SYDNEY TECHNICAL HIGH SCHOOL

YEAR 11 YEARLY EXAMINATION

MATHEMATICS EXTENSION 1

2005

Time allowed: 90 minutes

Directions to Candidates

- Attempt all questions
- Start each question on a new page
- All necessary working should be shown
- Unless otherwise specified, answers must be given in their simplest form
- Approved calculators may be used in all sections.
- Use a ruler when drawing straight lines
- Marks may be deducted for careless or poorly arranged work.
- Marks shown are approximate and may be varied.

Name:____________________________ Class:__________________

1	2	3	4	5	6	TOTAL

Question 1

Marks

a) Express in simplest form $6\times 3^{n}+3^{n+1}$. 2

b) If $(x-2)$ is a factor of $P(x)=x^3+ax+2$, find the value of a. 1

c) Form a quadratic equation with roots $1-\sqrt{5}$ and $1+\sqrt{5}$. Give your answer in general form. 2

d) Find the acute angle between the lines $x=3$ and $y=\frac{1}{2}x+1$ to the nearest minute. 2

e) If α,β,γ are the roots of the equation $x^3-2x+5=0$. Find the value of

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ 2

f) Draw a sketch of $P(x)=(x+3)(x-2)^2$ clearly indicating the x and y intercepts 2

Question 2

a) Let A and B be the points (0,1) and (2,3) respectively. 5

i) Find the equation of the perpendicular bisector of AB.

ii) The point P lies on the line $y=2x-9$ and is equidistant from A and B. Find the coordinates of P.

b) Find the quotient when $P(x)=2x^3+3x^2-8x-17$ is divided by $T(x)=x^2-4$. 3

c) Given $\sin 2\theta-\tan\theta\cos 2\theta=\tan\theta$.

Find the exact value, in rationalised form, of $\tan 67\frac{1}{2}°$. 3

Question 3

a) i) Sketch the graph of $x^2+y^2=25$ 5

ii) Explain why $x^2+y^2=25$ is not a function.

iii) By choosing the appropriate function, find the gradient of the tangent at the point (3,4) on the curve $x^2+y^2=25$.

b) Consider the parabola $8x = y^2 + 4y + 12$ 4

i) By completing the square find the coordinates of the vertex.

ii) Sketch the parabola showing the vertex, focus and any intercepts.

c) $P(x)$ is an odd monic polynomial of degree 3 with 2 as a zero. Write down the equation of $P(x)$. 2

Question 4

(a) i) If $x^4 - 6x^3 + 7x^2 + 6x - 8 \equiv (x^2 + ax)^2 + b(x^2 + ax) + c$, find the numerical values of *a*, *b* and *c*. 4

ii) Hence or otherwise, solve the equation $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$.

b) The points $P(12t, 6t^2)$ and $Q(36, 54)$ are points on a parabola 3

i) Find the Cartesian equation of the parabola

ii) If PQ is a focal chord find the value of *t*.

c) i) For what value of *k* does $x^2 - kx - k = 0$ have real roots 4

ii) Hence or otherwise find the range of $y = \dfrac{x^2}{1+x}$.

Question 5

a) i) Derive the equation of the tangent to the parabola $x = 2t, y = t^2$ at the point *P*, where $t = p$. 7

ii) If *Q* is the point on the parabola where $t = q$, and *OQ* is parallel to the tangent at *P* (*O* is the origin), show that $q = 2p$.

iii) *M* is the midpoint of *PQ*. Find the co ordinates of *M*.

iv) If *P* and *Q* move along the parabola so that *OQ* always remains parallel to the tangent at *P*, show that the equation of the locus of *M* is $5x^2 = 18y$.

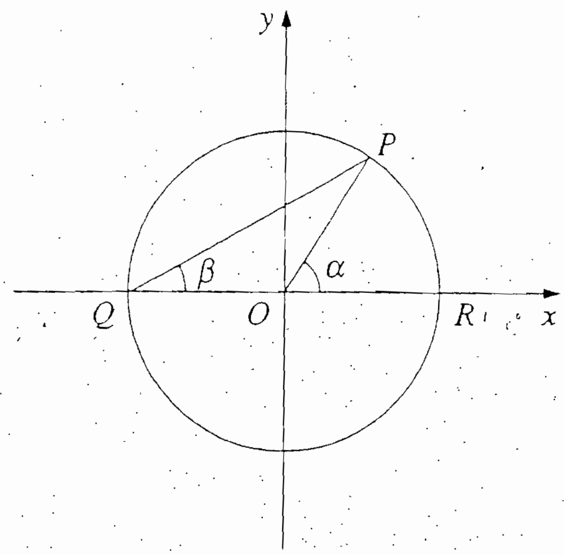
b) The formula for the area of a rectangle is given by $A = 8x\sin\theta - x^2\tan\theta$ where θ remains fixed 4

i) By treating *A* as a quadratic function show that the maximum value of *A* is $A = 8\sin 2\theta$

Question 6

a) The equation $|x^2 - 4x| = k$ has 3 solutions. Find the value of k. 1

b) 10



In the diagram, Q is the point (-1,0), R is the point (1,0), and P is another point on the circle with centre O and radius 1. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan\beta = m$.

(i) Explain why ΔOPQ is isosceles, and hence deduce that $\alpha = 2\beta$.

(ii) Find the equation of the line PQ in terms of m.

(iii) Show that the x coordinates of P and Q are solutions of the equation

$$(1+m^2)x^2 + 2m^2x + m^2 - 1 = 0.$$

(iv) Write an expression for the sum of the roots of this quadratic equation

(v) Hence or otherwise find the coordinates of P in terms of m.

(vi) By using a right angled triangle show that $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$

QUESTION 1

a)

$$6.3^n + 3^{n+1} = 6.3^n + 3^1.3^n$$
$$= 3^n(6+3)$$
$$= 9.3^n$$
$$= 3^2.3^n$$
$$= 3^{n+2}$$

b) $P(2)=0$

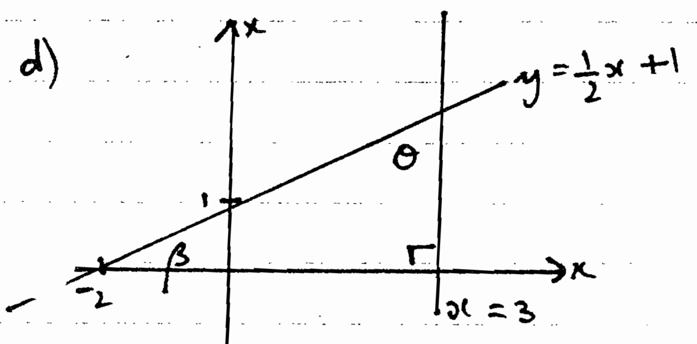
$$8+2a+2=0$$
$$2a=-10$$
$$a=-5$$

c)

$x^2 - (\text{sum roots})x + \text{product} = 0$

$x^2 - 2x - 4 = 0$

d)

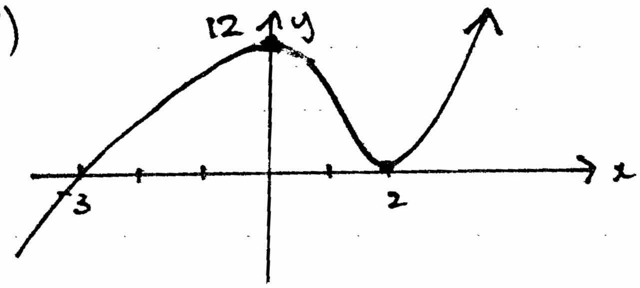


$$\tan\beta = 1/2$$
$$\beta = 26^\circ 34'$$
$$\therefore\ \theta = 63^\circ 26'$$

e)

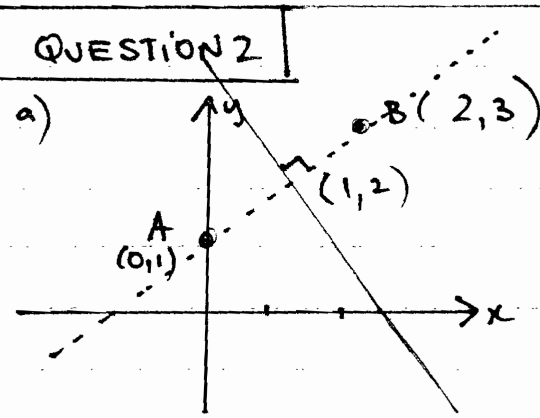
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{c/a}{-d/a}$$
$$= \frac{-2}{-5}$$

f)



QUESTION 2

a)



i) midpt AB $(1,2)$

$m_{AB} = \frac{2}{2} = 1 \quad \therefore$ perp $m = -1$

eqn perp bisector $y-2 = -1(x-1)$

$$x+y-3=0$$

ii) $P(x, 2x-9)$ lies on $x+y-3=0$

$$x+2x-9-3=0$$
$$3x-12=0$$
$$3x=12 \quad \therefore x=4$$

$\therefore P(4,-1)$

b)

$$\begin{array}{r} 2x+3 \\ x^2-0x-4 \overline{)\, 2x^3+3x^2-8x-17} \\ 2x^3+0x^2-8x \\ \hline 3x^2+0x-17 \\ 3x^2+0x-12 \\ \hline -5 \end{array}$$

c) $\sin 2\theta - \tan\theta . \cos 2\theta = \tan\theta$

$\sin 2\theta = \tan\theta + \tan\theta \cos 2\theta$

$\sin 2\theta = \tan\theta(1 + \cos 2\theta)$

$\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$

$\therefore \tan 67\tfrac{1}{2}^\circ = \dfrac{\sin 135^\circ}{1 + \cos 135^\circ}$

$= \dfrac{\sin 45^\circ}{1 - \cos 45^\circ}$

$= \dfrac{1}{\sqrt{2}} \div \left(1 - \dfrac{1}{\sqrt{2}}\right)$

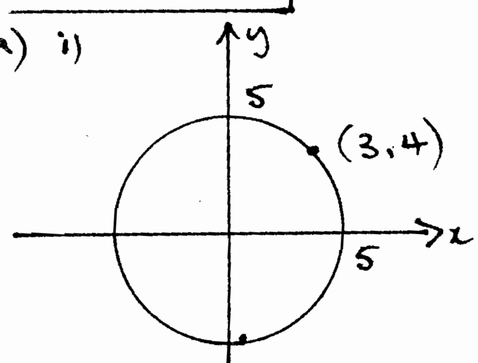
$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2} - 1}$

$= \dfrac{1}{\sqrt{2} - 1} \times \dfrac{\sqrt{2} + 1}{\sqrt{2} + 1}$

$= \underline{\underline{\sqrt{2} + 1}}$

QUESTION 3

a) i)



ii) Vertical line cuts more than once between $x = -5$ & $x = 5$

$\therefore$ each x value will not have a unique y value.

$\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{25 - x^2}}$

at $x = 3$ $m_T = \underline{\underline{-\dfrac{3}{4}}}$

b) i) $y^2 + 4y = 8x - 12$

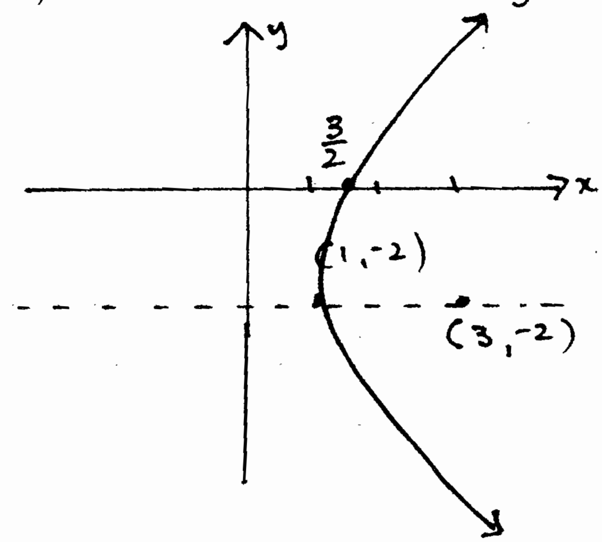
$(y^2 + 4 + 4) = 8x - 12 + 4$

$(y + 2)^2 = 8x - 8$

$(y + 2)^2 = 8(x - 1)$

Vertex $\underline{\underline{(1, -2)}}$

ii) $4a = 8$ $\therefore a = 2$ if $y = 0 \Rightarrow x =$



c) $P(x) = x^3 + ax + b$

$P(0) = 0$ passes throu' origin

$P(2) = 0$ given

$\therefore b = 0$

$8 + 2a = 0$

$2a = -8$

$a = -4$

$\therefore \underline{\underline{P(x) = x^3 - 4x}}$

$= x(x^2 - 4)$

QUESTION 4

a) i)

$x^4 - 6x^3 + 7x^2 + 6x - 8$

$\equiv x^4 + 2ax^3 + a^2x^2 + bx^2 + abx + c$

$2a = -6$ $\quad a^2 + b = 7$ $\quad ab = 6$ $\quad c = -8$

$a = -3$ $\quad 9 + b = 7$ ✓

$b = -2$ $\quad c = -8$

ii) $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8 = 0$

Let $u = x^2 - 3x$

$u^2 - 2u - 8 = 0$

$(u - 4)(u + 2) = 0$

$u = 4$ $\qquad u = -2$

$x^2 - 3x = 4$ $\qquad x^2 - 3x = -2$

$x^2 - 3x - 4 = 0$ $\qquad x^2 - 3x + 2 = 0$

$(x - 4)(x + 1) = 0$ $\qquad (x - 2)(x - 1) = 0$

$\therefore x = 4, -1, 2, 1$

b) i) $x = 12t$ $\qquad y = 6t^2$

$\frac{x}{12} = t$

$\therefore y = 6\left(\frac{x}{12}\right)^2$

$y = \frac{x^2}{24}$

ii) PQ passes throu' $(0, 6)$ focus

$m_{PQ} = \frac{54 - 6t^2}{36 - 12t} = \frac{6(9 - t^2)}{12(3 - t)}$

$= \frac{6(3 - t)(3 + t)}{12(3 - t)}$

$= \frac{3 + t}{2}$

eqn PQ

$y - 54 = \left(\frac{3 + t}{2}\right)(x - 36)$

sub pt $(0, 6)$

$6 - 54 = \left(\frac{3 + t}{2}\right)(-36)$

$-48 = -18(3 + t)$

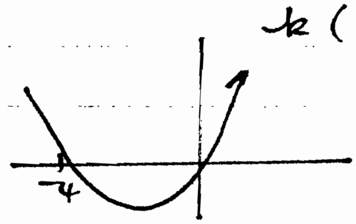
$\frac{48}{18} = 3 + t$

$t = -1/3$

c) i) $\Delta \geq 0$ $\qquad k^2 - 4.1 \times -k \geq 0$

$k^2 + 4k \geq 0$

$k(k + 4) \geq 0$



$\therefore k \geq 0$

$k \leq -4$

ii) $y = \frac{x^2}{1 + x}$ $\qquad y \neq 0$

Let $y = k$

$k(1 + x) = x^2$

$x^2 - xk - k = 0$

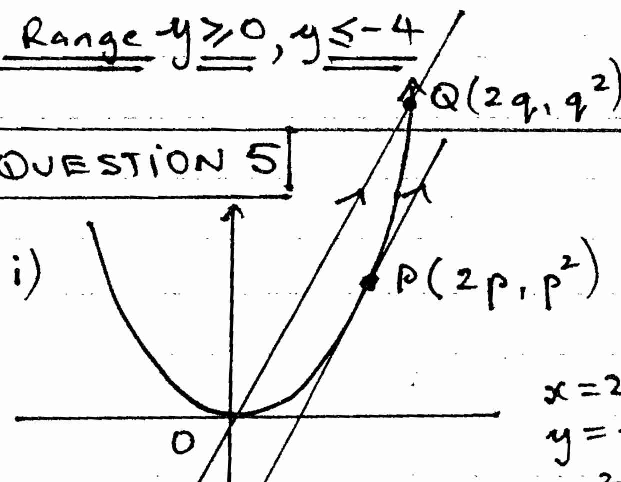
solutions for k if $\Delta \geq 0$

$\therefore k \geq 0 \quad k \leq -4$ from above

$\therefore$ Range $y \geq 0$, $y \leq -4$

QUESTION 5

a) i)



$x = 2t$

$y = t^2$

$y = \frac{x^2}{4}$ $\quad P(2p, p^2)$

$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$ $\quad Q(2q, q^2)$

$m_{T_P} = \frac{2p}{2} = p$

tang: $y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$y = px - p^2$

ii) gradient of $OQ = \frac{q^2}{2q} = \frac{q}{2}$

since $OQ \parallel$ to tangent at P

$\frac{q}{2} = p \quad \therefore \quad q = 2p$

iii) $M\left(p+q, \frac{p^2+q^2}{2}\right)$

iv) $x = p+q \qquad y = \frac{p^2+q^2}{2}$

$2y = p^2+q^2$

$(p+q)^2 = p^2+q^2+2pq$

$x^2 = 2y + 2(p.2p)$

$x^2 = 2y + 4p^2$

since $2y = p^2+q^2$

$2y = p^2+(2p)^2$

$2y = 5p^2$

$\frac{2y}{5} = p^2$

$\therefore x^2 = 2y + 4\left(\frac{2y}{5}\right)$

$5x^2 = 10y + 8y$

$5x^2 = 18y$

b) i) $A = 8x\sin\theta - x^2\tan\theta$

axis of sym: $x = \frac{-8\sin\theta}{-2\tan\theta}$

$x = \frac{4\sin\theta \cdot \cos\theta}{\sin\theta}$

$x = 4\cos\theta$

$\therefore$ max A

$= 8.4\cos\theta\sin\theta - (4\cos\theta)^2\tan\theta$

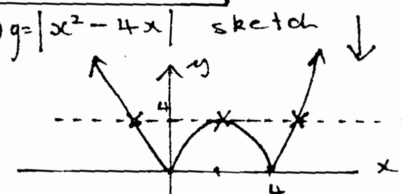
$= 32\cos\theta.\sin\theta - 16\cos^2\theta.\tan\theta$

$= 32\cos\theta.\sin\theta - 16\cos\theta.\sin\theta$

$= 16\cos\theta.\sin\theta$

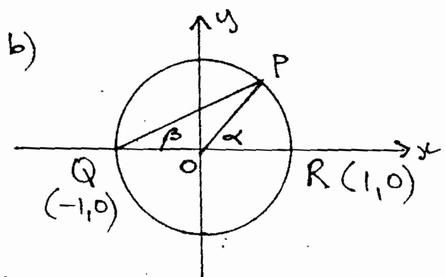
QUESTION 6

a) $y = |x^2 - 4x|$ sketch



$\therefore |x^2 - 4x| = k$ has 3 solutions if $k = 4$

b)



i) $OP = OQ$ (equal radii)

$\therefore \triangle OPQ$ is isosceles and

$O\hat{P}Q = \beta$ (opp. equal sides)

$\therefore \alpha = 2\beta$ (ext. angle of $\triangle$)

ii) eqn of PQ gradient $= m$ and $Q(-1,0)$

$y - 0 = m(x+1)$

$y = mx + m$

iii) eqn circle centre $(0,0)$ radius 1

$x^2+y^2 = 1$

PQ: $y = mx+m$ } sim. eq

$x^2 + (mx+m)^2 = 1$

$x^2 + m^2x^2 + 2m^2x + m^2 = 1$

$x^2(1+m^2) + 2m^2x + m^2 - 1 = 0$

iv) Sum Roots $(\alpha+\beta) = \frac{-2m^2}{1+m^2}$

v) if one root is -1

let other root be α

$\alpha - 1 = \frac{-2m^2}{1+m^2}$

$\alpha = \frac{-2m^2}{1+m^2} + 1$

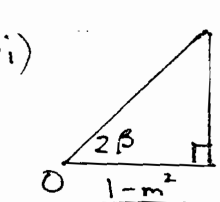
$\alpha = \frac{-2m^2+1+m^2}{1+m^2}$

$\alpha = \frac{1-m^2}{1+m^2}$ sub into line PQ

$y = m\left(\frac{1-m^2}{1+m^2}\right) + m$

$\therefore P\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$

vi)



$\frac{2m}{1+m^2}$

$\frac{1-m^2}{1+m^2}$

$\tan 2\beta = \frac{\frac{2m}{1+m^2}}{\frac{1-m^2}{1+m^2}}$

$\tan 2\beta = \frac{2m}{1-m^2}$

$m = \tan\beta$

$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$