SYDNEY TECHNICAL HIGH SCHOOL

YEAR 11 YEARLY EXAMINATION

MATHEMATICS EXTENSION 1

2006

Time allowed: 90 minutes

Directions to Candidates

- Attempt all questions
- Start each question on a new page
- All necessary working should be shown
- Unless otherwise specified, answers must be given in their simplest form
- Approved calculators may be used in all sections.
- Use a ruler when drawing straight lines
- Marks may be deducted for careless or poorly arranged work.
- Marks shown are approximate and may be varied.

Name:______________________________ Class:____________________

1	2	3	4	5	6	TOTAL

Question 1

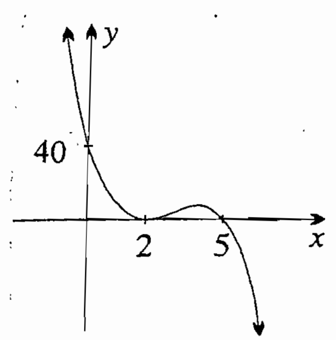
a) Fully factorise $2a^3 - 128$. (1)

b) i) Find the remainder when $P(x) = x^4 + 2x^2 - 5$ is divided by $(x-2)$.

ii) Explain why there is a zero in the domain $0 < x < 2$. (3)

c) If $4 + \sqrt{b} = \sqrt{19 + \sqrt{m}}$, find the values of b and m. (2)

d) This is the graph of $y = P(x)$. (3)



i) Write down the equation of $y = P(x)$.

ii) Write down the domain of $y = \sqrt{P(x)}$.

e) Find the gradient of the tangent to the curve $y = \sqrt{5 + x^2}$ at $x = -2$ as a fraction. (2)

Question 2 (Start on the next page)

a) Use long division to find the remainder when $x^4 - x^2 - x + 8$ is divided by $x^2 - 3$. (2)

b) If $f(x) = \dfrac{x^2}{x+4}$ (3)

i) Find $f'(x)$.

ii) Find the values of x if $f'(x) > 0$.

c) i) Explain why $|xy| = 4$ is not a function.

ii) Sketch the graph of $|xy| = 4$. (2)

Q2 (cont.)

d) i) Write down the expansion for $\tan 2A$. (4)

ii) Hence find the exact value of $\tan 22\frac{1}{2}°$.

Question 3 (Start on the next page)

a) Two roots of the cubic equation $x^3 + mx + n = 0$ are -3 and 4.

i) Find the third root. (2)

ii) Find the value of n.

b) The points $P\left(t, \frac{t^2}{2}\right)$ and Q (-4, 8) are points on a parabola. (3)

i) Find the cartesian equation of the parabola.

ii) If PQ is a focal chord, what are the co-ordinates of P.

c) If $\sec\theta - \tan\theta = x$, show that $x = \dfrac{1-t}{1+t}$ where $t = \tan\frac{\theta}{2}$. (3)

d) i) Find the value of c if $P(x) = x^3 - 3x^2 - 4x + c$ is divisible by $x - 3$. (3)

ii) Hence evaluate $\lim\limits_{x\to 3} \dfrac{x^3 - 3x^2 - 4x + c}{x-3}$.

Question 4 (Start on the next page)

a) i) Find the locus of the point P which is equidistant from the points $A(0, 2)$ and $B(6, 0)$. (5)

ii) A point Q is closer to B than A and less than 4 units from A. Write inequalities which would describe the region where Q could be located.

b) $T\ (6t, 3t^2)$ is a point on the parabola $x^2 = 12y$. The point D is at the intersection of the directrix and the y axis. (6)

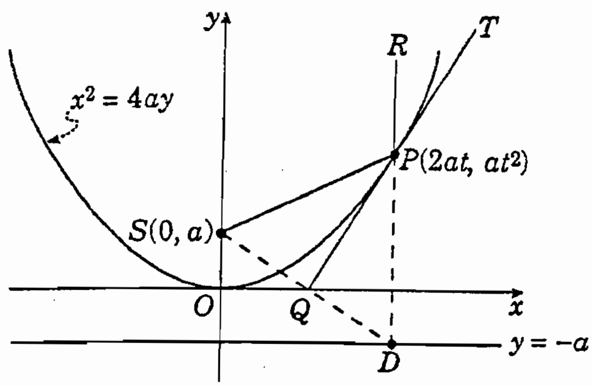
i) The point P divides TD internally in the ratio 2:1. Write down the co-ordinates of P in terms of t.

ii) Show that as T moves on the parabola $x^2 = 12y$ the locus of P is $x^2 = 4y+8$.

iii) Write down the focus and directrix of the locus of P.

Question 5 (Start on the next page)

a) (8)



The diagram shows the parabola $x^2 = 4ay$ with focus $S(0, a)$ and directrix $y = -a$. The point $P(2at, at^2)$ is an arbitrary point on the parabola and the line RP is drawn parallel to the y axis, meeting the directrix at D. The tangent QPT to the parabola at P intersects SD at Q.

i) Explain why $SP = PD$.

ii) Derive the gradient m_1 of the tangent at P.

iii) Find the gradient m_2 of the line SD.

iv) Prove that PQ is perpendicular to SD.

v) Prove that $\angle RPT = \angle SPQ$.

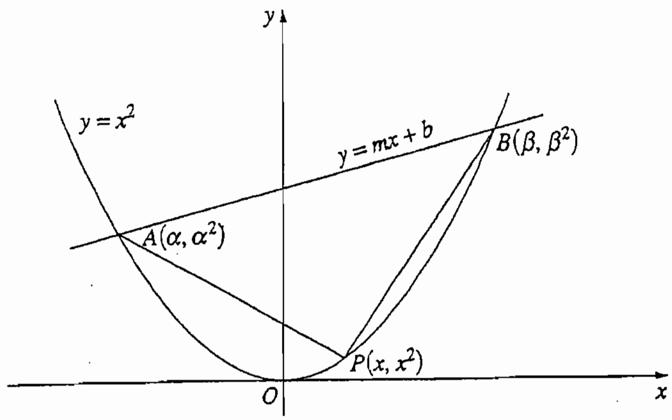
b) The sum of two roots of the equation $x^3 + kx^2 + mx + n = 0$ is zero. (3)

Show that $km = n$.

Question 6 (Start on the next page)

a) Solve $\cos 2x = \sin x$ for $0° \le x \le 360°$. (3)

b) (8)



The parabola $y = x^2$ and the line $y = mx + b$ intersect at the points $A(\alpha, \alpha^2)$ and $B(\beta, \beta^2)$ as shown in the diagram.

i) Explain why $\alpha + \beta = m$ and $\alpha\beta = -b$.

ii) Factorise the expression $(\alpha - \beta)^2 + (\alpha^2 - \beta^2)^2$

iii) Hence or otherwise, using the fact that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$, show that the distance AB is given by

$$AB = \sqrt{(m^2 + 4b)(1 + m^2)}$$

iv) The point $P(x, x^2)$ lies on the parabola between A and B. Show that the area of the triangle ABP is given by $\frac{1}{2}\left(mx - x^2 + b\right)\sqrt{m^2 + 4b}$.

v) By treating $\frac{1}{2}\left(mx - x^2 + b\right)\sqrt{m^2 + 4b}$ as a quadratic expression in terms of x, show that the maximum area of the triangle ABP is $\dfrac{(m^2 + 4b)^{\frac{3}{2}}}{8}$

End of Exam

1

a) $2a^3-128 = 2(a^3-64)$

$= 2(a-4)(a^2+4a+16)$

b) i) $P(2) = 16+8-5$

$= 19$

ii) $P(0) = -5$

$P(x)$ changes sign between 0 and 2. ∴ at least one zero

c) $(4+\sqrt{b})^2 = 19+\sqrt{m}$

$16+8\sqrt{b}+b = 19+\sqrt{m}$

$\therefore b=3$

and $8\sqrt{3} = \sqrt{m}$

$\therefore m = 192$

d) i) $P(x) = -2(x-2)^2(x-5)$

ii) $x \leq 5$

e) $y' = \frac{1}{2}\cdot 2x\cdot(5+x^2)^{-1/2}$

$= x(5+x^2)^{-1/2}$

at $x=-2$

$y' = \frac{-2}{\sqrt{9}}$

$= -2/3$

2

a)

$$\begin{array}{r} x^2 \qquad +2 \\ x^2-3 \,\big)\overline{\, x^4 \qquad -x^2-x+8} \\ x^4 \qquad -3x^2 \qquad\quad \\ \hline 2x^2-x \quad\; \\ 2x^2 \qquad -6 \\ \hline -x+14 \end{array}$$

∴ remainder $-x+14$

b) i) $f(x) = \frac{x^2}{x+4}$

$f'(x) = \frac{(x+4).2x - x^2.1}{(x+4)^2}$

$= \frac{x^2+8x}{(x+4)^2}$

ii) $f'(x) > 0$

$\frac{x^2+8x}{(x+4)^2} > 0$

$\frac{x(x+8)}{(x+4)^2} > 0$

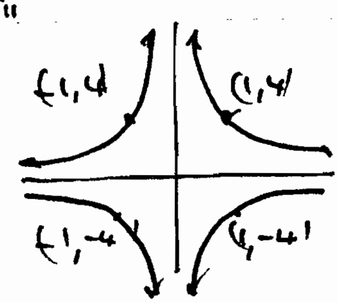
$-8 \qquad 0$

∴

$x<-8, \; x>0$

c) i) each value of x there is more than 1 value of y

ii



d) i) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

ii) Let $x = \tan 22\frac{1}{2}$

$\therefore \tan 45^\circ = \dfrac{2x}{1-x^2}$

$1 = \dfrac{2x}{1-x^2}$

$1-x^2 = 2x$

$x^2+2x-1=0$

$x = \dfrac{-2 \pm \sqrt{4+4}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2}$

$= -1 \pm \sqrt{2}$

$\tan 22\frac{1}{2} > 0$

$\therefore$

$\tan 22\frac{1}{2} = \sqrt{2}-1$

③ a) i) Let roots be $-3, 4, \alpha$

$\therefore -3+4+\alpha = 0$

$\alpha = -1$

ii) $-3 \times 4 \times -1 = -n$

$12 = -n$

$n = -12$

b) i) $x = t$

$y = \dfrac{t^2}{2}$

$\therefore y = \dfrac{x^2}{2}$

ii) At Q $t = -4$

$\therefore$ at P $t = \frac{1}{4}$

$\therefore P\left(\frac{1}{4}, \frac{1}{32}\right)$

c) $\tan\theta/2 = t$

$\sec\theta - \tan\theta = \dfrac{1+t^2}{1-t^2} - \dfrac{2t}{1-t^2}$

$= \dfrac{t^2-2t+1}{1-t^2}$

$= \dfrac{(1-t)^2}{(1-t)(1+t)}$

$= \dfrac{1-t}{1+t}$

$\therefore x = \dfrac{1-t}{1+t}$

d) i) For factor

$P(3) = 0$

$\therefore P(3) = 27-27-12+c$

$\therefore c = 12$

ii) $\lim\limits_{x\to 3} \dfrac{x^3-3x^2-4x+12}{x-3}$

$= \lim\limits_{x\to 3} \dfrac{x^2(x-3)-4(x-3)}{(x-3)}$

$= \lim\limits_{x\to 3} \dfrac{(x^2-4)(x-3)}{(x-3)}$

$= \lim\limits_{x\to 3} x^2-4$

$= 5$

4 a) i) $PA^2 = PB^2$

$\therefore x^2+(y-2)^2 = (x-6)^2+y^2$

$x^2+y^2-4y+4 = x^2-12x+36+y^2$

$12x-4y-32=0$

$3x-y-8=0$

ii) Locus 4 units from A

$x^2+(y-2)^2 = 16$

at B

$\therefore 3x-y-8 > 0$

$\therefore$ region: intersection of

$3x-y-8>0$ and

$x^2+(y-2)^2<16$

b)

i) $D(0,-3)$

$(6t, 3t^2) \quad P \quad (0,-3)$

$2 \quad 1$

$P = \dfrac{6t}{3}, \dfrac{3t^2-6}{3}$

$=$

$= (2t, t^2-2)$

ii) $x = 2t$ (1)

$y = t^2-2$ (2)

From (1) $t = \frac{x}{2}$

Sub in (2)

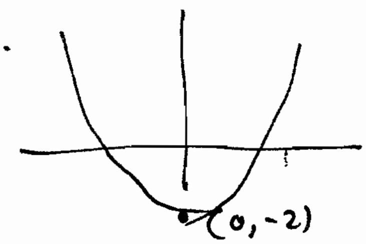
$y = \left(\frac{x}{2}\right)^2-2$

$4y = x^2-8$

$x^2 = 4y+8$

iii $x^2 = 4(y+2)$

$\therefore$



$\therefore$ focus $(0,-1)$

directrix $y=-3$

5 i) $SP = PD$ - definition of parabola

ii) $x = 2at \qquad y = at^2$

$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{2at}{2a}$

$= t$

iii $D = (2at, -a)$

$m_{SD} = \dfrac{a-(-a)}{0-2at}$

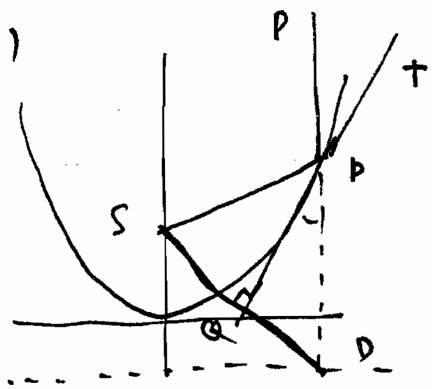
$= \dfrac{2a}{-2at}$

iv) $m_1 = t$
$m_2 = -1/t$

$\therefore m_1 \cdot m_2 = t \times -\frac{1}{t}$
$= -1$

$\therefore PQ \perp SQ$

v)



$\angle RPT = \angle QPD$
(vertically opposite $\angle$)

Now $\cos QPD = \frac{PQ}{PD}$

$\cos QPS = \frac{PQ}{SP}$

but $PD = SD$ proven in (i)
$\therefore \angle QPD = \angle QPS$

$\therefore \angle RPT = \angle QPS$

b) Let roots be
$\alpha, -\alpha, \beta$.

$\therefore \beta = -K$

but β is a root of the polynomial

$\therefore P(-K) = 0$

$(-K)^3 + K(-K)^2 - mK + n = 0$

$\therefore mK = n$.

⑥ $\cos 2x = \sin x$
$1 - 2\sin^2 x = \sin x$
$2\sin^2 x + \sin x - 1 = 0$
$(2\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin x = 1/2$, $\sin x = -1$
$\therefore x = 30°, 150°$, $x = 270°$

$\therefore x = 30°, 150°, 270$

b) i) $y = x^2$
$y = mx + b$
$\therefore x^2 = mx + b$
$x^2 - mx - b = 0$
α and β are the roots of this equation.
$\therefore \alpha + \beta = -\frac{b}{a}$
$= m$
$\alpha\beta = \frac{c}{a}$
$= -b$

ii) $(\alpha - \beta)^2 + (\alpha^2 - \beta^2)^2$
$= (\alpha - \beta)^2 + [(\alpha - \beta)(\alpha + \beta)]^2$
$= (\alpha - \beta)^2 [1 + (\alpha + \beta)^2]$

iii) $AB = \sqrt{(\alpha - \beta)^2 + (\alpha^2 - \beta^2)^2}$

$= \sqrt{(\alpha - \beta)^2 [1 + (\alpha + \beta)^2]}$

from results

$= \sqrt{[(\alpha + \beta)^2 - 4\alpha\beta][1 + (\alpha + \beta)^2}$

$$= \sqrt{(m^2+4b)(1+m^2)}$$

iv)

Distance P to line AP

$$d = \frac{mx - x^2 + b}{\sqrt{1+m^2}}$$

$$\therefore \text{Area} = \frac{1}{2} \cdot \sqrt{(m^2+4b)(1+m^2)} \cdot \frac{mx-x^2+b}{\sqrt{1+m^2}}$$

$$= \frac{1}{2}(mx - x^2+b)\sqrt{m^2+4b}$$

v) maximum value occurs when

$$x = \frac{-b}{2a}$$

$$= \frac{-m}{-2}$$

$$= \frac{m}{2}.$$

$$\therefore \text{Area} = \frac{1}{2} \cdot \left[\frac{m^2}{2} - \frac{m^2}{4} + b\right]\sqrt{m^2+4b}$$

$$= \frac{1}{2}\left[\frac{m^2}{4} + b\right]\sqrt{m^2+4b}$$

$$= \frac{1}{8}\left[m^2+4b\right]\sqrt{m^2+4b}$$

$$= \frac{(m^2+4b)^{3/2}}{8}$$