SYDNEY TECHNICAL HIGH SCHOOL

YEAR 11 YEARLY EXAMINATION

MATHEMATICS EXTENSION 1

2004

Time allowed: 90 minutes

Directions to Candidates

- Attempt all questions
- Start each question on a new page
- All necessary working should be shown
- Unless otherwise specified, answers must be given in their simplest form
- Approved calculators may be used in all sections.
- Use a ruler when drawing straight lines
- Marks may be deducted for careless or poorly arranged work.
- Marks shown are approximate and may be varied.

Name:______________________________ Class:__________________

1	2	3	4	5	6	TOTAL
$\overline{12}$	$\overline{10}$	$\overline{12}$	$\overline{10}$	$\overline{9}$	$\overline{10}$	$\overline{63}$

QUESTION 1

a) If $\sin \alpha = \dfrac{2}{3}$ and α is acute find the exact value of $\sin 2\alpha$ — 2

b) If $x^2 - y^2 = 36$ and $x + y = 8$ find the value of $x - y$ — 2

c) Solve $2^{2x} - 6(2^x) + 8 = 0$ — 3

d) Solve $\dfrac{x-2}{x} \geq 1$ — 3

e) Find the exact value of $\tan\alpha$ if α is the acute angle between the lines

$y = \dfrac{1}{2}x$ and $y = \dfrac{-1}{\sqrt{3}}x + 1$ (do not rationalize the denominator) — 2

QUESTION 2

a) Find the coordinates of the points on the curve $y = x^3 + 3x^2$ where the tangent is parallel to the line $9x - y - 5 = 0$ — 3

b) A parabola has equation $y^2 - 6y - 3 = 12x$

i) Write the equation of the parabola in the form $(y-k)^2 = 4a(x-h)$ — 2

Find the:

ii) Coordinates of the vertex — 1

iii) Coordinates of the focus — 1

iv) Equation of the directrix — 1

c) Copy and complete the following to define a parabola as a locus — 2

"A parabola is the locus of all points that............."

QUESTION 3

a) Show the equation of the chord PQ joining the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$ is $y = \frac{1}{2}(p+q)x - apq$ 3

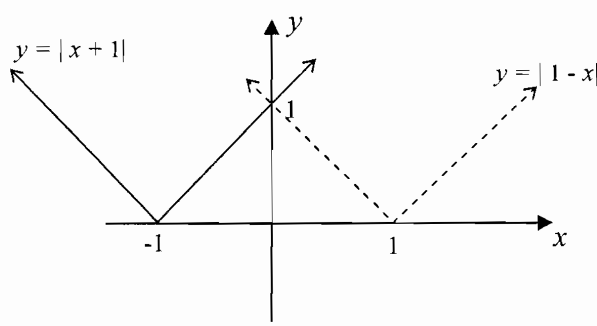
b) Consider the function $f(x) = 6x^3 - 19x^2 + 11x + 6$

i) Use the factor theorem to show that $x - 2$ is a factor of $f(x)$ 1

ii) Completely factorise $f(x)$ 3

c) Solve $\sin\theta - \cos 2\theta = 0$ for $0° \le \theta \le 360°$ 3

d) The graphs of $y = |x+1|$ and $y = |1-x|$ are shown 2



Hence sketch the graph of $y = |x+1| - |1-x|$

QUESTION 4

a) Differentiate $y = x\sqrt{x+1}$. Give your answer as a single fraction 3

b) Given $\cos 2A = 2\cos^2 A - 1$ evaluate p, q and r if $\cos 4x = p\cos^4 x + q\cos^2 x + r$ 3

c) Consider the polynomial $P(x) = (2-x)(x+1)^2$

i) Sketch $y = P(x)$ indicating where it crosses the axes 2

ii) Hence solve $P(x) > 0$ 2

QUESTION 5

a) When the polynomial $P(x)$ is divided by $(x+1)(x-2)$, the result can be written as $P(x) = (x+1)(x-2)Q(x) + R(x)$, where $R(x) = ax+b$

i) Given that $P(-1) = 3$, find the value of $R(-1)$ 1

ii) Given also that the remainder is -2 when $P(x)$ is divided by $x-2$, find the values of a and b 3

b) i) Find the value of k if the equation $x^3 + kx^2 - 4x - 8 = 0$ has two roots equal in size but opposite in sign. 3

ii) Hence solve the equation 2

QUESTION 6

$P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$

(a) Show that the equation of the normal to the parabola at the point P is $x + py = 2ap + ap^3$ 2

(b) If the normal at P cuts the y-axis at Q show that the coordinates of Q are $(0, 2a + ap^2)$. 1

(c) Show that the coordinates of R which divided the interval PQ externally in the ratio 2:1 are $(-2ap, 4a + ap^2)$. 2

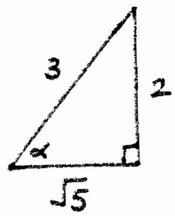
(d) Find the Cartesian equation of the locus of R. 2

(e) Show that if the normal at P passes through a given point (h,k) then p must be a root of the equation $ap^3 + (2a-k)p - h = 0$ 2

(f) Hence state the maximum number of normals to the parabola $x^2 = 4ay$ which can pass through any given point. 1

2004 YEAR 11 YEARLY
EXTENSION 1

Question 1



a) $\sin\alpha = \dfrac{2}{3}$

$\cos\alpha = \dfrac{\sqrt{5}}{3}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2 \times \frac{2}{3} \times \frac{\sqrt{5}}{3}$$
$$= \underline{\underline{\frac{4\sqrt{5}}{9}}}$$

b) $(x+y)(x-y) = x^2 - y^2$

$8\,(x-y) = 36$

$x - y = \underline{\underline{4 \cdot 5}}$

c) $2^{2x} - 6(2^x) + 8 = 0$

let $a = 2^x$

$a^2 - 6a + 8 = 0$

$(a-4)(a-2) = 0$

$a = 4$ or $a = 2$

$2^x = 4$ $\quad 2^x = 2$

$\therefore \underline{\underline{x = 2, 1}}$

d) $x^2 \times \dfrac{x-2}{x} \geq 1 \times x^2$

$x(x-2) \geq x^2$

$x^2 - 2x \geq x^2$

$-2x \geq 0$

$x \leq 0$

$\therefore \underline{\underline{x < 0}} \quad (x \neq 0)$

e) $m_1 = \frac{1}{2}$ and $m_2 = -\frac{1}{\sqrt{3}}$

$$\tan\alpha = \left| \frac{\frac{1}{2} - \left(-\frac{1}{\sqrt{3}}\right)}{1 + \frac{1}{2} \times -\frac{1}{\sqrt{3}}} \right|$$
$$= \left| \frac{\frac{1}{2} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{2\sqrt{3}}} \right|$$
$$= \frac{\frac{\sqrt{3}+2}{2\sqrt{3}}}{\frac{2\sqrt{3}-1}{2\sqrt{3}}}$$
$$= \underline{\underline{\frac{\sqrt{3}+2}{2\sqrt{3}-1}}}$$

Question 2

a) $y = x^3 + 3x^2 \qquad m_{line} = 9$

$\dfrac{dy}{dx} = 3x^2 + 6x$

$3x^2 + 6x = 9$

$3x^2 + 6x - 9 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

$\therefore$ points are $\underline{\underline{(-3, 0) \ \& \ (1, 4)}}$

b) $y^2 - 6y - 3 = 12x$

i. $y^2 - 6y + 9 = 12x + 3 + 9$

$\underline{\underline{(y-3)^2 = 12(x+1)}}$

ii. $\underline{\underline{(-1, 3)}}$ $\qquad$ iv. $\underline{\underline{x = -4}}$

iii. $(2, 3)$

c) A parabola is the locus of all points that are equidistant from a fixed point (focus) and a fixed line (directrix).

Question 3

a) $$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p+q)(p-q)}{2a(p-q)} = \frac{1}{2}(p+q)$$

$$y - y_1 = m(x - x_1)$$
$$y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$$
$$y - ap^2 = \frac{1}{2}(p+q)x - ap^2 - apq$$
$$y = \frac{1}{2}(p+q)x - apq$$

b) $f(x) = 6x^3 - 19x^2 + 11x + 6$

i. $x - 2$ is a factor if $f(2) = 0$

$$f(2) = 6(2^3) - 19(2^2) + 11.2 + 6 = 0$$

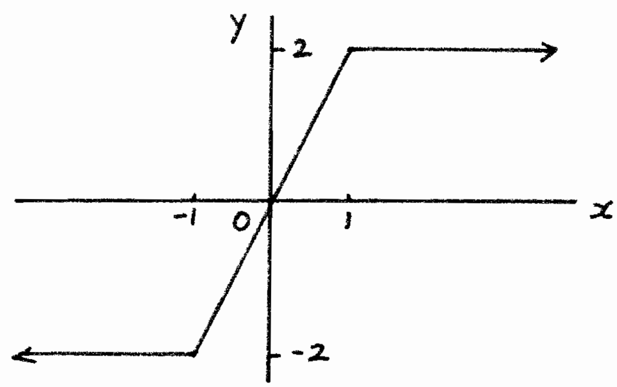
$\therefore$ $x - 2$ is a factor

ii.
$$\begin{array}{r|l} & 6x^2 - 7x - 3 \\ \hline x - 2 & 6x^3 - 19x^2 + 11x + 6 \\ & \underline{6x^3 - 12x^2} \\ & -7x^2 + 11x \\ & -7x^2 + 14x \end{array}$$

$$f(x) = (x-2)(6x^2 - 7x - 3)$$
$$\therefore f(x) = (x-2)(3x+1)(2x-3)$$

c) $$\sin\theta - \cos 2\theta = 0$$
$$\sin\theta - (1 - 2\sin^2\theta) = 0$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(\sin\theta + 1)(2\sin\theta - 1) = 0$$
$$\sin\theta = -1 \quad \text{or} \quad \sin\theta = \frac{1}{2}$$
$$\therefore \theta = 30,\ 150,\ 270$$

d)



Question 4

a) $y = x\sqrt{x+1}$

$$u = x \qquad v = (x+1)^{\frac{1}{2}}$$
$$u' = 1 \qquad v' = \frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot 1 = \frac{1}{2\sqrt{x+1}}$$

$$\frac{dy}{dx} = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = \frac{2(x+1) + x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

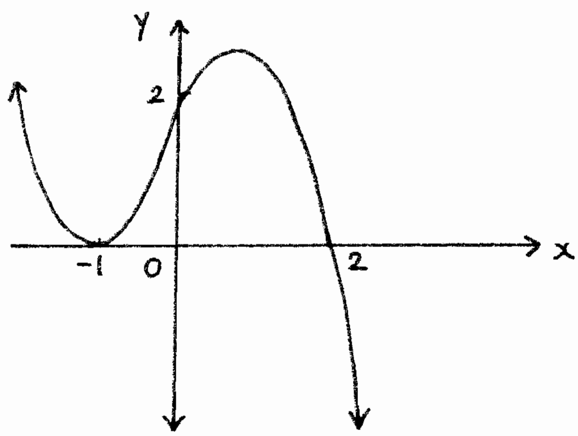
b) $\cos 2A = 2\cos^2 A - 1$

$$\begin{aligned}\cos 4x &= 2\cos^2 2x - 1\\ &= 2(\cos 2x)^2 - 1\\ &= 2(2\cos^2 x - 1)^2 - 1\\ &= 2(4\cos^4 x - 4\cos^2 x + 1) - 1\\ &= 8\cos^4 x - 8\cos^2 x + 1\end{aligned}$$

$\therefore\ p = 8 \quad q = -8 \quad r = 1$

c) $P(x) = (2-x)(x+1)^2$

i.



ii. $(2-x)(x+1)^2 > 0$

$x < 2,\ x \neq -1$

or $x < -1,\ -1 < x < 2$

Question 5

a) $P(x) = (x+1)(x-2)\,Q(x) + R(x)$

i. $P(-1) = 0 + R(-1)$

$\therefore\ R(-1) = 3$

ii. $P(2) = 0 + R(2)$

$R(2) = -2$

Using $R(x) = ax + b$

$3 = -a + b$

$-2 = 2a + b$

Solve simultaneously

$-5 = 3a$

$a = -\frac{5}{3}$

$\frac{5}{3} + b = 3$

$b = \frac{4}{3}$

$\therefore\ a = -\frac{5}{3} \quad b = \frac{4}{3}$

b) $x^3 + kx^2 - 4x - 8 = 0$

i. let the roots be $\alpha, -\alpha, \beta$

sum of roots : $\alpha + -\alpha + \beta = -k$

$\beta = -k$

sum of roots 2 at a time : $-\alpha^2 + \alpha\beta - \alpha\beta = -4$

$\alpha^2 = 4$

$\alpha = \pm 2$

product of roots : $-\alpha^2\beta = 8$

$-4\beta = 8$

$\beta = -2$

$\therefore\ k = 2$

ii. $x = 2, -2$

Question 6

i. $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$m_{\text{tangent at } P} = \frac{2ap}{2a}$

$= p$

$m_{\text{normal at } P} = -\frac{1}{p}$

Equation of normal :

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\underline{\underline{x + py = 2ap + ap^3}}$

ii. At Q, $x = 0$

$0 + py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore \underline{\underline{Q(0, 2a + ap^2)}}$

iii. ratio of $-2 : 1$

$R\left(\frac{1 \times 2ap + -2 \times 0}{-2+1}, \frac{1 \times ap^2 + -2(2a + ap^2)}{-2+1}\right)$

$= R\left(\frac{2ap}{-1}, \frac{ap^2 - 4a - 2ap^2}{-1}\right)$

$= \underline{\underline{R(-2ap, 4a + ap^2)}}$

iv. $x = -2ap \Rightarrow p = \frac{-x}{2a}$

$y = 4a + ap^2$

$y = 4a + a\left(\frac{-x}{2a}\right)^2$

$y = 4a + \frac{x^2}{4a}$

$\underline{\underline{4ay = 16a^2 + x^2}}$

v. (h, k) satisfies equation of normal

$h + pk = 2ap + ap^3$

$ap^3 + 2ap - pk - h = 0$

$\underline{\underline{ap^3 + (2a - k)p - h = 0}}$

vi. $\underline{\underline{3}}$

(a cubic can have at most three distinct solutions).