NAME MASTER

SYDNEY GRAMMAR SCHOOL

2016 Annual Examination

FORM V
MATHEMATICS EXTENSION 1

Monday 5th September 2016

General Instructions
- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks
- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your name, class and Master on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your name, class and Master on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

5A: DNW	5B: PKH	5C: LRP	5D: FMW
5E: WJM	5F: GMC	5G: NL	5H: SO

Checklist
- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 141 boys

Examiner
FMW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

What is the value of $20e^{-2}$, correct to 3 significant figures?

(A) 2·71

(B) 2·70

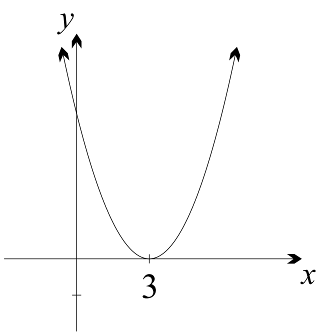
(C) 2·707

(D) 2·706

QUESTION TWO

Which graph below best represents $y = (x+3)^2$?

A



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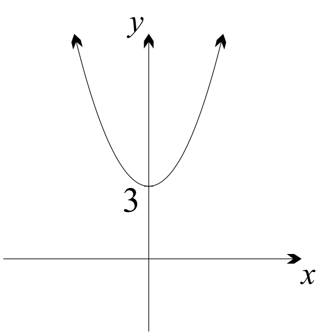

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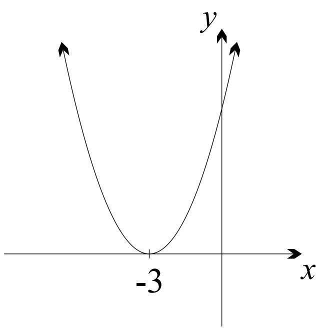
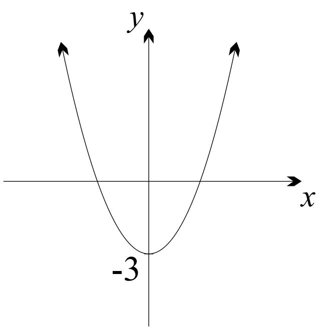


D



Examination continues next page ...

QUESTION THREE

What are the solutions of $3x^2 - 6x + 2 = 0$?

(A) $x = \dfrac{3 \pm \sqrt{15}}{3}$

(B) $x = \dfrac{-3 \pm \sqrt{3}}{3}$

(C) $x = 1 \pm 2\sqrt{3}$

(D) $x = \dfrac{3 \pm \sqrt{3}}{3}$

QUESTION FOUR

Given $y = \dfrac{x^2}{x-1}$, then $\dfrac{dy}{dx}$ is equal to:

(A) $2x$

(B) $\dfrac{x-2}{x}$

(C) $\dfrac{x(2-x)}{(x-1)^2}$

(D) $\dfrac{x(x-2)}{(x-1)^2}$

QUESTION FIVE

In relation to the function $y = |2x - 3|$, which of the following statements is NOT true?

(A) The domain of the function is all real x.

(B) The range of the function is $y \geq 0$.

(C) The function is continuous for all values of x.

(D) The function is differentiable for all values of x.

Examination continues overleaf ...

QUESTION SIX

What is the solution of $7^x = 2$?

(A) $x = \log_2 7$

(B) $x = \log_{10} \frac{2}{7}$

(C) $x = \dfrac{\log_{10} 2}{\log_{10} 7}$

(D) $x = \dfrac{\log_e 7}{\log_e 2}$

QUESTION SEVEN

Which of the following is a primitive of $\dfrac{1}{x\sqrt{x}}$?

(A) $\log_e x\sqrt{x}$

(B) $-\dfrac{2}{\sqrt{x}}$

(C) $-\dfrac{3}{2}x^{-\frac{5}{2}}$

(D) $-\dfrac{2}{5}x^{-\frac{5}{2}}$

QUESTION EIGHT

How many solutions does the equation $\cos 2\theta = 1$ have in the domain $0^\circ \leq \theta \leq 360^\circ$?

(A) 1

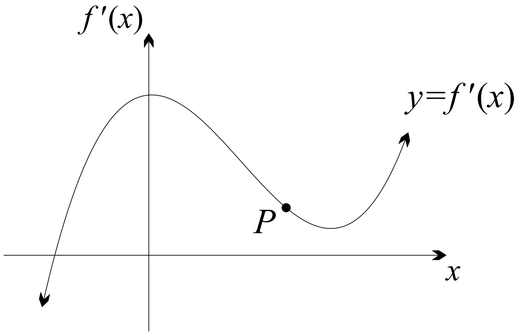
(B) 2

(C) 3

(D) 4

Examination continues next page ...

QUESTION NINE



The GRADIENT FUNCTION $y = f'(x)$ of a curve is sketched above. Which of the following statements is true at the point P?

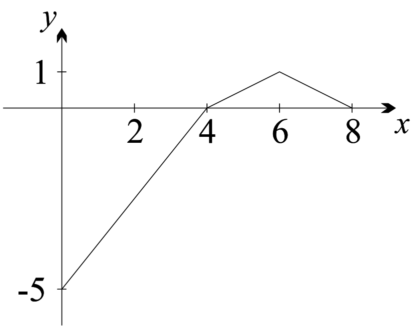
(A) $f'(x) < 0$ and $f''(x) > 0$

(B) $f'(x) > 0$ and $f''(x) < 0$

(C) $f'(x) > 0$ and $f''(x) > 0$

(D) $f'(x) < 0$ and $f''(x) < 0$

QUESTION TEN



The function $y = f(x)$ is sketched above. Which of the following integrals yields the greatest value?

(A) $\displaystyle\int_0^2 f(x)\,dx$

(B) $\displaystyle\int_0^4 f(x)\,dx$

(C) $\displaystyle\int_0^6 f(x)\,dx$

(D) $\displaystyle\int_0^8 f(x)\,dx$

End of Section I

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Expand and simplify $(2\sqrt{3}-5)(2\sqrt{3}+5)$. **1**

(b) Write down the equation of the line which is parallel to $3x - y + 7 = 0$ and has a y-intercept of -5. **1**

(c) Write down a quadratic equation with roots 5 and -2. **1**

(d) Simplify:

(i) $\log_e e^3$ **1**

(ii) $e^{\log_e 5}$ **1**

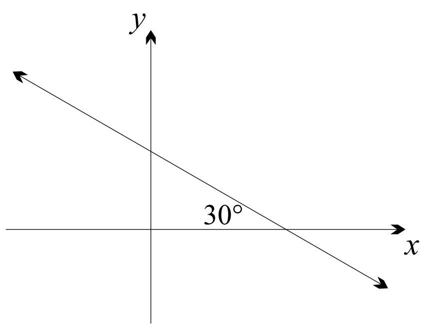
(e) Write down the equations of the vertical asymptotes of the graph of the function $f(x) = \dfrac{1}{x^2 - 16}$. **1**

(f) Solve $\cos\theta = \dfrac{1}{\sqrt{2}}$, for $0° \le \theta \le 360°$. **1**

(g) Find the gradient of the interval joining the points $(1, 2)$ and $(-2, -4)$. **1**

(h) Find the sum of the first 100 terms of the arithmetic sequence $12, 6, 0, \ldots$. **2**

(i) **1**



Write down the gradient of the line sketched above.

(j) Find the gradient of the tangent to the curve $y = e^{2x+1}$ at the point on the curve where $x = -1$. **2**

(k) (i) Sketch $y = e^x$. **1**

(ii) Evaluate $\displaystyle\int_0^1 e^x \, dx$. **1**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

(i) $y = \dfrac{3}{x}$ **1**

(ii) $y = \ln(5x - 1)$ **1**

(iii) $y = xe^x$ **1**

(b) Find:

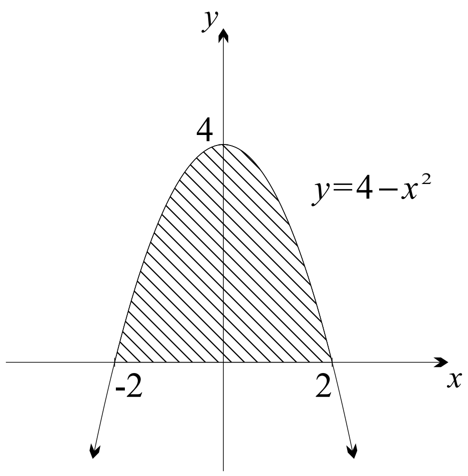
(i) $\displaystyle\int (2x-5)^4\,dx$ **1**

(ii) $\displaystyle\int \frac{2}{x-1}\,dx$ **1**

(iii) $\displaystyle\int \frac{x-1}{x^2}\,dx$ **2**

(c) Find the equation of the curve passing through the point $(-1, 4)$ with gradient function $\dfrac{dy}{dx} = 3x^2 + 4x - 5$. **3**

(d) **2**

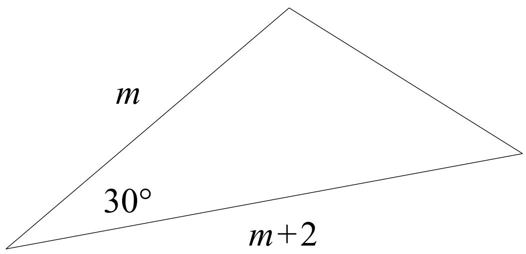


Find the area bounded by the curve $y = 4 - x^2$ and the x-axis, as shaded above.

(e) Find the values of k for which the quadratic equation $x^2 + kx + 36 = 0$ has two distinct, real roots. **3**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) $\boxed{\mathbf{3}}$



The area of the triangle above is 12 square units. Find the value of m.

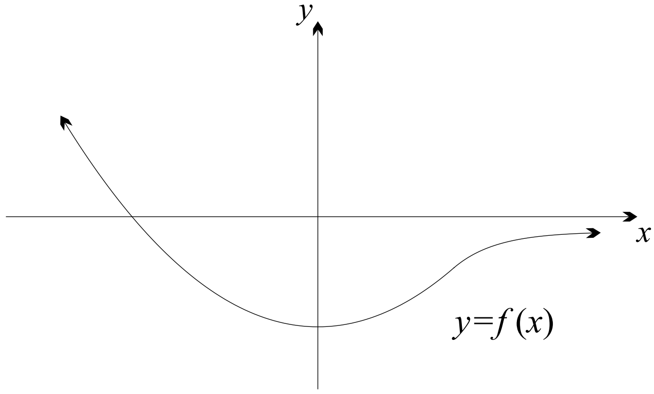
(b) (i) Copy and complete the following table of values for $y = \log_e x$, giving your answers correct to 2 decimal places where necessary. $\boxed{\mathbf{1}}$

x	1	2	3
y			

(ii) Use the trapezoidal rule with all the values from your table to find an approximation for $\displaystyle\int_1^3 \log_e x\,dx$. $\boxed{\mathbf{2}}$

(iii) Explain, with the use of a diagram, whether the approximation in (ii) is less than or greater than the exact value. $\boxed{\mathbf{1}}$

(c) $\boxed{\mathbf{2}}$



The diagram above shows the curve $y = f(x)$. Sketch a possible graph of $y = f'(x)$.

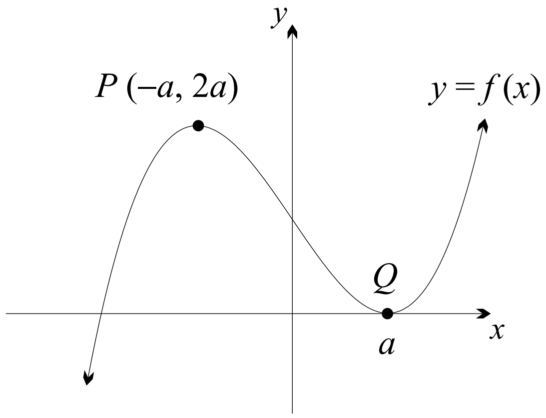
Examination continues next page ...

(d) (i) Given the function $g(x) = 2x^3 - 3x^2 - 12x + 4$, find $g'(x)$ and $g''(x)$. $\boxed{1}$

(ii) For what values of x is $y = g(x)$ decreasing? $\boxed{2}$

(iii) For what values of x is $y = g(x)$ concave up? $\boxed{1}$

(e)



The graph of $y = f(x)$ is sketched above with points P and Q as marked.

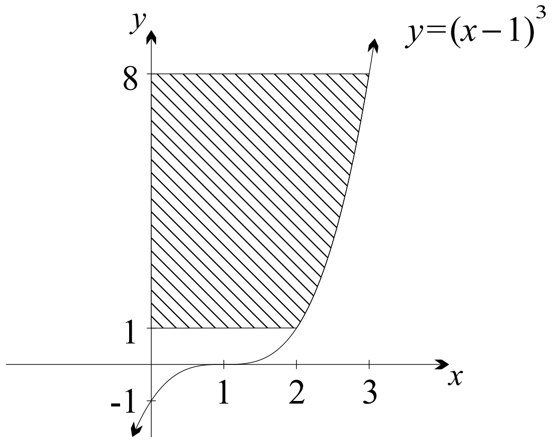
(i) Write down the coordinates of point P under the transformation $y = -2f(x)$. $\boxed{1}$

(ii) Write down the coordinates of point Q under the transformation $y = f(-x)$. $\boxed{1}$

Examination continues overleaf ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) **3**



The diagram above shows the curve $y = (x-1)^3$. Find the shaded area.

(b) Let α and β be the roots of the quadratic equation $2x^2 - 4x + 7 = 0$. Find:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ **2**

(c) Show algebraically that the curve $y = (2-x)^3$ has a point of inflexion at $(2, 0)$. **2**

(d) Find the coordinates of any stationary points on the curve $y = x^2 + \dfrac{16}{x}$ and determine their nature. **3**

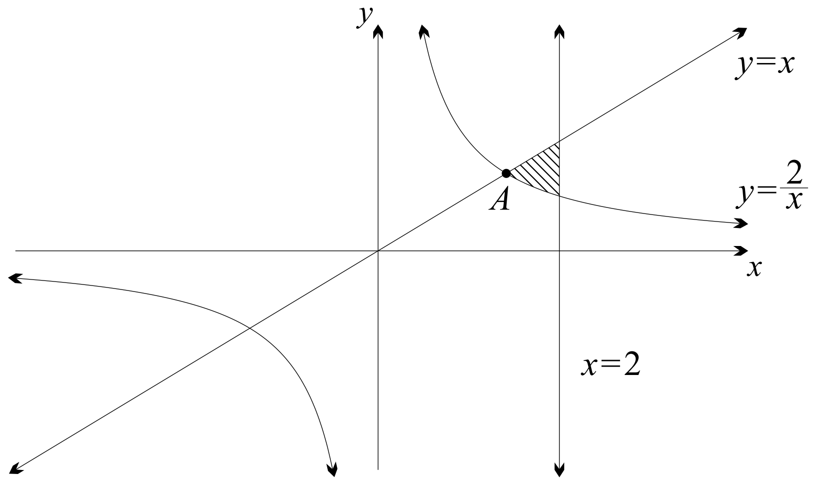
(e) Use a suitable substitution to solve the equation $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$. **3**

Examination continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Factorise completely $3b^3 - 24$. **2**

(b)



The graphs of $y = x$, $y = \dfrac{2}{x}$ and $x = 2$ are sketched above.

(i) Show that A has coordinates $(\sqrt{2}, \sqrt{2})$. **1**

(ii) Show that the area of the region bounded by the hyperbola and the two lines, shaded above, is $1 - \ln 2$. **2**

(iii) Find the volume of the solid formed when the region is rotated around the x-axis. **3**

(c) Solve $\sin x \tan x - 4\sin x - \tan x + 4 = 0$, for $0^\circ \le x \le 360^\circ$. Give your answer correct to the nearest degree, where necessary. **3**

(d) A geometric series has first term 35 and common ratio 2^b.

(i) For what values of b does the series have a limiting sum? **1**

(ii) Find the value of b for which the limiting sum is equal to 40. **2**

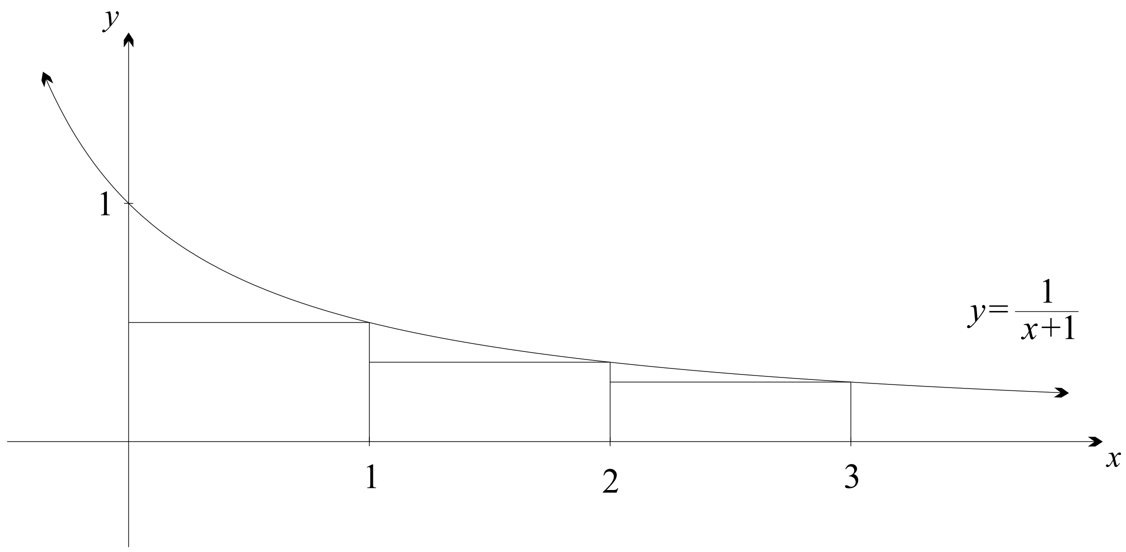
(e) Determine the range of the function $y = \ln(x^2 + e)$. **1**

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The tangent at P on the hyperbola $xy = 4$ cuts the x and y axes at the points A and B respectively. Find all points P such that $OA : OB = 2 : 1$. **2**

(b) Consider the quadratic $y = mx^2 + nx + mn^2$, where m and n are non-zero real numbers. For what values of m and n is the quadratic positive definite? **3**

(c)



The diagram above shows part of the curve $y = \dfrac{1}{x+1}$ together with three rectangles of unit width.

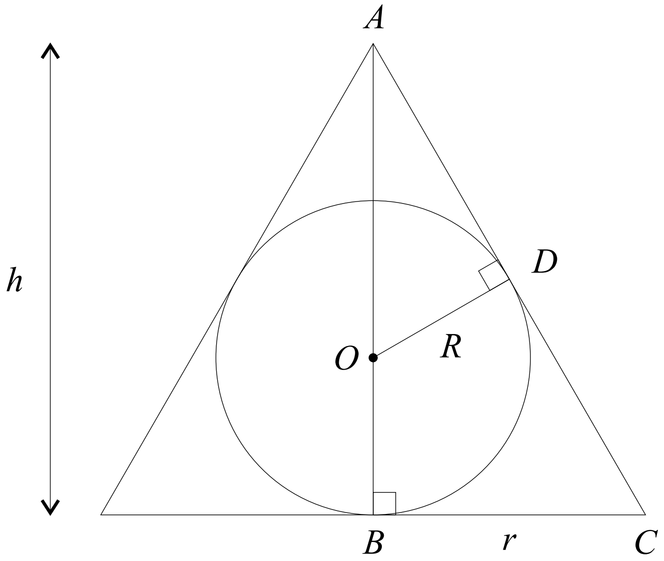
(i) Explain how the diagram shows that $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} < \displaystyle\int_0^3 \frac{1}{x+1}\,dx$. **1**

(ii) The curve $y = \dfrac{1}{x+2}$ passes through the top left-hand corner of each of the three rectangles shown. By considering the rectangles drawn above in relation to this curve, write down a second inequality involving $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}$ and a different definite integral. **1**

(iii) By similar arguments show that $3{\cdot}92 < \displaystyle\sum_{r=2}^{100} \frac{1}{r} < 4{\cdot}61$. **2**

Examination continues next page ...

(d)



A cone of height h and radius r is circumscribed about a sphere with centre O and fixed radius R. The diagram above shows a cross-section of the situation through an axis of the cone AB.

(i) Show that $r^2 = \dfrac{R^2 h}{h - 2R}$. **2**

(ii) Show that the volume of the cone is given by $V = \dfrac{\pi R^2 h^2}{3(h - 2R)}$. **1**

(iii) Find the dimensions of the cone with the least volume. You need not test that a minimum value occurs. **2**

(iv) Given the cone has minimum volume, find the ratio of the volume of the cone to the volume of the sphere. **1**

End of Section II

END OF EXAMINATION

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Name: ....

Class: ....Master: ....

Sydney Grammar School

2016
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FORM V
MATHEMATICS EXTENSION 1
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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Ext 1 Solutions

MC

① $20e^{-2} = 2.706\ldots$
$= 2.71 \quad (2\text{ d.p.})$.

$\boxed{A}$

② $\boxed{C}$

③ $3x^2 - 6x + 2 = 0$

$$x = \frac{6 \pm \sqrt{6^2 - 4(3)(2)}}{2\times 3}$$
$$= \frac{6 \pm \sqrt{12}}{6}$$
$$= \frac{6 \pm 2\sqrt{3}}{6}$$
$$= \frac{3 \pm \sqrt{3}}{3}$$

$\boxed{D}$

④ $y = \dfrac{x^2}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)\times 2x - x^2 \times 1}{(x-1)^2}$$
$$= \frac{2x^2 - 2x - x^2}{(x-1)^2}$$
$$= \frac{x^2 - 2x}{(x-1)^2}$$
$$= \frac{x(x-2)}{(x-1)^2}$$

$\boxed{D}$

5 $\boxed{D}$

6 $7^x = 2$

$\boxed{C}$

$x = \log_7 2$

$= \dfrac{\log_{10} 2}{\log_2 7}$

7 $\int \frac{1}{x\sqrt{x}}\,dx = \int \frac{1}{x \times x^{\frac{1}{2}}}\,dx$

$= \int x^{-\frac{3}{2}}\,dx$

$= \dfrac{x^{-\frac{1}{2}}}{-\frac{1}{2}}$

$\boxed{B}$

$= -\dfrac{2}{\sqrt{x}} + c$

8

3 solutions

$\boxed{C}$

9 $\boxed{B}$

10

$$\int_0^2 f(x)\,dx = -\frac{1}{2} \times 2 \times (5 + 2\tfrac{1}{2})$$

$$= -7\tfrac{1}{2}$$

*

$$\int_0^4 f(x)\,dx = -\frac{1}{2} \times 5 \times 4$$

$$= -10$$

$$\int_0^6 f(x)\,dx = -10 + \frac{1}{2} \times 2 \times 1$$

$$= -9$$

$$\int_0^8 f(x)\,dx = -10 + \frac{1}{2} \times 4 \times 1$$

$$= -8$$

$\boxed{A}$

(11)

(a) $(2\sqrt{3}-5)(2\sqrt{3}+5) = 4\times 3 - 5\times 5$
$= -13$

(b) $3x - y + 7 = 0$
$y = 3x + 7$

equation: $y = 3x - 5$

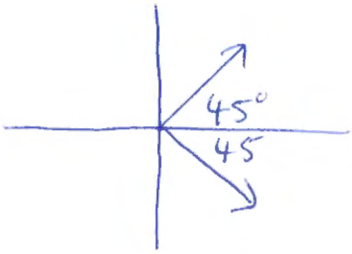
(c) $(x-5)(x+2) = 0$

or $x^2 - (5-2)x + (5\times -2) = 0$
$x^2 - 3x - 10 = 0$

(d) (i) $\log_e e^3 = 3$
(ii) $e^{\log_e 5} = 5$

(e) $x = 4$ and $x = -4$

(f) $\cos\theta = \frac{1}{\sqrt{2}}$



$\theta = 45°$ or $315°$

(g) $m = \frac{2-(-4)}{1-(-2)}$
$= \frac{6}{3}$
$= 2$

(h) $a = 12$, $d = -6$, $n = 100$

$$S_{100} = \frac{100}{2}\left(2(12) + 99 \times -6\right)$$

$$= -28500$$

(i) $m = -\tan 30^\circ$

$$= -\frac{1}{\sqrt{3}}$$

(j) $y = e^{2x+1}$

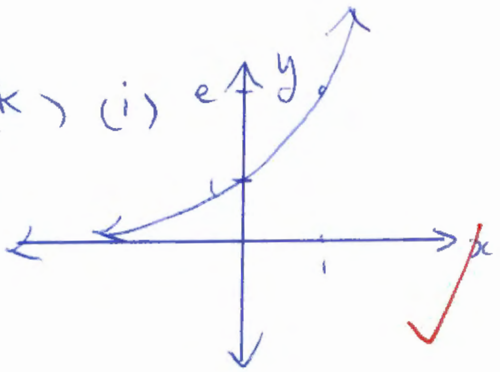
$y' = 2e^{2x+1}$

at $x = -1$, $y' = 2e^{2(-1)+1}$

$$= 2e^{-1}$$

$$= \frac{2}{e}$$

(k) (i)



(ii) $\int_0^1 e^x\,dx = \left[e^x\right]_0^1$

$$= e^1 - e^0$$

$$= e - 1$$

15

(12)

(a) (i) $y = 3x^{-1}$

$y' = -3x^{-2}$

$= \frac{-3}{x^2}$

(ii) $y = \ln(5x-1)$

$y' = \frac{5}{5x-1}$

(iii) $y = xe^x$

$= x \times e^x + 1 \times e^x$

$= xe^x + e^x$

$= e^x(x+1)$

(b) (i) $\int (2x-5)^4 \, dx = \frac{(2x-5)^5}{5 \times 2}$

$= \frac{1}{10}(2x-5)^5 + c$

(ii) $\int \frac{2}{x-1} \, dx = 2\log_e(x-1) + c$

(iii) $\int \frac{x-1}{x^2} \, dx = \int \frac{1}{x} - \frac{1}{x^2} \, dx$

$= \int \frac{1}{x} - x^{-2} \, dx$

$= \ln x + x^{-1}$

$= \ln x + \frac{1}{x} + c$

(c) $\frac{dy}{dx} = 3x^2 + 4x - 5$

$y = x^3 + 2x^2 - 5x + c$

substituting $(-1, 4)$

$4 = (-1)^3 + 2(-1)^2 - 5(-1) + c$

$c = -2$

so $y = x^3 + 2x^2 - 5x - 2$

(d) $A = 2\int_0^2 (4 - x^2)\,dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\left(4(2) - \frac{2^3}{3} - 0\right)$

$= \frac{32}{3}$ $\left(10\frac{2}{3}\right)$ u^2

(e) $x^2 + kx + 36 = 0$

$\Delta = k^2 - 4(1)(36)$

$= k^2 - 144$

If $\Delta > 0$, $k^2 - 144 > 0$

$k^2 > 144$

$k < -12$ or $k > 12$

15

Q13

(a) $\frac{1}{2} \times m \times (m+2) \times \sin 30^\circ = 12$

$\frac{1}{2} m(m+2) \times \frac{1}{2} = 12$

$m^2 + 2m = 48$

$m^2 + 2m - 48 = 0$

$(m+8)(m-6) = 0$

$m = -8$ or $m = 6$

as $m > 0$ choose $m = 6$

(b)

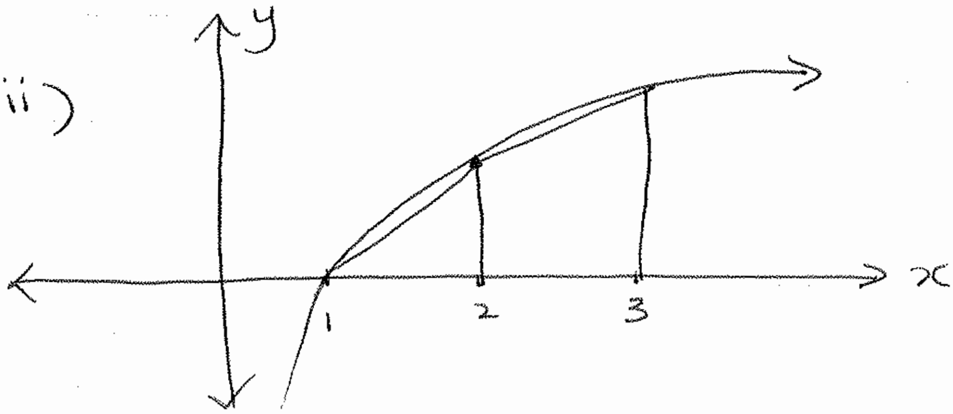
(i)

x	1	2	3
y	0	0.69	1.10

(ii) $\int_1^3 \log_e x \, dx \doteqdot \frac{1}{2} \times 1 (0 + 0.69) + \frac{1}{2} \times 1 (0.69 + 1.10)$

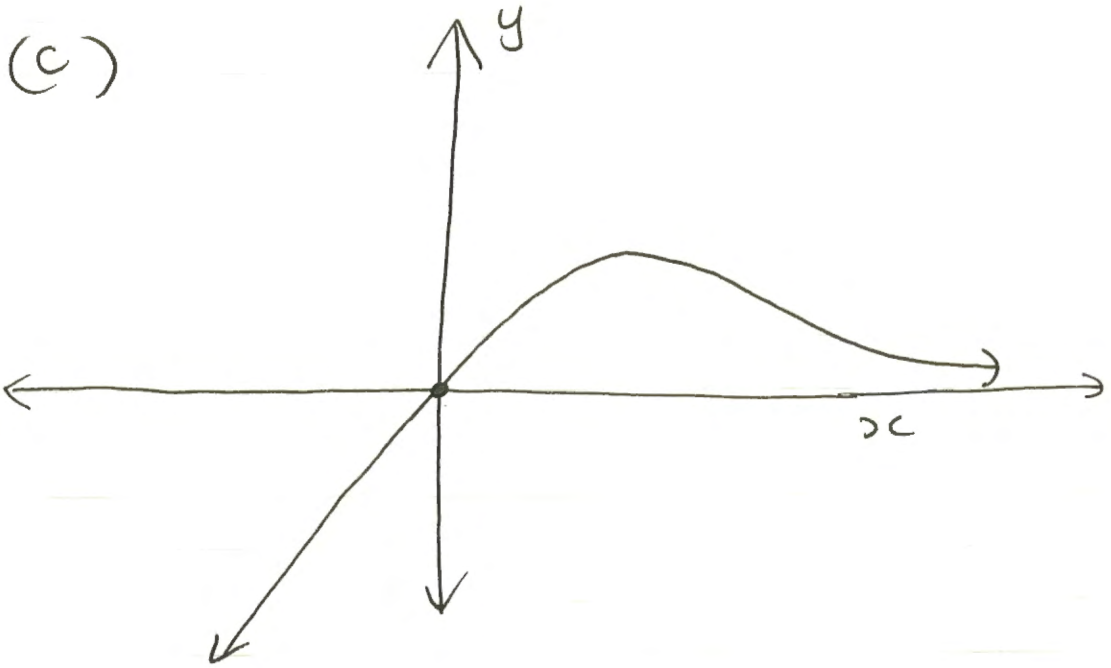
$= 1.24$

(iii)



The triangle & trapezium underestimate the area as the curve is concave down. So the approximation will be less than the exact value.

(c)



(d)(i) $f(x) = 2x^3 - 3x^2 - 12x + 4$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

| chaser ✓ (both)

(ii) decreasing when $f'(x) < 0$ ✓

that is, $6x^2 - 6x - 12 < 0$

$x^2 - x - 2 < 0$

$(x-2)(x+1) < 0$

$-1 < x < 2$ ✓

(iii) concave up when $f''(x) > 0$ ✓

that is $12x - 6 > 0$

$12x > 6$

$x > \frac{1}{2}$ ✓

(e) (i) $(-a, -4a)$ ✓

(ii) $(-a, 0)$ ✓

(14)

(a) $y = (x-1)^3$

$\sqrt[3]{y} = x - 1$

$x = \sqrt[3]{y} + 1$

$$A = \int_1^8 (\sqrt[3]{y} + 1)\,dy$$

$$= \int_1^8 (y^{\frac{1}{3}} + 1)\,dy$$

$$= \left[\frac{3}{4}y^{\frac{4}{3}} + y\right]_1^8$$

$$= \left(\frac{3}{4} \times 8^{\frac{4}{3}} + 8\right) - \left(\frac{3}{4} \times 1 + 1\right)$$

$$= \frac{73}{4}$$

$$= 18\tfrac{1}{4}$$

(b) $2x^2 - 4x + 7 = 0$

(i) $\alpha + \beta = \frac{4}{2}$ (ii) $\alpha\beta = \frac{7}{2}$

$= 2$

(iii) $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{2^2 - 2 \times \frac{7}{2}}{\frac{7}{2}}$$

$$= -\frac{6}{7}$$

(c) $y = (2-x)^3$

$y' = 3(2-x)^2 \times -1$

$= -3(2-x)^2$

$y'' = -6(2-x)^1 \times -1$

$= 6(2-x)$

$y'' = 0$ at $x = 2$, $y = 0$

possible point of inflexion at $(2,0)$

test:

x	1	2	3
y''	6	0	-6
	$\cup$		$\cap$

there is a concavity change so there is a point of inflexion at $(2,0)$

(d) $y = x^2 + \dfrac{16}{x}$

$= x^2 + 16x^{-1}$

$y' = 2x - 16x^{-2}$

$= 2x - \dfrac{16}{x^2}$

$= \dfrac{2x^3 - 16}{x^2}$

$= \dfrac{2(x^3-8)}{x^2}$

$y'' = 2 + \dfrac{32}{x^3}$

$y' = 0$ at $x^3 = 8$

$x = 2$

$y = 12$

$y'' = 6$

> 0

there is a minimum turning point at $(2,12)$

(e) $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$

let $m = x^{\frac{1}{3}}$

the equation becomes

$m^2 + m - 6 = 0$

$(m + 3)(m - 2) = 0$

$m = -3$ or $m = 2$

$x^{\frac{1}{3}} = -3$ $\quad$ $x^{\frac{1}{3}} = 2$

$x = -27$ $\quad$ $x = 8$

(15)

(a) $3b^3 - 24 = 3(b^3 - 8)$

$= 3(b-2)(b^2 + 2b + 4)$

(b)(i) if $\frac{2}{x} = x$

$x^2 = 2$

$x = \sqrt{2} \quad (x > 0)$

$y = \sqrt{2}$

So A is the pt $(\sqrt{2}, \sqrt{2})$

(ii) $A = \int_{\sqrt{2}}^{2} (x - \frac{2}{x})\, dx$

$= \left[\frac{x^2}{2} - 2\ln x \right]_{\sqrt{2}}^{2}$ } either

$= \frac{2^2}{2} - 2\ln 2 - \left(\frac{(\sqrt{2})^2}{2} - 2\ln(\sqrt{2}) \right)$

$= 2 - 2\ln 2 - 1 + 2\ln 2^{\frac{1}{2}}$

$= 1 - 2\ln 2 + \ln 2$

$= 1 - \ln 2$

(iii) $V = \pi \int_{\sqrt{2}}^{2} x^2 - \left(\frac{2}{x}\right)^2 dx$

$= \pi \int_{\sqrt{2}}^{2} x^2 - \frac{4}{x^2}\, dx$

$= \pi \left[\frac{x^3}{3} + \frac{4}{x} \right]_{\sqrt{2}}^{2}$

$= \pi \left(\frac{2^3}{3} + \frac{4}{2} - \left(\frac{\sqrt{2}^3}{3} + \frac{4}{\sqrt{2}} \right) \right)$

$= \pi \left(\frac{14}{3} - \frac{2\sqrt{2}}{3} - 2\sqrt{2} \right)$

$= \frac{\pi}{3}(14 - 8\sqrt{2})\ u^3$

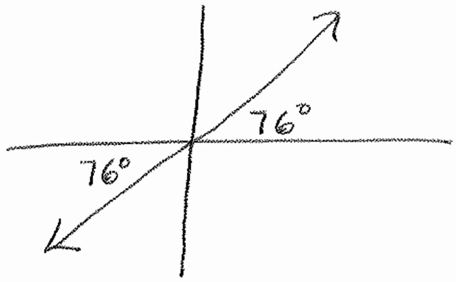
(c) $\sin x \tan x - 4\sin x - \tan x + 4 = 0$

$\sin x(\tan x - 4) - 1(\tan x - 4) = 0$

$(\sin x - 1)(\tan x - 4) = 0$ ✓

✓ $\sin x = 1$ $\tan x = 4$

$x = 90°$ $x = 76°$ or $256°$



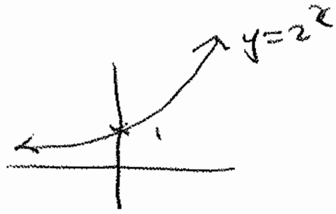
✓ but $\tan 90°$ is not defined so

$x = 76°$ or $256°$ are the only solutions

(d) $a = 35$, $r = 2^x$

(i) $-1 < 2^b < 1$

$b < 0$ ✓



(ii) $\frac{35}{1-2^b} = 40$ ✓

$1 - 2^b = \frac{35}{40}$

$2^b = \frac{1}{8}$

$b = -3$ ✓

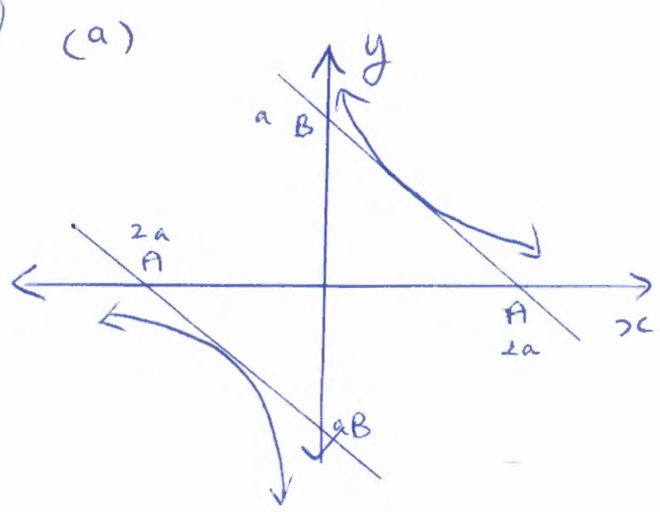
(e) $y = \ln(x^2 + e)$

if $x = 0$, $y = \ln e$
$= 1$

for any other value of x, $x^2 > 0$
and $\therefore \ln(x^2 + e) > 1$

so range is $y \geq 1$. ✓

16 (a)



gradient of tangent is $-\frac{1}{2}$

$xy = 4$

$y = 4x^{-1}$

$y' = -4x^{-2}$

so $-\frac{4}{x^2} = -\frac{1}{2}$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$y = \frac{\pm 4}{2\sqrt{2}}$

$= \pm\sqrt{2}$

the points are $(2\sqrt{2}, \sqrt{2})$ and $(-2\sqrt{2}, -\sqrt{2})$

(b) $y = mx^2 + nx + mn^2$

for the quadratic to be positive definite we need $m > 0$ and $n^2 - 4(m)(mn^2) < 0$

$n^2 - 4m^2n^2 < 0$

$n^2(1 - 4m^2) < 0$

now $n^2 > 0$ for all $n \neq 0$

$1 - 4m^2 < 0$

$4m^2 > 1$

$m^2 > \frac{1}{4}$

$m < -\frac{1}{2}$ or $m > \frac{1}{2}$

but $m > 0$ so choose $m > \frac{1}{2}$

So the quadratic is positive definite for all $n \neq 0$, $m > \frac{1}{2}$

(c)(i) the area of the first rectangle is $\frac{1}{2} \times 1 = \frac{1}{2}$

" " second " is $\frac{1}{3} \times 1 = \frac{1}{3}$

" " third $\frac{1}{4} \times 1 = \frac{1}{4}$

the rectangles underestimate the area under the curve which is given by

$$\int_0^3 \frac{1}{x+1}\,dx$$

so $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} < \int_0^3 \frac{1}{x+1}\,dx$

(ii) This time the rectangles will over-estimate the area

so $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \int_0^3 \frac{1}{x+2}\,dx$

(iii) $\sum_{r=2}^{100} \frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{100}$

from (i) + (ii)

$$\int_0^{99} \frac{1}{x+2}\,dx < \sum_{r=2}^{100} \frac{1}{r} < \int_0^{99} \frac{1}{x+1}\,dx$$

$$\Big[\ln(x+2)\Big]_0^{99} < \sum_{r=2}^{100} \frac{1}{r} < \Big[\ln(x+1)\Big]_0^{99}$$

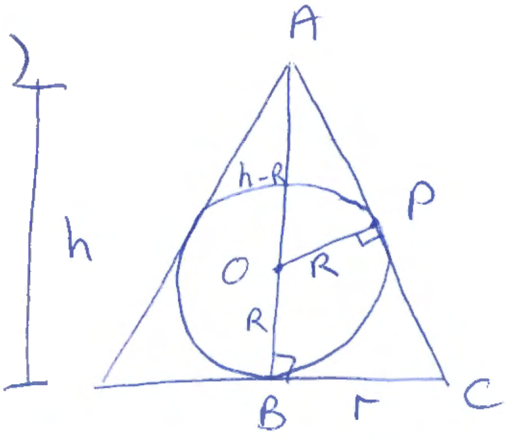
$$\ln 101 - \ln 2 < \sum_{r=2}^{100} \frac{1}{r} < \ln 100 - \ln 1$$

$$\ln\left(\frac{101}{2}\right) < \sum_{r=2}^{100} \frac{1}{r} < \ln 100$$

$$3.92 < \sum_{r=2}^{100} \frac{1}{r} < 4.61$$

to 2 d.p.

(d)

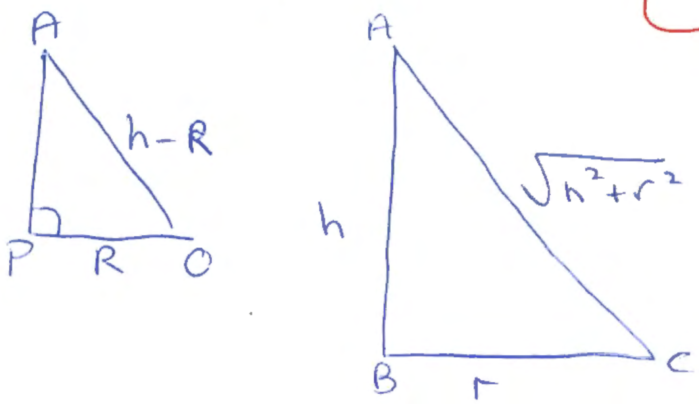


(i) In Δ's AOP and ABC

$\angle APO = \angle ABC$ (given)

$\angle A$ is common

$\Delta AOP \; ||| \; \Delta ACB$ (angle angle similarity test)



So $\dfrac{r}{R} = \dfrac{\sqrt{h^2+r^2}}{h-R}$

$$\frac{r^2}{R^2} = \frac{h^2+r^2}{(h-R)^2}$$

$$r^2(h-R)^2 = R^2(h^2+r^2)$$

$$r^2(h^2-2hR+R^2) = R^2h^2 + R^2r^2$$

$$r^2(h^2-2hR+R^2-R^2) = R^2h^2$$

$$r^2 = \frac{R^2h^2}{h^2-2hR}$$

$$= \frac{R^2h}{h-2R}$$

(ii) $V = \dfrac{\pi}{3} r^2 h$

$$= \frac{\pi}{3} \times \frac{R^2h}{h-2R} \times h$$

$$= \frac{\pi R^2 h^2}{3(h-2R)}$$

(iii) $$\frac{dV}{dh} = \frac{1}{3}\pi\left(\frac{(h-2R)\times 2R^2h - R^2h^2(1)}{(h-2R)^2}\right)$$

$$= \frac{1}{3}\pi\left(\frac{2R^2h^2 - 4R^3h - R^2h^2}{(h-2R)^2}\right)$$

$$= \frac{1}{3}\pi\left(\frac{R^2h^2 - 4R^3h}{(h-2R)^2}\right)$$

$\frac{dV}{dh} = 0$ at $R^2h^2 - 4R^3h = 0$

$hR^2(h - 4R) = 0$

$h = 4R$, $h \neq 0$ as $h > 2R$

if $h = 4R$, $r^2 = \frac{R^2 \times 4R}{4R - 2R}$

$= \frac{4R^3}{2R}$

$= 2R^2 \rightarrow r = \sqrt{2}R$

least volume with $h = 4R$ and $r = \sqrt{2}R$

(iv) $V_C = \frac{1}{3}\pi r^2 h$ $\qquad V_S = \frac{4}{3}\pi R^3$

$= \frac{1}{3}\pi\, 2R^2 \times 4R$

$= \frac{8\pi R^3}{3}$

ratio is $2:1$