
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2019 Annual Examination

FORM V

MATHEMATICS EXTENSION 1

Friday 6th September 2019

General Instructions

- Writing time — 2 hours
- Write using black pen.
- NESA-approved calculators may be used.

Total — 80 Marks

- All questions may be attempted.

Section I — 8 Marks

- Questions 1 – 8 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II — 72 Marks

- Questions 9–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Nine.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Reference Sheet
- Candidature — 155 boys

Examiner

WJM

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which of the following is the solution to $|x-2|<4$?

(A) $x<4$

(B) $x<2$

(C) $x<-2$ or $x>6$

(D) $-2<x<6$

QUESTION TWO

A bunch of flowers is formed using single flowers from five different types: daffodils, tulips, roses, lilies, and petunias. What is the minimum number of flowers in a bunch required to ensure that it contains at least four flowers of any one type?

(A) 15

(B) 16

(C) 20

(D) 21

QUESTION THREE

The polynomial $2x^3+4x^2-6x+5$ has zeros α, β and γ.

What is the value of $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$?

(A) $-\dfrac{6}{5}$

(B) $-\dfrac{4}{5}$

(C) $\dfrac{4}{5}$

(D) $\dfrac{6}{5}$

QUESTION FOUR

A curve is defined parametrically by the equations $x = \sin\theta$, $y = 2\cos\theta$. What is the Cartesian equation of the curve?

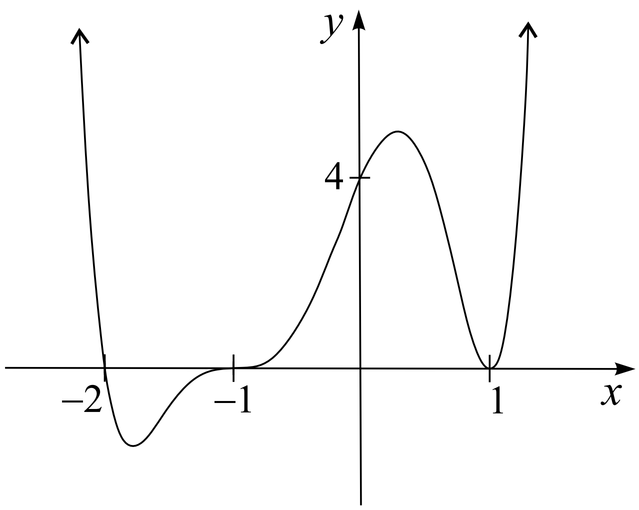
(A) $4x^2 + y^2 = 4$

(B) $4x^2 + y^2 = 1$

(C) $x^2 + y^2 = 1$

(D) $x^2 + y^2 = 4$

QUESTION FIVE



The diagram above shows the graph of $y = a(x+b)(x+c)^2(x+d)^3$. What are possible values of a, b, c and d?

(A) $a = 1,\ b = 1,\ c = -2,\ d = -1$

(B) $a = 1,\ b = 2,\ c = -1,\ d = 1$

(C) $a = 2,\ b = 2,\ c = -1,\ d = 1$

(D) $a = 2,\ b = 2,\ c = 1,\ d = -1$

QUESTION SIX

A certain multiple-choice quiz contains ten questions, each with three possible answers: A, B, or C. Bob has been told that there are three questions with correct answer 'A', five questions with correct answer 'B', and two questions with correct answer 'C'.

Bob decides to allocate his answers randomly, ensuring that he puts three 'A's, five 'B's and two 'C's. What is the probability that he gets every question correct?

(A) $\dfrac{1}{3\,628\,800}$

(B) $\dfrac{1}{2520}$

(C) $\dfrac{3}{100}$

(D) $\dfrac{4}{7}$

QUESTION SEVEN

A function is defined by $f(x) = -\sqrt{4 - x^2}$ for $0 \le x \le 2$. Which of the following correctly represents the inverse function of $f(x)$?

(A) $f^{-1}(x) = -\sqrt{4 - x^2}$ for $0 \le x \le 2$

(B) $f^{-1}(x) = \sqrt{4 - x^2}$ for $0 \le x \le 2$

(C) $f^{-1}(x) = -\sqrt{4 - x^2}$ for $-2 \le x \le 0$

(D) $f^{-1}(x) = \sqrt{4 - x^2}$ for $-2 \le x \le 0$

QUESTION EIGHT

Which of the following is NOT an even function?

(A) $f(x) = x^3 \sin x$

(B) $f(x) = \tan(\sin x)$

(C) $f(x) = \log_e(x \tan x)$

(D) $f(x) = \cos(\tan x)$

End of Section I

Examination continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) Find the remainder when the polynomial $P(x) = x^3 - 2x^2 + x + 5$ is divided by $(x - 2)$. **1**

(b) Differentiate the following with respect to x:

(i) $y = \sqrt[3]{x}$ **1**

(ii) $y = (x^3 + 4)^5$ **2**

(c) Solve the inequality $x^2 + x - 6 \geq 0$. **2**

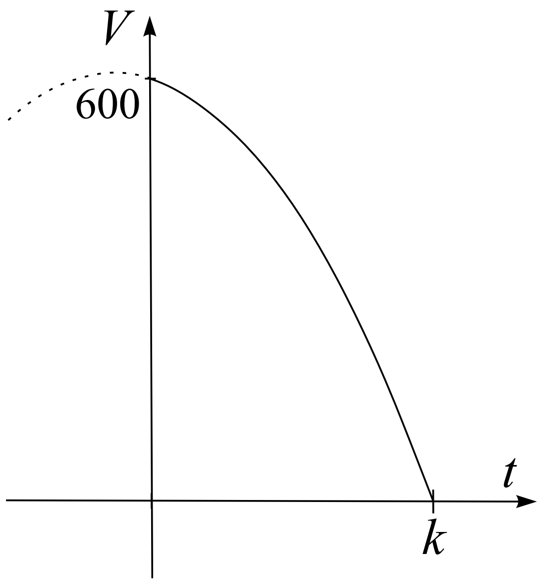
(d) Solve the equation $2\sin\theta + 1 = 0$, where $0 \leq \theta \leq 2\pi$. **2**

(e) A committee of four people is to be chosen from a group of eight women and five men. How many committees are possible that consist of two women and two men? **2**

(f) Simplify $\dfrac{1 + \tan x}{1 + \cot x}$. **2**

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above represents the volume of water in a water tank as it is being drained. The volume of water in the tank is initially 600 L, and it takes k minutes for the water to completely drain from the tank. It is known that the volume of water in litres after t minutes is given by $V = -30t^2 - 30t + 600$ for $0 \leq t \leq k$.

(i) Find the value of k. **1**

(ii) Find the average rate of change of the volume of water over the time that the tank is drained. **1**

(iii) Find the time at which the instantaneous rate of change in volume, $\dfrac{dV}{dt}$, is equal to the average rate of change of the volume for the time it takes the tank to drain. **2**

(b) Eight friends are seated in a row of eight at a cinema. The seats on each end of the row are aisle seats. How many ways can they be seated:

(i) with no restrictions? **1**

(ii) if Alice, Barney and Chris insist on sitting together as a group of three? **2**

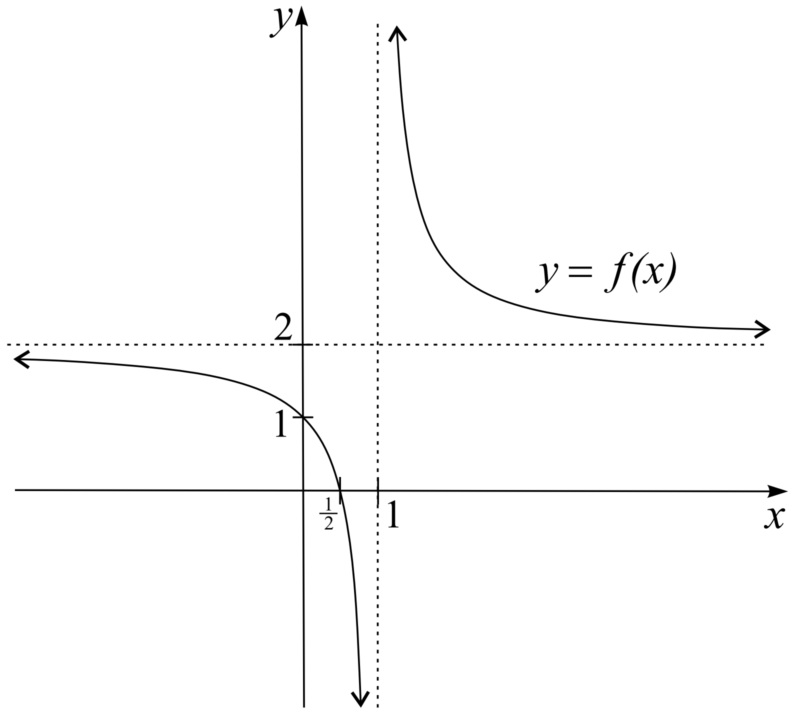
(iii) if Daisy and Eric won't sit together? **2**

(iv) if Feng and George insist on sitting in aisle seats, and Hannah sits the same number of seats away from Alice as she does from Barney? **3**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequality $\dfrac{x-6}{x-2} \le -1$. **3**

(b)



The diagram above shows the graph of $y = f(x)$, which has a horizontal asymptote of $y = 2$ and a vertical asymptote of $x = 1$.

On three separate diagrams, each one-third of a page, draw clear sketches of:

(i) $y = \left|f(x)\right|$ **2**

(ii) $y = f\big(|x|\big)$ **2**

(iii) $|\,y\,| = f(x)$ **3**

(c) Suppose that $x^3 + ax^2 + 2x + b = (x+2)(x-1)Q(x) + 1$ for all x, where $Q(x)$ is a polynomial and a and b are real numbers. **2**

Find the values of a and b.

QUESTION TWELVE (12 marks) Use a separate writing booklet. **Marks**

(a) A six-sided die with faces labelled 1, 2, 3, 4, 5, and 6 is constructed such that the probability of rolling a 6 is larger than the probability of rolling a 1. For some constant a, the probability distribution table is as follows: **3**

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{6} - a$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6} + a$

It is known that the expected value of rolling this die is 4.

Find the value of a.

(b) The polynomial $P(x) = x^4 - 10x^3 + 24x^2 + 32x + k$ has a triple zero.

(i) Determine the value of the triple zero. **3**

(ii) Hence find the value of k. **1**

(iii) Fully factorise $P(x)$. **2**

(c) Consider the function $y = \dfrac{x}{x^2 + 1}$.

(i) Find $\dfrac{dy}{dx}$. **1**

(ii) Using your answer to part (i), determine the domain for which the function is increasing. **2**

QUESTION THIRTEEN (12 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x) = x^3 + 8x^2 + cx - 48$ has a zero that is the sum of the other two. That is, the zeros of $P(x)$ are α, β, and $\alpha + \beta$.

(i) Write down the value of $\alpha + \beta$. **1**

(ii) Find the value of c. **1**

(iii) Fully factorise $P(x)$. **2**

(iv) Let the function $f(x)$ be defined by $f(x) = \log_e \big(P(x)\big)$. **2**

What is the domain of $f(x)$?

(b) A polynomial $Q(x)$ is defined by $Q(x) = x^3 - b^3$, where b is a constant. Divide $Q(x)$ by $(x - b)$, and hence express $Q(x)$ as the product of a linear and a quadratic factor. **1**

(c) Using your answer to part (b) or otherwise, simplify **2**

$$\frac{(\tan^2\theta - 1)(\sin\theta\cos\theta + 1)}{\cos^2\theta(\tan^3\theta - 1)}.$$

(d) A piecewise function is defined by:

$$f(x) = \begin{cases} x^2 + 2x\,, & \text{for } x \le 1 \\ -\dfrac{2}{2x-1} + 5\,, & \text{for } x > 1. \end{cases}$$

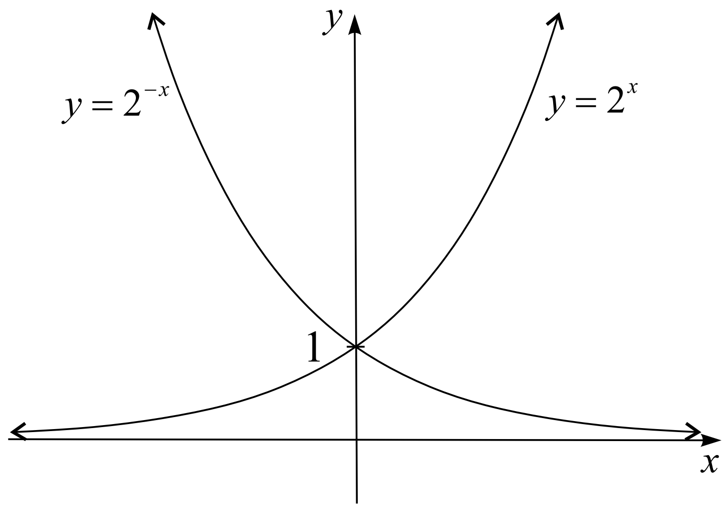
Providing justification for your answer:

(i) determine if $f(x)$ is continuous at $x = 1$. **1**

(ii) determine if $f(x)$ is differentiable at $x = 1$. **2**

QUESTION FOURTEEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the graphs of $y = 2^x$ and $y = 2^{-x}$. Let $g(x) = 2^x + 2^{-x}$ for all real values of x.

(i) By first copying the graphs of $y = 2^x$ and $y = 2^{-x}$, sketch the graph of $y = g(x)$. **1**

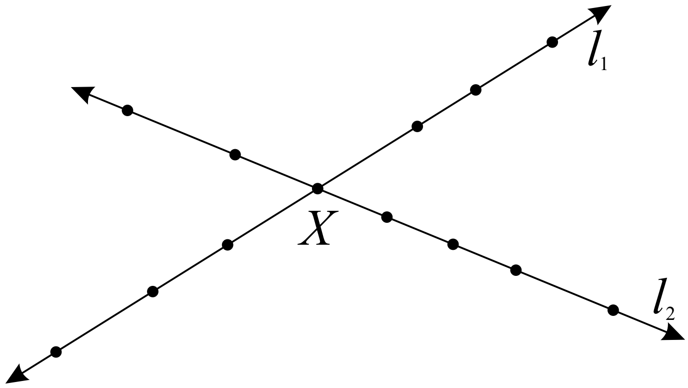
(ii) Explain why the inverse of $g(x)$ is not an inverse function. **1**

(iii) Let $f(x)$ be the restriction $f(x) = 2^x + 2^{-x}$, $x \geq 0$. On a separate diagram, sketch the graph of $y = f^{-1}(x)$. **1**

(iv) Find an expression for $y = f^{-1}(x)$ in terms of x. **2**

QUESTION FOURTEEN (Continued)

(b)



The diagram above shows two distinct intersecting lines l_1 and l_2 lying on a plane. Each line has n distinct points marked on it, with the point X lying on both lines. The case above shows a possible situation when $n = 7$, and there are a total of thirteen points marked.

Consider the general case where there are n distinct points marked on each line, for $n \geq 3$.

(i) In how many ways can three points be chosen from l_1? **1**

(ii) By considering the total number of points on both lines and your answer to part (i), determine the number of triangles that can be formed using the marked points on either line as vertices. **2**

(iii) Of the total number of possible triangles in part (ii):

(α) how many have X as a vertex? **1**

(β) how many have two vertices that lie on l_1? **1**

(iv) Use your answers to part (ii) and part (iii) to show that **2**

$$^{2n-1}\mathrm{C}_3 = 2 \times {}^{n}\mathrm{C}_3 + (n-1)^3 \, .$$

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Form V Ext. 1 Annual Exam 2019

1. D

2. B

3. $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\gamma+\alpha\beta+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{-\frac{6}{2}}{-\frac{5}{2}}$

$= \frac{6}{5} \Rightarrow D$

4. $x^2 = \sin^2\theta$

$\frac{y^2}{4} = \cos^2\theta$

$x^2 + \frac{y^2}{4} = 1$

$4x^2 + y^2 = 4 \Rightarrow A$

5. $b=2,\ c=-1,\ d=1$

$y = a(x+2)(x-1)^2(x+1)^3$

When $x=0$, $y=4$: $4 = a \times 2 \times 1 \times 1$

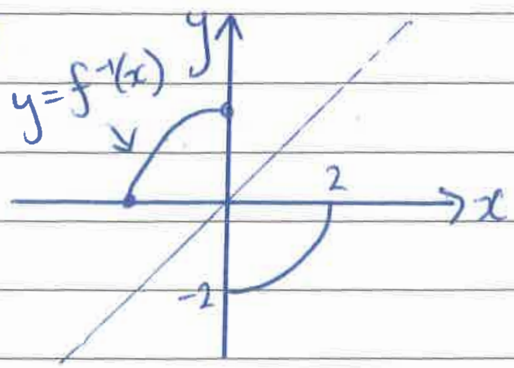
$a = 2 \Rightarrow C$

6. Number of words with $3\times A$, $5\times B$, $2\times C$:

$\frac{10!}{3!5!2!} = 2520$

$\therefore \frac{1}{2520} \Rightarrow B$

7.



D

(8) $f(x) = \tan(\sin x)$

$f(-x) = \tan(\sin(-x))$

$= \tan(-\sin x)$

$= -\tan(\sin x)$

$= -f(x) \quad \Rightarrow \quad B$

Question 9

(a) $P(2) = 2^3 - 2 \times 2^2 + 2 + 5$

$= 7$

$\therefore$ the remainder is 7

(b) (i) $y = \sqrt[3]{x}$

$= x^{1/3}$

$\frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}}$

$= \frac{1}{3\sqrt[3]{x^2}}$

(ii) $y = (x^3 + 4)^5$

$\frac{dy}{dx} = 5(x^3 + 4)^4 \times 3x^2$

$= 15x^2(x^3 + 4)^4$

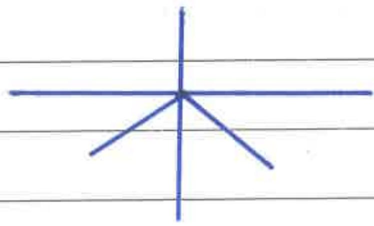
(c) $x^2 + x - 6 \geq 0$

$(x+3)(x-2) \geq 0$

$x \leq -3$ or $x \geq 2$

(d) $2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2}$



$\alpha = \frac{\pi}{6}$

$\theta = \pi + \alpha, \ 2\pi - \alpha$

$= \pi + \frac{\pi}{6}, \ 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}, \ \frac{11\pi}{6}$

(e) $^8C_2 \times {}^5C_2 = 280$ ✓

(f) $\dfrac{1+\tan x}{1+\cot x} = \dfrac{1+\tan x}{1+\frac{1}{\tan x}} \times \dfrac{\tan x}{\tan x}$ ✓ ~~(any reasonable progress)~~

$= \dfrac{\tan x(\cancel{1+\tan x})}{\cancel{\tan x+1}}$

$= \tan x$ ✓

Question 10

(a) (i) $-30t^2 - 30t + 600 = 0$

$t^2 + t - 20 = 0$

$(t+5)(t-4) = 0$

$t = -5, \quad t = 4$

$\therefore \; k = 4$

(ii) Average rate of change $= -\dfrac{600}{4}$

$= -150 \; L/min$

(iii) $\dfrac{dV}{dt} = -60t - 30$

$-60t - 30 = -150$

$-60t = -120$

$t = 2$

$\therefore$ instantaneous rate of change = average rate of change after 2 minutes.

(b) (i) $8! = 40\,320$

(ii) • $3!$ ways to order A, B, C

• $6!$ ways to arrange $\boxed{ABC}$, D, E, F, G, H

$3! \times 6! = 4320$

(iii) There are $2! \times 7! = 10080$ ways D & E can sit together.

$\therefore$ there are $8! - 2!7! = 30240$ ways that they are separated.

(iv) • 2 ways to seat Feng

• 1 way to seat George

Consider possible seats for A, H, B:

$\times$ $\underline{A}$ $\underline{H}$ $\underline{B}$ _ _ _ $\times$

$\underline{A}$ $\underline{H}$ $\underline{B}$

$\underline{A}$ $\underline{H}$ $\underline{B}$

$\underline{A}$ $\underline{H}$ $\underline{B}$

$2! \times 4$ ways if they all sit together

$\times$ $\underline{A}$ _ $\underline{H}$ _ $\underline{B}$ _ $\times$

$\underline{A}$ $\underline{H}$ $\underline{B}$

$2! \times 2$ ways if they are 1 seat apart.

- $2! \times 4 + 2! \times 2 = 12$ ✓

- $3!$ ways to seat remaining people.

Total: $2 \times 1 \times 12 \times 3! = 144$ ✓

Question 11

(a) $\dfrac{x-6}{x-2} \leq -1$

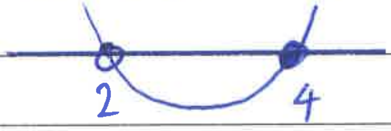
$(x-6)(x-2) \leq -(x-2)^2 \quad , \quad x \neq 2$ ✓ (multiplying by $(x-2)^2$, or valid alternative method)

$(x-2)^2 + (x-6)(x-2) \leq 0$

$(x-2)[x-2+x-6] \leq 0$

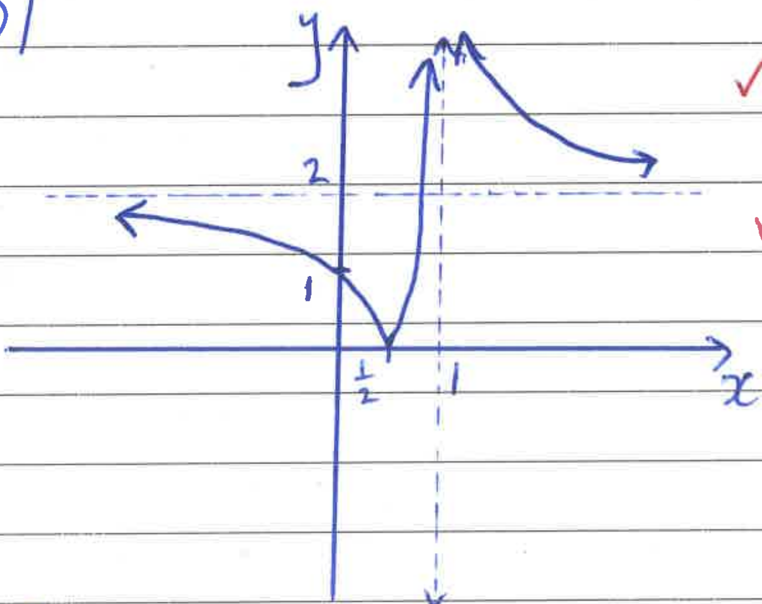
$(x-2)(2x-8) \leq 0$

$2(x-2)(x-4) \leq 0$ ✓ (obtaining values of interest: $x=2$ & $x=4$)



$\therefore \quad 2 < x \leq 4$ ✓

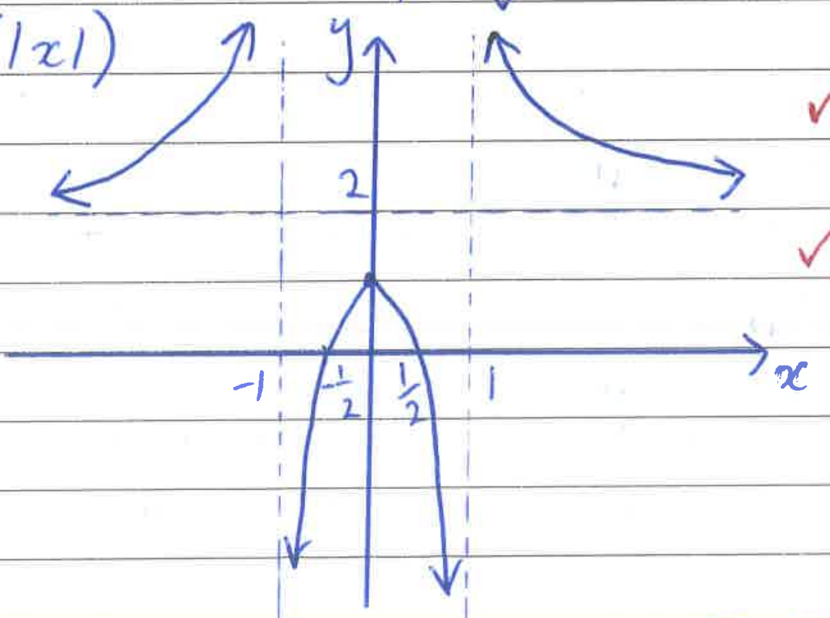
(b) (i) $y = |f(x)|$



✓ positive section remains

✓ negative reflected about x-axis

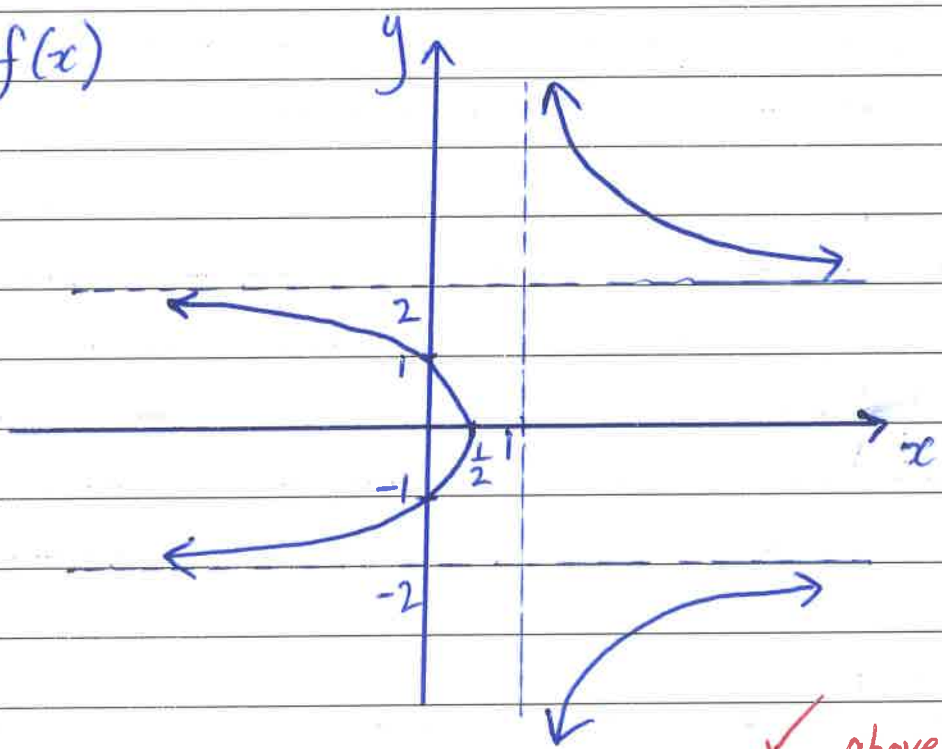
(ii) $y = f(|x|)$



✓ section when $x>0$ remains.

✓ reflection of section where $x>0$ about y-axis.

(iii) $|y| = f(x)$



✓ above x-axis remains

✓ below x-axis is a reflection of where $y > 0$

✓ All features labelled

(c) $x^3 + ax^2 + 2x + b = (x+2)(x-1)Q(x) + 1$

Let $x = -2$: $\quad -8 + 4a - 4 + b = 1$

$4a + b = 13 \quad (1)$ ✓ (recognising correct substitution $x=-2$ or $x=1$)

Let $x = 1$: $\quad 1 + a + 2 + b = 1$

$a + b = -2 \quad (2)$

(1) − (2): $\quad 3a = 15$

$a = 5$

sub. into (2): $\quad 5 + b = -2$

$b = -7$ ✓

Question 12

(a)

x	1	2	3	4	5	6
$xp(x)$	$\frac{1}{6}-a$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$1+6a$

$\sum xp(x) = 4,$

$$\frac{1}{6}-a+\frac{2}{6}+\frac{3}{6}+\frac{4}{6}+\frac{5}{6}+1+6a = 4$$

$$\frac{15}{6}+1+5a = 4$$

$$5a = \frac{1}{2}$$

$$a = \frac{1}{10}$$

(b) $P(x) = x^4 - 10x^3 + 24x^2 + 32x + k$

(i) $P'(x) = 4x^3 - 30x^2 + 48x + 32$

$P''(x) = 12x^2 - 60x + 48$

Let $P''(x) = 0$: $12x^2 - 60x + 48 = 0$

$$x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

$\therefore$ the triple zero is either 1 or 4.

$P'(1) = 4 - 30 + 48 + 32$

$= 54 \neq 0$

Since it is given that $P(x)$ has a triple zero, it can be deduced that it is 4.

(ii) $P(4)=0$: $4^4 - 10\times 4^3 + 24\times 4^2 + 32\times 4 + k = 0$

$$128 + k = 0$$

$$k = -128$$

(iii) Let the zeros be $4, 4, 4, \alpha$.

Sum of zeros: $4+4+4+\alpha = -\frac{-10}{1}$

$$\alpha + 12 = 10$$

$$\alpha = -2$$

$\therefore P(x) = (x-4)^3(x+2)$

(c) $y = \dfrac{x}{x^2+1}$

(i) $\dfrac{dy}{dx} = \dfrac{(x^2+1).1 - x \times 2x}{(x^2+1)^2}$

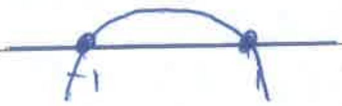
$= \dfrac{1-x^2}{(x^2+1)^2}$

(ii) Increasing when $\dfrac{dy}{dx} \geq 0$

$\dfrac{1-x^2}{(x^2+1)^2} \geq 0$

$1-x^2 \geq 0$ since $(x^2+1)^2 > 0$ for all x

$(1+x)(1-x) \geq 0$



$\therefore$ increasing when $-1 \leq x \leq 1$

Question 13

$P(x) = x^3 + 8x^2 + cx - 48$

$\alpha, \beta, \alpha+\beta$

(a)(i) Sum of zeros:

$$\alpha + \beta + (\alpha+\beta) = -\frac{8}{1}$$

$$2\alpha + 2\beta = -8$$

$$\alpha + \beta = -4$$

(ii) -4 is a zero of $P(x)$.

$$P(-4) = (-4)^3 + 8(-4)^2 + c(-4) - 48$$

$$= -64 + 128 - 4c - 48$$

$$= 16 - 4c$$

$$\therefore \quad 16 - 4c = 0$$

$$c = 4$$

(iii)

$$\begin{array}{r|l} & x^2 + 4x - 12 \\ \hline x+4 & x^3 + 8x^2 + 4x - 48 \\ & \underline{x^3 + 4x^2} \\ & 4x^2 + 4x \\ & \underline{4x^2 + 16x} \\ & -12x - 48 \\ & \underline{-12x - 48} \\ & 0 \end{array}$$

$$\therefore \quad P(x) = (x+4)(x^2 + 4x - 12)$$

$$= (x+4)(x+6)(x-2)$$

Or// Zeros are $\alpha, \beta, -4$

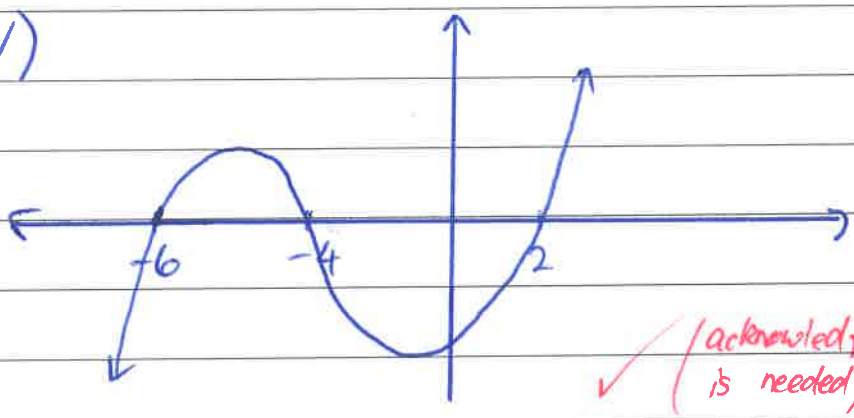
Product of zeros: $-4\alpha\beta = -\frac{-48}{1}$

$$\alpha\beta = -12$$

Since $\alpha+\beta = -4$, by inspection $\alpha = -6, \ \beta = 2$

$$\therefore \quad P(x) = (x+4)(x+6)(x-2)$$

(iv)



(acknowledgement that $P(x)>0$ is needed, through graph or otherwise)

$f(x)$ defined when $P(x) > 0$

i.e. domain of $f(x)$ is $-6 < x < -4$ or $x > 2$

(b)

$$\begin{array}{r l}
 & x^2 + bx + b^2 \\
x - b \,\big) & x^3 + 0x^2 + 0x - b^3 \\
 & x^3 - bx^2 \\
\hline
 & bx^2 + 0x \\
 & bx^2 - b^2x \\
\hline
 & b^2x - b^3 \\
 & b^2x - b^3 \\
\hline
 & 0
\end{array}$$

$\therefore\ Q(x) = (x-b)(x^2+bx+b^2)$

(c) $$\frac{(\tan^2\theta - 1)(\sin\theta\cos\theta + 1)}{\cos^2\theta(\tan^3\theta - 1)} = \boxed{\frac{(\tan\theta+1)\cancel{(\tan\theta-1)}(\sin\theta\cos\theta+1)}{\cos^2\theta\cancel{(\tan\theta-1)}(\tan^2\theta+\tan\theta+1)}}$$

$$= \frac{(\tan\theta+1)(\sin\theta\cos\theta+1)}{\cos^2\theta(\tan\theta + \sec^2\theta)}$$

$$= \frac{(\tan\theta+1)\cancel{(\sin\theta\cos\theta+1)}}{\cancel{\sin\theta\cos\theta + 1}}$$

$$= \tan\theta + 1$$

Correct factorisation using answer from (b), and cancelling a factor.

(d)(i) $f(1) = 1^2 + 2\times 1$

$= 3$

As $x \to 1^+$, $f(x) \to -\frac{2}{2\times 1-1} + 5$

$\to -\frac{2}{1} + 5$

$\to 3$

$\therefore$ $f(x)$ is continuous at $x=1$, as $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

(ii) $\frac{d}{dx}(x^2+2x) = 2x+2$

When $x=1$, $2x+2 = 2\times 1+2$

$= 4$

$\frac{d}{dx}\left(-\frac{2}{2x-1}+5\right) = \frac{d}{dx}\left(-2(2x-1)^{-1}+5\right)$

$= 2(2x-1)^{-2}\times 2$

$= \frac{4}{(2x-1)^2}$

when $x=1$, $\frac{4}{(2x-1)^2} = \frac{4}{(2\times 1-1)^2}$

$= 4$

* For 2 marks, pupils were required to correctly evaluate both limits and show that they were equal

$\therefore$ As $x \to 1^-$, $f'(x) \to 4$

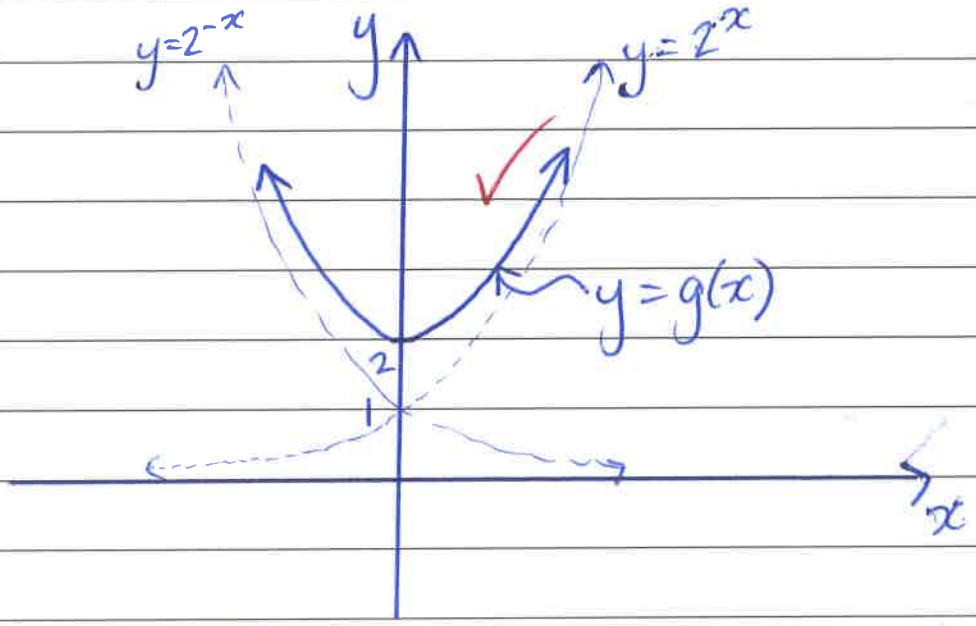
as $x \to 1^+$, $f'(x) \to 4$

Also, $f(x)$ is continuous at $x=1$ ← Not required

$\therefore$ $f(x)$ is differentiable at $x=1$.

Question 14

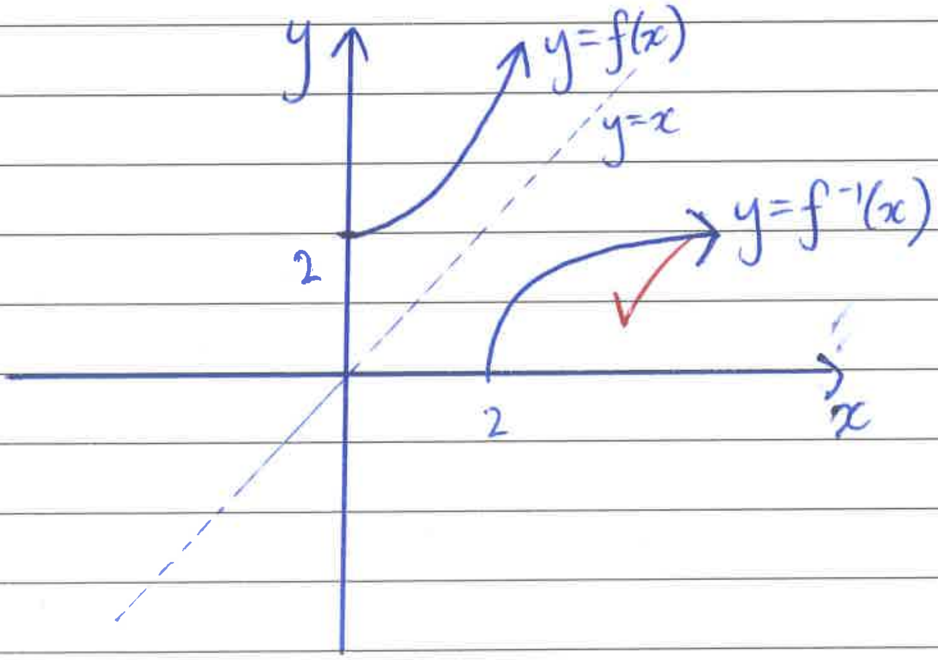
(a) (i)



(ii) $y=g(x)$ fails the horizontal line test.

or// $g(x)$ is many-to-one, so its inverse would be one-to-many, which is not a function.

(iii)



(iv) $x = 2^y + 2^{-y}$

$= 2^y + \frac{1}{2^y}$

$x \cdot 2^y = 2^{2y} + 1$

(exchanging x & y, and making some progress rearranging)

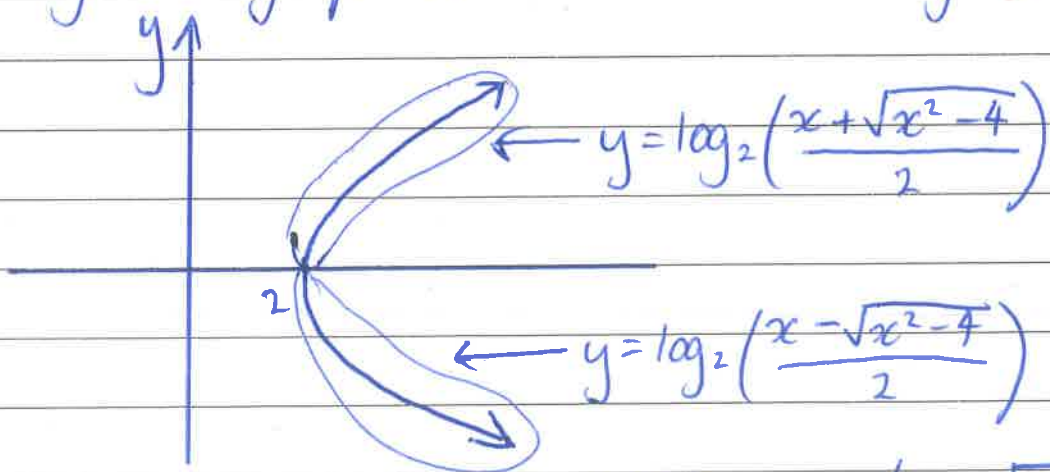
$$2^{2y} - x.2^y + 1 = 0$$
$$(2^y)^2 - x.2^y + 1 = 0$$
$$\left(2^y - \frac{x}{2}\right)^2 = -1 + \frac{x^2}{4} \quad \text{(completing the square in } 2^y\text{)}$$
$$2^y - \frac{x}{2} = \pm\sqrt{\frac{x^2-4}{4}}$$
$$2^y = \frac{x \pm \sqrt{x^2-4}}{2}$$
$$y = \log_2\left(\frac{x \pm \sqrt{x^2-4}}{2}\right)$$

Now for $y = f^{-1}(x)$: Domain: $x \geq 2$
Range: $y \geq 0$

$\therefore$ the numerator of $\frac{x \pm \sqrt{x^2-4}}{2}$ must be greater than or equal to 2. Testing values of x suggests $x + \sqrt{x^2-4} \geq 2$ for $x \geq 2$, but $x - \sqrt{x^2-4} \not\geq 2$.

Also, $\log_2\left(\frac{x+\sqrt{x^2-4}}{2}\right) \geq \log_2\left(\frac{x-\sqrt{x^2-4}}{2}\right)$.

Considering the graph of the inverse of $g(x)$:



$$\therefore f^{-1}(x) = \log_2\left(\frac{x+\sqrt{x^2-4}}{2}\right)$$

(b) (i) ${}^nC_3$

(ii) There are $2n-1$ points in total, and any 3 non-collinear points can be chosen to form the vertices of a triangle.

$\therefore$ There are ${}^{2n-1}C_3 - 2 \times {}^nC_3$ possible triangles.

(iii) (α) If X is assigned as a vertex:

- 1 vertex must be chosen from l_1 ($n-1$ ways)
- 1 vertex must be chosen from l_2 ($n-1$ ways)

$\therefore$ $(n-1)^2$ triangles have X as a vertex.

(β)

- ${}^nC_2$ ways to choose the vertices on l_1
- $(n-1)$ ways to choose the remaining vertex on l_2

$\therefore$ ${}^nC_2(n-1)$ triangles have 2 vertices on l_1.

(iv) The total number of triangles that can be formed can be considered as:

Triangles with 2 vertices on l_1
+ Triangles with 2 vertices on l_2
− Triangles that have X as a vertex (as they have been counted twice).

reasoning and using results from part (iii).

$\therefore$ From (iii), Total triangles $= {}^nC_2(n-1) + {}^nC_2(n-1) - (n-1)^2$

$$= 2 \times \frac{n!}{2!\,(n-2)!}(n-1) - (n-1)^2$$

$$= n(n-1)^2 - (n-1)^2$$

$$= (n-1)^2(n-1)$$

$= (n-1)^3$

Equating this result with result from (b)(ii):

$$ {}^{2n-1}C_3 - 2 \times {}^{n}C_3 = (n-1)^3 $$

$$ {}^{2n}C_3 = 2 \times {}^{n}C_3 + (n-1)^3 $$