SYDNEY GRAMMAR SCHOOL

2011 Annual Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 31st August 2011

General Instructions

- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 135
- All nine questions may be attempted.
- All nine questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the nine questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5A: BDD	5B: PKH	5C: FMW
5D: MK	5E: SJE	5F: RCF
5G: LJF	5H: SO	5I: MLS

Checklist

- Writing leaflets: 9 per boy.
- Candidature — 142 boys

Examiner

RCF

QUESTION ONE (15 marks) Start a new leaflet. **Marks**

(a) Write down the exact value of $\sin 315^\circ$. **1**

(b) Express $\dfrac{1-\sqrt{3}}{\sqrt{2}}$ with a rational denominator. **1**

(c) Factorise $x^2 - 7x + 12$. **1**

(d) Evaluate $\log_3 \frac{1}{9}$. **1**

(e) Differentiate:

(i) $x^5 + 2x$ **1**

(ii) e^{2x+1} **1**

(f) Write down a primitive of $x^3 - 2$. **1**

(g) What is the nature of a stationary point if $\dfrac{d^2y}{dx^2} > 0$ at that point. **1**

(h) Simplify $\ln(e^3)$. **1**

(i) What is the natural domain of the function $f(x) = \ln(x+2)$? **1**

(j) Write as a single logarithm $2\log_{10} a + \log_{10} 3b$. **1**

(k) Sketch the graph of $(x-2)^2 + y^2 = 4$, clearly indicating any intercepts with the axes. **2**

(l) Solve $\tan\theta + \sqrt{3} = 0$, for $0^\circ \le \theta \le 360^\circ$. **2**

Exam continues next page ...

QUESTION TWO (15 marks) Start a new leaflet. **Marks**

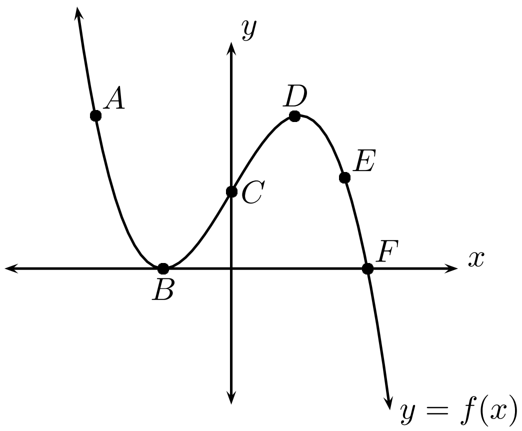
(a) (i) Sketch the parabola $y^2 = 4x$. **1**

(ii) State the co-ordinates of the focus. **1**

(b) Find the monic quadratic equation with roots 2 and -4. **2**

(c) The limiting sum of a GP with first term $\frac{1}{2}$ is 2. Find the common ratio. **2**

(d)



Which point labelled in the diagram above has:

(i) $f'(x) < 0$ and $f''(x) > 0$, **1**

(ii) $f''(x) = 0$. **1**

(e) Given points $A(-2, 4)$ and $B(3, -11)$, find the co-ordinates of the point P which divides the interval AB internally in the ratio $2 : 3$. **2**

(f) (i) Complete the square on the expression $x^2 + 6x - 1$. **2**

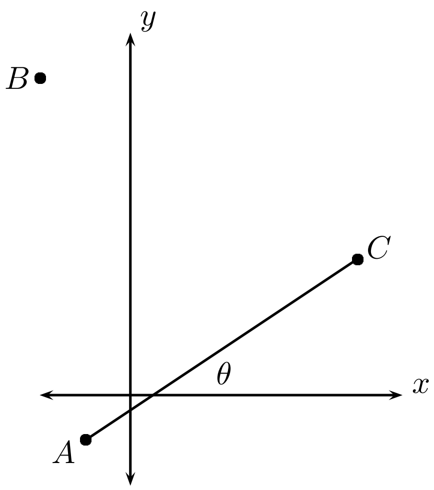
(ii) Hence, or otherwise, find the co-ordinates of the vertex of the parabola $y = x^2 + 6x - 1$. **1**

(g) (i) Graph the function $f(x) = \sqrt{4 - x^2}$. **1**

(ii) Hence evaluate the definite integral $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ without using calculus. **1**

QUESTION THREE (15 marks) Start a new leaflet. **Marks**

(a)



The points A, B and C, shown in the diagram above, have co-ordinates (-1,-1), (-2,7) and (5,3) respectively. The angle between AC and the x-axis is θ.

(i) Find the gradient of AC. **1**

(ii) Calculate the size of angle θ to the nearest degree. **1**

(iii) Find the equation of line AC. Give your answer in general form. **1**

(iv) Find the co-ordinates of D, the midpoint of AC. **1**

(v) Show that AC is perpendicular to BD. **1**

(vi) Explain why $\triangle ABC$ is isosceles. **1**

(vii) Find the area of $\triangle ABC$. **2**

(viii) Write down the co-ordinates of point E such that $ABCE$ is a rhombus. **1**

(b) (i) Differentiate $y = \log_e(2x - 1)$. **1**

(ii) Hence find the gradient of the tangent to the curve $y = \log_e(2x - 1)$ at the point where $x = 3$. **1**

(c) (i) Graph $y = |2x - 2|$. Use at least one third of a page in your answer booklet for the graph. **2**

(ii) Hence, or otherwise, solve the inequation $|2x - 2| \geq 4$. **2**

QUESTION FOUR (15 marks) Start a new leaflet. **Marks**

(a) Differentiate:

(i) $y = (5 - 2x)^7$ **2**

(ii) $y = \dfrac{x^4 + 2x^2}{x}$ **2**

(b) Find:

(i) $\displaystyle\int x\sqrt{x}\, dx$ **2**

(ii) $\displaystyle\int \frac{1}{2x^3}\, dx$ **2**

(iii) $\displaystyle\int \frac{1}{x - 3}\, dx$ **1**

(iv) $\displaystyle\int e^{4x-2}\, dx$ **1**

(c) Evaluate $\displaystyle\int_{-1}^{3} x(x - 2)\, dx$. **2**

(d) Shade the region where the inequalities $x \geq 3$, $y \leq 10$ and $2x - y > 0$ are simultaneously satisfied. Clearly indicate on your diagram the inclusion or exclusion of points on the boundary lines and corners. **3**

QUESTION FIVE (15 marks) Start a new leaflet. **Marks**

(a) Solve $\sin(\alpha + 45^\circ) = \frac{1}{2}$, for $0^\circ \leq \alpha \leq 360^\circ$. **3**

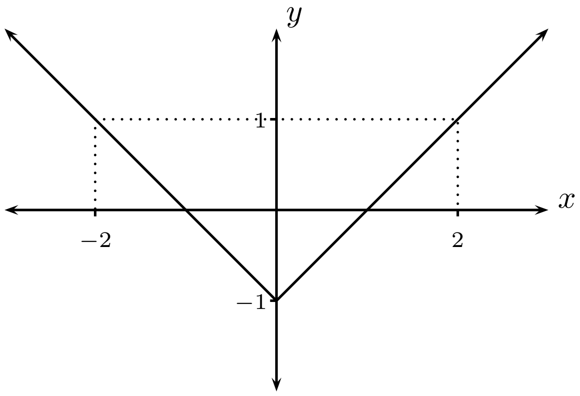
(b) The second term of a geometric series is 16 and the fifth term is 2.

(i) Find the common ratio. **1**

(ii) Find the first term. **1**

(iii) Find the sum of the first fifteen terms. Write your answer as a rational number in simplest form. **2**

(c) **1**



The diagram above shows the graph of $y = |x| - 1$. Find the value of $\int_0^2 |x| - 1\,dx$.

(d) Solve $\dfrac{2x - 6}{x} \leq 1$ **3**

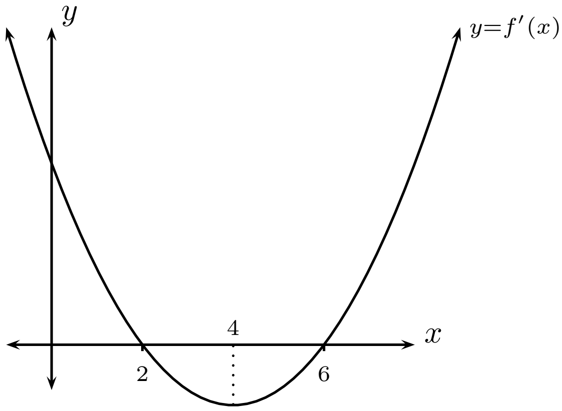
(e) Consider the parabola with equation $(y - 4)^2 = -16(x + 2)$.

(i) What is the focal length? **1**

(ii) Sketch the curve, clearly showing the focus, the directrix and any intercepts with the axes. **3**

QUESTION SIX (15 marks) Start a new leaflet. **Marks**

(a) **3**



The diagram above shows the graph of the gradient function $y = f'(x)$ of the curve $y = f(x)$. Given that $f(0) = 0$, sketch the curve $y = f(x)$.

(b) Let $f(x) = \ln(x^2 + 1)$.

(i) Use the trapezoidal rule with three function values to estimate $\displaystyle\int_0^8 f(x)\,dx$. Write your answer correct to one decimal place. **2**

(ii) Use Simpson's rule with five function values to estimate $\displaystyle\int_0^8 f(x)\,dx$. Write your answer correct to three decimal places. **2**

(c) Differentiate:

(i) $y = \dfrac{\ln x}{x^2}$ **2**

(ii) $y = e^{2x}(2 - x^2)$ **2**

(d) Evaluate $\displaystyle\int_{-1}^{0} \frac{4x}{x^2 + 3}\,dx$. **2**

(e) Solve the equation $9^x - 10 \times 3^x + 9 = 0$. **2**

Exam continues overleaf ...

QUESTION SEVEN (15 marks) Start a new leaflet. **Marks**

(a) Suppose that the equation $2x^2 + 4x - 1 = 0$ has roots α and β. Without solving the equation, find the values of:

(i) $\alpha\beta$ **1**

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **1**

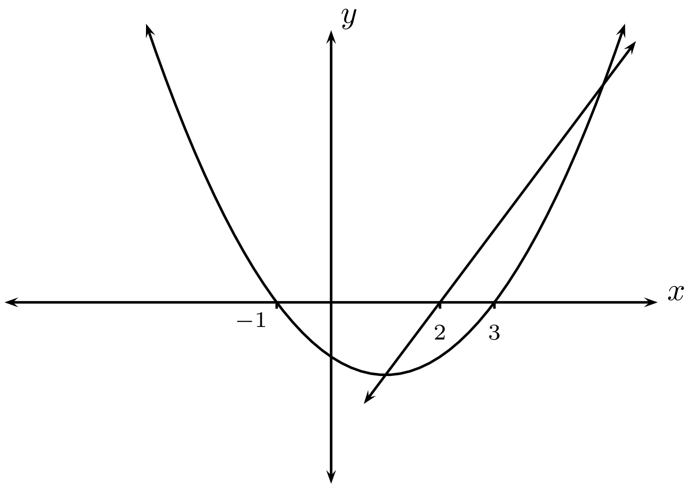
(iii) $(\alpha - \beta)^2$ **2**

(iv) $|\alpha - \beta|$ **1**

(b) (i) By forming a suitable arithmetic sequence, find how many multiples of 7 lie between 100 and 1000? **2**

(ii) What is the sum of these multiples? **1**

(c)



Consider the diagram above showing the parabola $y = 2x^2 - 4x - 6$ and the straight line $y = 8x - 16$.

(i) Find the x co-ordinates of the points of intersection of the parabola and the line. **1**

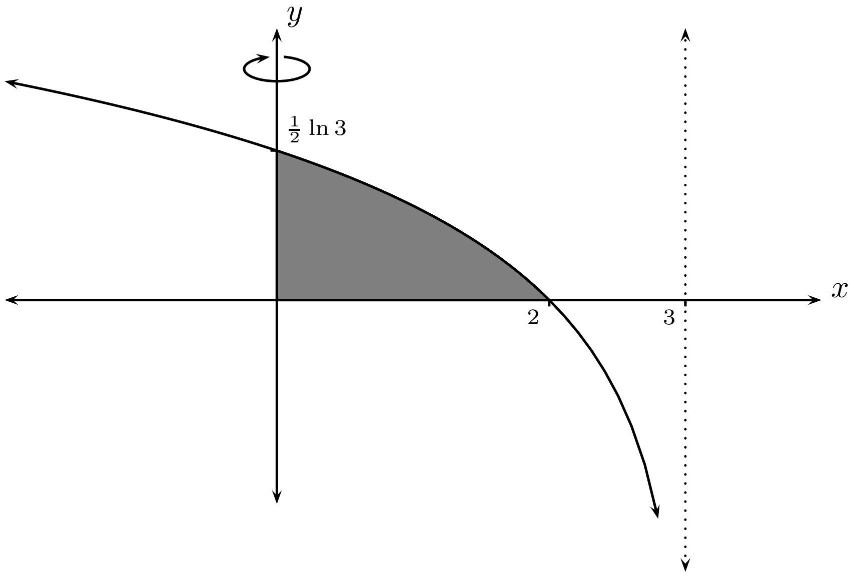
(ii) Find the area enclosed between the parabola and the line. **3**

(d) (i) Show that the x co-ordinates of the points of intersection of the line $y = mx$ and the circle $(x + 3)^2 + y^2 = 1$ satisfy the equation $(1 + m^2)x^2 + 6x + 8 = 0$. **1**

(ii) Hence find the equations of the two tangents to the circle $(x + 3)^2 + y^2 = 1$ that pass through the origin. **2**

QUESTION EIGHT (15 marks) Start a new leaflet. **Marks**

(a)



In the diagram above the shaded region is bounded by the curve $y = \frac{1}{2}\ln(3-x)$ and the co-ordinate axes.

(i) Find x as a function of y. **1**

(ii) Hence find the volume of the solid of revolution formed when the shaded region is rotated about the y-axis. **3**

(b) Consider the curve $y = (x-2)e^{-x}$.

(i) Find any x or y intercepts. **2**

(ii) Find any stationary points and determine their nature. **3**

(iii) Find any points of inflection. **3**

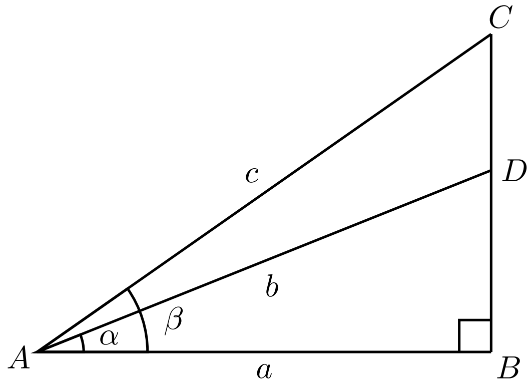
(iv) By considering the behaviour of the function for large values of x, determine the equation of the horizontal asymptote. **1**

(v) Sketch the curve $y = (x-2)e^{-x}$, clearly showing all the above information. **2**

Exam continues overleaf ...

QUESTION NINE (15 marks) Start a new leaflet. **Marks**

(a)



In the diagram above, $\angle BAD = \alpha$, $\angle BAC = \beta$ and sides AB, AD and AC have lengths a, b and c respectively.

(i) Write down an expression for the area of $\triangle ACD$ in terms of b, c, α and β. **1**

(ii) Hence show that $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$. **2**

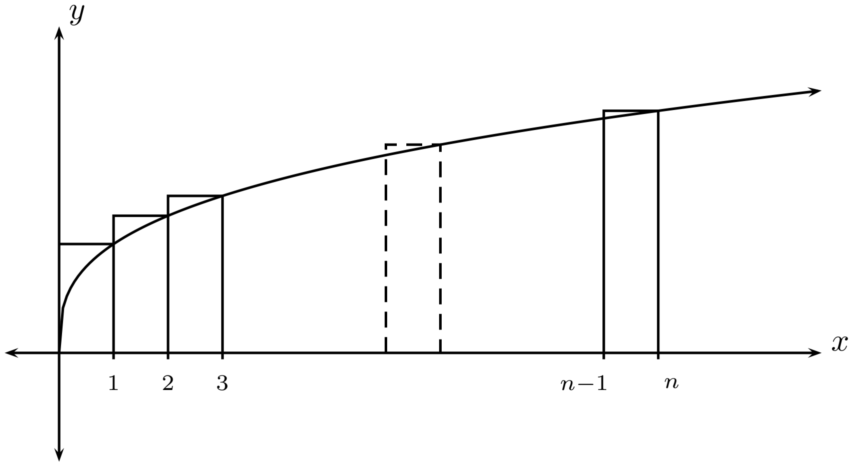
(b) (i) Differentiate $\sqrt{x}e^{\sqrt{x}}$. **1**

(ii) Hence find $\displaystyle\int e^{\sqrt{x}}\,dx$. **2**

(iii) Evaluate $\displaystyle\int_0^4 e^{\sqrt{x}}\,dx$. **1**

Exam continues next page ...

(c)



The diagram above shows the curve $y = \sqrt[3]{x}$, together with a set of n rectangles of unit width.

(i) By considering the areas of these rectangles, explain why **1**

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + ... + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\, dx\,.$$

(ii) By drawing another set of rectangles and considering their areas, show that **2**

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + ... + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\, dx\,.$$

(iii) Using parts (i) and (ii) find the value of $\displaystyle\sum_{n=1}^{100} \sqrt[3]{n}$, giving your answer accurate to as many significant figures as can be justified. **2**

(d) Given that e^x is the limiting sum of the infinite series **3**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$,

find an expression for the limiting sum of the infinite series

$$1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \cdots + \frac{(n+1)x^n}{n!} + \cdots$$

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

FIFTH FORM ANNUAL 2011

Question 1

a) $\sin 315° = (-\sin 45°) = (-\frac{1}{\sqrt{2}})$ ✓

b) $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{2}$ ✓

c) $x^2-7x+12 = (x-3)(x-4)$ ✓

d) $\log_3 \frac{1}{9} = (-2)$ ✓

e) (i) $y = x^5+2x$
$\frac{dy}{dx} = 5x^4+2$ ✓

(ii) $y = e^{2x+1}$
$\frac{dy}{dx} = 2e^{2x+1}$ ✓

f) $f(x) = x^3-2$
$F(x) = \frac{x^4}{4} - 2x + C$ ✓

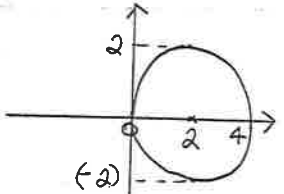
g) Minimum turning point ✓

h) $\ln e^3 = 3$ ✓

i) $x+2 > 0$
$x > (-2)$ ✓

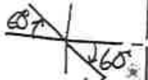
j) $2\log_{10} a + \log_{10} 3b$
$= \log_{10}(a^2 \times 3b) = \log_{10} 3a^2b$ ✓

k)



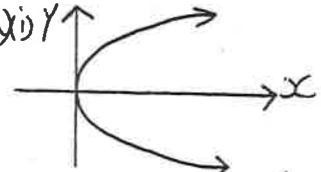
✓SHAPE
✓INTS

l) $\tan\theta = (-\sqrt{3})$
$\theta = 120°, 300°$ ✓✓



Question 2

a) (i)



✓ ORIENTATION & VERTEX

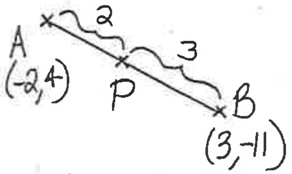
(ii) $S(1,0)$ ✓

b) $(x-2)(x+4) = 0$ ✓LHS
or $x^2+2x-8=0$ ✓RHS

c) $S_\infty = \frac{a}{1-r}$ $2 = \frac{1/2}{1-r}$ ✓
$1-r = \frac{1}{4}$
$r = 3/4$ ✓

d) (i) Point A ✓
(ii) Point C ✓

e) $A(-2,4)$ $B(3,-11)$ $2:3$



$x = \frac{2\times 3 + 3\times(-2)}{2+3}$ ✓
$= 0$

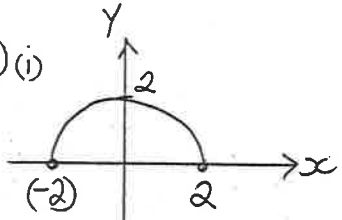
$Y = \frac{2\times(-11) + 3\times 4}{2+3}$
$= (-2)$

P is $(0,-2)$ ✓

f) (i) x^2+6x-1
$= (x+3)^2 - 10$ ✓✓

(ii) $V(-3,-10)$ ✓

2g) (i)



✓

(ii) $\int^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi r^2$
$= \frac{1}{2}\pi \times 2^2$
$= 2\pi$ ✓

Question 3

a) (i) $A(-1,-1)$ $C(5,3)$ ✓
$m_{AC} = \frac{3-(-1)}{5-(-1)} = \frac{4}{6} = \frac{2}{3}$ ✓

(ii) $\tan\theta = \frac{2}{3}$ $\theta = \tan^{-1}\frac{2}{3}$
$\doteqdot 34°$ ✓

(iii) $y+1 = \frac{2}{3}(x+1)$
$3y+3 = 2x+2$
$0 = 2x-3y-1$ ✓

(iv) $D\left(\frac{-1+5}{2}, \frac{-1+3}{2}\right) = (2,1)$ ✓

(v) $m_{BD} = \frac{7-1}{-2-2} = \frac{6}{-4} = -\frac{3}{2}$ SHOW ✓
$\therefore m_{AC} \times m_{BD} = \frac{2}{3} \times \left(-\frac{3}{2}\right) = (-1)$
$\therefore AC \perp BD$

(vi) $\triangle ABD \equiv \triangle CBD$ (SAS) ✓
hence $AB = CB$ (Corresponding sides in congruent triangles) ✓

OR

$\triangle ABC$ is Isosceles since median BD is also altitude

OR

SHOW $AB = BC = \sqrt{65}$

(vii) Area $= \frac{1}{2} \times AC \times BD$
$= \frac{1}{2} \times \sqrt{4^2+6^2} \times \sqrt{4^2+6^2}$ ✓
$= \frac{1}{2} \times 52$
$= 26u^2$ ✓

(viii) $B \to A$ $+1$, -8
hence $C \to E$ $+1$, -8
$E(6,-5)$ ✓

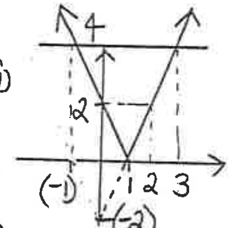
OR

Use D as midpoint of BE since diagonals of a rhombus bisect.

b) (i) $y = \log_e(2x-1)$
$\frac{dy}{dx} = \frac{1}{2x-1} \times 2$
$= \frac{2}{2x-1}$ ✓

(ii) $\left(\frac{dy}{dx}\right)_{x=3} = \frac{2}{6-1} = \frac{2}{5}$ ✓
Gradient of tangent at $x=3$ is $2/5$.

c) (i)



✓SHAPE
✓INTS

(ii) $x \leq (-1)$ ✓ or $x \geq 3$ ✓

Question 4

a)(i) $y=(5-2x)^7$

$\frac{dy}{dx}=7(5-2x)^6\times(-2)$ ✓✓

$=-14(5-2x)^6$

(ii) $y=\frac{x^4+2x^2}{x}$

$=x^3+2x$ ✓

$\frac{dy}{dx}=3x^2+2$ ✓

b)(i) $\int x^{\frac{3}{2}}dx=\frac{x^{5/2}}{5/2}+C$ ✓

$=\frac{2x^{5/2}}{5}+C$ ✓

(ii) $\int \frac{1}{2}x^{-3}dx=\frac{1}{2}\frac{x^{-2}}{-2}+C$ ✓

$=\frac{-1}{4x^2}+C$ ✓

(iii) $\int \frac{1}{x-3}dx=\ln|x-3|+C$ ✓

(iv) $\int e^{4x-2}dx=\frac{e^{4x-2}}{4}+C$ ✓

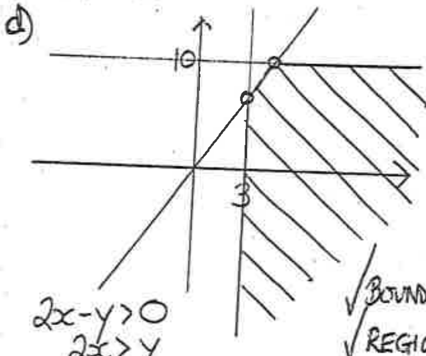
c) $\int_{-1}^{3}x(x-2)dx=\int_{-1}^{3}x^2-2x\,dx$

$=\left[\frac{x^3}{3}-x^2\right]_{-1}^{3}$ ✓

$=(9-9)-(-\frac{1}{3}-1)$

$=\frac{4}{3}$ ✓

d)



$2x-y>0$

$2x>y$

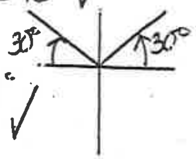
✓ BOUNDARIES
✓ REGION
✓ CORNERS

Question 5

a) $\sin(\alpha+45°)$ $\quad 0\le\alpha\le360°$

$45°\le\alpha+45°\le405°$ ✓

$\alpha+45°=150°, 390°$ ✓



$\alpha=105°, 345°$ ✓

b)(i) GP $\quad t_2=ar=16$ ①

$t_5=ar^4=2$ ②

② ÷ ①

$r^3=\frac{2}{16}=\frac{1}{8}$

$r=\frac{1}{2}$ ✓

(ii) $ar=16$

$\frac{a}{2}=16$

$a=32$ $\quad t_1=32$ ✓

(iii) $S_{15}=\frac{a(1-r^{15})}{1-r}$

$=\frac{32(1-(\frac{1}{2})^{15})}{\frac{1}{2}}$ ✓

$=64(1-(\frac{1}{2})^{15})=\frac{32767}{512}$ ✓

$(\doteqdot 63.998)$

Approx decimal earns one mark only.

c) $\int_0^2 |x-1|\,dx=0$ ✓

(since triangles of equal area but one above & one below x axis)

d) $\frac{2x-6}{x}\le1$ $\quad$ Domain $x\ne0$.

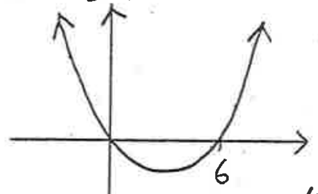
$(\times x^2)$

$(2x-6)x\le x^2$ ✓

$2x^2-6x-x^2\le0$

$x^2-6x\le0$

$x(x-6)\le0$



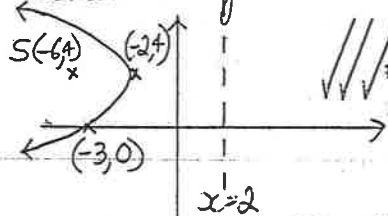
Hence $0<x\le6$ ✓✓

e)(i) $4a=16$

$a=4$. Focal length is 4 ✓

(ii) Vertex $(-2,4)$ $\quad\therefore$ Focus $(-6,4)$

Concave to left



✓✓✓ SHAPE, FOCUS & DIRECTRIX, INT.

Intercepts $y=0$ $\quad 16=-16(x+2)$

$(x+2)=-1$

$x=-3$

Question 6

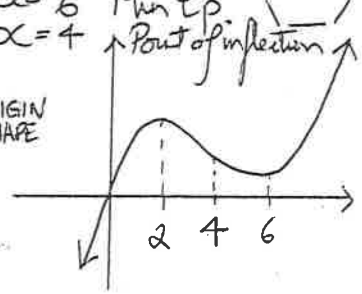
a) Stat pts @ $x=2$ and $x=6$

✓ $x=2$ Max t.p.

✓ $x=6$ Min t.p.

✓ $x=4$ Point of inflection

✓ ORIGIN & SHAPE



b)(i)

x	0	2	4	6	8
$f(x)$	$\ln 1=0$	$\ln 5$	$\ln 17$	$\ln 37$	$\ln 65$

$\int_0^8 f(x)dx\doteqdot\frac{1}{2}\times4[\ln1+\ln17]+\frac{1}{2}\times4[\ln17+\ln65]$ ✓

$\doteqdot2[2\ln17+\ln65]$

$\doteqdot19.7$ (1dp) ✓

(ii) $\frac{1}{6}\times4\times(0+4\ln5+\ln17)+\frac{1}{6}\times4(\ln17+4\ln37+\ln65)$ ✓

$\doteqdot20.481$ (3dp) ✓

c)(i) $y=\frac{\ln x}{x^2}$

$\frac{dy}{dx}=\frac{x^2(\frac{1}{x})-\ln x(2x)}{x^4}$ ✓

$=\frac{x-2x\ln x}{x^4}$

$=\frac{1-2\ln x}{x^3}$ ✓

(ii) $y = e^{2x}(2-x^2)$

$\frac{dy}{dx} = 2e^{2x}(2-x^2) + e^{2x}(-2x)$ ✓

$= 2e^{2x}[2-x-x^2]$ ✓

$= -2e^{2x}(x^2+x-2)$

$= -2e^{2x}(x+2)(x-1)$

d) $\int_{-1}^{0} \frac{4x}{x^2+3}dx = 2\int_{-1}^{0} \frac{2x}{x^2+3}dx$

$= [2\ln(x^2+3)]_{-1}^{0}$ ✓

$= 2\ln 3 - 2\ln 4$

$= 2\ln\frac{3}{4}$ or $-2\ln\frac{4}{3}$ ✓

e) $9^x - 10\times 3^x + 9 = 0$

$(3^x)^2 - 10\times 3^x + 9 = 0$

$(3^x - 9)(3^x - 1) = 0$ ✓

$3^x = 9$ or $3^x = 1$

$\therefore x = 2$ or $x = 0$ ✓

Question 7

a) $2x^2 + 4x - 1 = 0$

$a = 2 \quad b = 4 \quad c = (-1)$

(i) $\alpha\beta = c/a = (-\frac{1}{2})$ ✓

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$

$= \frac{-b/a}{+c/a}$

$= -b/c$

$= 4$ ✓

(iii) $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta$

$= (\alpha+\beta)^2 - 4\alpha\beta$ ✓

$= (-2)^2 - 4\times(-\frac{1}{2})$

$= 4 + 2 = 6$ ✓

(iv) $|\alpha-\beta| = \sqrt{(\alpha-\beta)^2}$

$= \sqrt{6}$ ✓

b)(i) AP $a = 105 \quad d = 7$

$t_n = a + (n-1)d$

$= 105 + 7(n-1)$ ✓

$t_n < 1000$

$105 + 7(n-1) < 1000$

$98 + 7n < 1000$

$7n < 902$

$n < \frac{902}{7}$

$n < 128\frac{6}{7}$

$t_{128} = 105 + 7\times 127$

$= 105 + 889$

$= 994$ ✓

$\therefore$ 128 multiples of 7 between 100 and 1000.

(ii) $S_{128} = \frac{n}{2}(a+l)$

$= \frac{128}{2}(105 + 994)$

$= 70336$ ✓

c) (i) $2x^2 - 4x - 6 = 8x - 16$

$2x^2 - 12x + 10 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1$ or $x = 5$ ✓

(ii) Area $= \int_1^5 (8x-16) - (2x^2-4x-6)\,dx$ ✓

$= \int_1^5 12x - 10 - 2x^2\,dx$

$= \left[6x^2 - 10x - \frac{2x^3}{3}\right]_1^5$ ✓

$= (150 - 50 - \frac{250}{3}) - (6 - 10 - \frac{2}{3})$

$= 104 - \frac{248}{3}$

$= 104 - 82\frac{2}{3}$

$= 21\frac{1}{3}u^2$ ✓

d)(i) $y = mx$ ①

$(x+3)^2 + y^2 = 1$ ②

sub ① into ②

$x^2 + 6x + 9 + m^2x^2 = 1$ show ✓

$(1+m^2)x^2 + 6x + 8 = 0$ ✓

$\Delta = 36 - 4\times(1+m^2)\times 8$

$= 36 - 32(1+m^2)$

$= 4 - 32m^2$ ✓

For tangency $\Delta = 0$

$m^2 = 1/8$

$m = \pm\frac{1}{2\sqrt{2}}$

$\therefore$ Eqns of tangents ✓

$y = \frac{\sqrt{2}x}{4}$ or $y = -\frac{\sqrt{2}x}{4}$

Question 8

a)(i) $y = \frac{1}{2}\ln(3-x)$ ①

$V_y = \pi\int x^2\,dy$

from ① $2y = \ln(3-x)$

$e^{2y} = (3-x)$

$x = 3 - e^{2y}$ ✓

(ii) $x^2 = 9 - 6e^{2y} + e^{4y}$

$V = \pi\int_0^{\frac{1}{2}\ln 3} 9 - 6e^{2y} + e^{4y}\,dy$ ✓

$= \pi\left[9y - \frac{6e^{2y}}{2} + \frac{e^{4y}}{4}\right]_0^{\frac{1}{2}\ln 3}$ ✓

$= \pi[(\frac{9}{2}\ln 3 - 9 + \frac{9}{4}) - (0 - 3 + \frac{1}{4})]$

$= \pi(\frac{9}{2}\ln 3 - 4)\ u^3$ ✓

b) $y = (x-2)e^{-x}$

(i) y ints. $x = 0$

$y = -2\times e^{-0} = -2 \quad (0,-2)$ ✓

x into $y = 0$

$e^{-x} \neq 0 \therefore x - 2 = 0$

$x = 2$

$(2,0)$ ✓

(ii) $\frac{dy}{dx} = -e^{-x}(x-2) + 1\times e^{-x}$

$= e^{-x}[3-x]$ ✓

Stat pts $\frac{dy}{dx} = 0$.

$e^{-x} \neq 0 \quad \therefore x = 3 \quad (3, 1/e^3)$ ✓

$y = 1/e^3$

$\frac{d^2y}{dx^2} = -e^{-x}(3-x) + e^{-x}(-1)$

$= e^{-x}[x-4]$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=3} = -1/e^3 < 0 \therefore$ Max turning pt @ $(3, 1/e^3)$ ✓

(iii) Possible POI $\frac{d^2y}{dx^2} = e^{-x}(x-4)$ ✓

$\frac{d^2y}{dx^2} = 0 \quad e^{-x} \neq 0 \quad \therefore x = 4$ ✓

$y = \frac{2}{e^4}$

x	3	4	5
$\frac{d^2y}{dx^2}$	$-\frac{1}{e^3}$	0	$\frac{1}{e^5}$
	∩	•	∪

✓

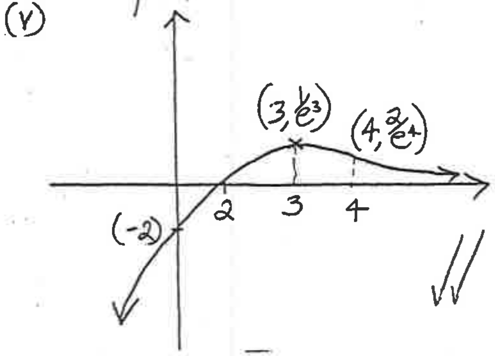
There is a change in concavity hence $(4, \frac{2}{e^4})$ is a point of inflection

(iv) $y = (x-2)e^{-x}$

as $x \to \infty \quad xe^{-x} \to 0^+$

$\therefore$ (x axis) $y = 0$ is an asymptote ✓

(v)



✓✓

Question 9

a) i) Area $\triangle ACD = \frac{1}{2}bc\sin(ACD)$

$= \frac{1}{2}bc\sin(\beta - \alpha)$ ✓

OR

Area $\triangle ABD = \frac{1}{2}ab\sin\alpha$

Area $\triangle ABC = \frac{1}{2}ac\sin\beta$.

$\therefore$ Area $\triangle ACD = \frac{1}{2}ac\sin\beta - \frac{1}{2}ab\sin\alpha$ ①

Equating area expressions

$\frac{1}{2}bc\sin(\beta-\alpha) = \frac{1}{2}ac\sin\beta - \frac{1}{2}ab\sin\alpha$

in $\triangle ABD$

$\cos\alpha = \frac{a}{b}$

$\therefore a = b\cos\alpha$ ② ✓

in $\triangle ABC$

$\cos\beta = \frac{a}{c}$

$\therefore a = c\cos\beta$ ③

sub ② and ③ into ①

$\frac{1}{2}bc\sin(\beta-\alpha) = \frac{1}{2}(b\cos\alpha)c\sin\beta - \frac{1}{2}(c\cos\beta)b\sin\alpha$ ✓

$(\div \frac{1}{2}bc)$

$\sin(\beta-\alpha) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$

b) $y = \sqrt{x}e^{\sqrt{x}}$

$= x^{\frac{1}{2}}e^{x^{\frac{1}{2}}}$

(i) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}} + x^{\frac{1}{2}} \times \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{e^{\sqrt{x}}}{2}$ (*) ✓

(ii) $\int e^{\sqrt{x}}\,dx$ Hence integrate result (*)

$\sqrt{x}e^{\sqrt{x}} + C = \int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx + \int \frac{e^{\sqrt{x}}}{2}\,dx$ ✓

$\sqrt{x}e^{\sqrt{x}} + C = e^{\sqrt{x}} + \frac{1}{2}\int e^{\sqrt{x}}\,dx$

$\therefore \frac{1}{2}\int e^{\sqrt{x}}\,dx = e^{\sqrt{x}}(\sqrt{x}-1) + C$ ✓

$\int e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}(\sqrt{x}-1) + d$ ✓

(iii) $\int_0^4 e^{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}(\sqrt{x}-1)\right]_0^4$

$= 2e^2 + 2 = 2(e^2+1)$ ✓

c) Area under cube root curve is OVERestimated by upper rectangles since $\sqrt[3]{x}$ is concave down.

Height of each rectangle is x co-ord at right hand edge cube rooted, width of each rectangle is one

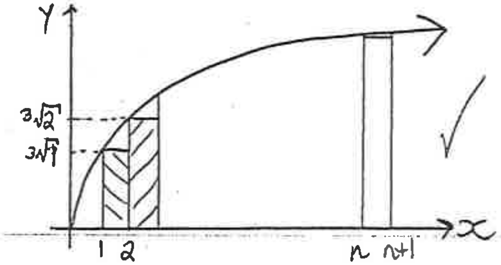
ie first rectangle $A_1 = 1 \times \sqrt[3]{1}$

last rectangle $A_n = 1 \times \sqrt[3]{n}$ ✓

hence

$\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx$

(ii) This set of rectangles must underestimate area under curve hence take height as cube root of x co-ordinate at left hand end of rectangle to get lower rectangles as illustrated below ✓



✓

hence

$\int_1^{n+1} \sqrt[3]{x}\,dx > \sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}$

note change to limits of integral $1 \to n+1$

(iii) $\int_0^{100} \sqrt[3]{x}\,dx < \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{100} < \int_1^{101} \sqrt[3]{x}\,dx$

$\left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^{100} < \sum_{n=1}^{100}\sqrt[3]{n} < \left[\frac{3x^{\frac{4}{3}}}{4}\right]_1^{101}$ ✓

$\frac{3}{4}(100^{\frac{4}{3}}) < \sum_{n=1}^{100}\sqrt[3]{n} < \frac{3}{4}(101^{\frac{4}{3}} - 1)$

$348.1\ldots < \sum_{n=1}^{100}\sqrt[3]{n} < 352.0\ldots$

$\sum_{n=1}^{100}\sqrt[3]{n} \doteqdot 350$ (2sf) ✓

(Must consider both bounds for two marks)

d) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

$xe^x = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots + \frac{x^{n+1}}{n!} + \ldots$

$\frac{d}{dx}(xe^x) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots + \frac{(n+1)x^n}{n!} + \ldots$ ✓

LHS $= xe^x + 1 \times e^x$

$= e^x(x+1)$

Hence required result is $e^x(x+1)$ ✓