln b$

Then $\int_0^{\pi} \pi^x\,dx$

$= \frac{1}{\ln\pi}\int_0^{\pi} \pi^x \ln\pi\,dx$

$= \frac{1}{\ln\pi}\left[\pi^x\right]_0^{\pi}$ $\frac{1}{2}$

$= \frac{1}{\ln\pi}\left[\pi^{\pi} - \pi^0\right]$

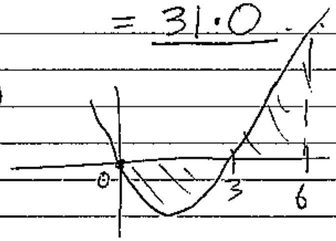
$= \frac{1}{\ln\pi}\left[\pi^{\pi} - 1\right]$

$= 30.978$ $\frac{1}{2}$

$= 31.0$

2

(b)



$y = x^2 - 3x$
$= x(x-3)$

(a) This is incorrect because some of the area is below the x-axis so has a negatively signed area and some is above which has a positively signed area.
Finding $\int_0^6 f(x)\,dx$ finds the difference in these 2 areas.

1

(b) Area $= \left|\int_0^3 (x^2-3x)\,dx\right| + \int_3^6 (x^2-3x)\,dx$

$= \left|\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3\right| + \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^6$

Q4 (cont)

(b)(b) cont

$$\text{Area} = \left|\left(9-\frac{27}{2}\right)-0\right| + \left(\frac{216}{3}-\frac{108}{2}\right)-\left(9-\frac{27}{2}\right)$$

$$|-4.5| \quad + \quad 18 + 4.5$$

$= 27$ units^2.

2

c) $f(x) = \frac{\ln\sqrt{x}}{x} = \frac{\ln x^{\frac{1}{2}}}{x} = \frac{\frac{1}{2}\ln x}{x} = \frac{\ln x}{2x}$.

Domain $x > 0$ ✓ $\frac{1}{2}$

(i) $f'(x) = \frac{2x.\left(\frac{1}{x}\right) - \ln x . 2}{4x^2}$

$= \frac{2-2\ln x}{4x^2}$

$f'(x) = \frac{1-\ln x}{2x^2}$

For stat pts, $f'(x) = 0 \Rightarrow 1-\ln x = 0$

$\ln x = 1$

$\Rightarrow x = e$

When $x = e$, $y = \frac{1}{2e}$

$\Rightarrow$ Stat pt at $\left(e, \frac{1}{2e}\right)$ ✓

Type?

x	2	e	3
y'	0.17	0	-0.05
	/	—	\

max at $\left(e, \frac{1}{2e}\right)$ $\frac{1}{2}$

(ii) $x \neq 0, y \neq 0$ As $x \to \infty$, $y \to 0$ $\frac{1}{2}$

Critical Value at $x=0 \Rightarrow$ vertical asymptote $\frac{1}{2}$

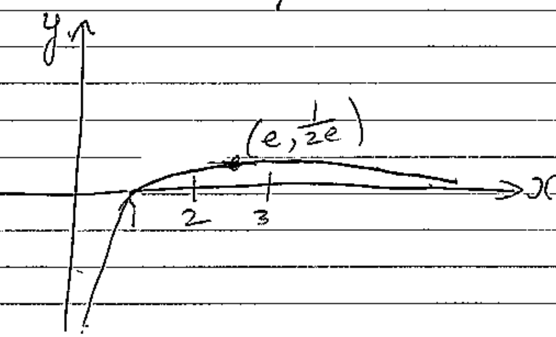
As $x \to 0$, $y \to -\infty$ $\frac{1}{2}$

x	0	$\frac{1}{2}$
y'	N.D.	3.038 +ve

$\Rightarrow$ /

4(c)(ii) cont Intercept when $y=0 \Rightarrow \ln x = 0$

$x = 1$



5

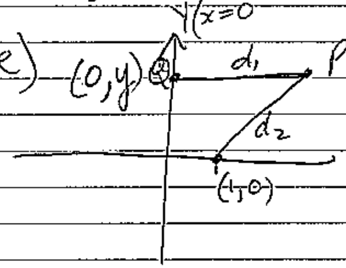
(d) $y = \log_2 x = \frac{\ln x}{\ln 2} = \frac{1}{\ln 2}\ln x$ ✓

$y' = \frac{1}{\ln 2} \cdot \frac{1}{x}$

$y' = \frac{1}{x\ln 2}$. ✓

2

(e)



Let $Q = (0, y)$ = point on Y axis s.t. $PQ \perp$ Y axis ✓

Then $d_1^2 = d_2^2$

$\Rightarrow x^2 = y^2 + (x-1)^2$ ✓

Then $y^2 = x^2 - x^2 + 2x - 1$

$y^2 = 2x - 1$

$y^2 = 2\left(x - \frac{1}{2}\right)$. ✓

3

4. (f) $\quad V = \int \pi x^2 \, dy$ for rotation about Y-axis (4)

For $y = 4x \Rightarrow x = \frac{y}{4}$ and for $y = 5 - x^2$

$x^2 = 5 - y$

$x = \sqrt{5-y}$ since $x > 0$

Then $V = \int_0^4 \pi \frac{y^2}{16} \, dy + \int_4^5 \pi (5-y) \, dy$.

$= \frac{\pi}{16}\left[\frac{y^3}{3}\right]_0^4 + \pi\left[5y - \frac{y^2}{2}\right]_4^5$

$= \frac{\pi}{16}\left[\frac{64}{3} - 0\right] + \pi\left[\left(25 - \frac{25}{2}\right) - (20 - 8)\right]$

$= \frac{4}{3}\pi + \frac{\pi}{2}$

$= \frac{11\pi}{6}$ cubic units

5