Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

MANNERS MAKYTH MAN

YEAR 11 YEARLY EXAMINATION

Mathematics Extension 1

September 2007

TIME ALLOWED: 90 minutes

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- **START EACH QUESTION ON A NEW PAGE**

(FOR MARKERS USE ONLY)

Q1	Q2	Q3	Q4	Q5	TOTAL
/15	/14	/14	/13	/14	/70

QUESTION 1: (15 MARKS)

Marks

1 (a) Find $\lim_{x\to\infty}\frac{2x^2-x+1}{3x^2+2}$

2 (b) If θ is an acute angle and $\sin\theta=\frac{2t}{1+t^2}$ find an expression for $\cos\theta$ in terms of t if $-1<t<1$

(c) Find the derivatives of the following, leaving your answer in simplest terms and including no negative indices:

2 (i) $y=\sqrt{1-3x^2}$

2 (ii) $y=\frac{1}{1+x^2}$

2 (iii) $y=x\sqrt{1+x}$

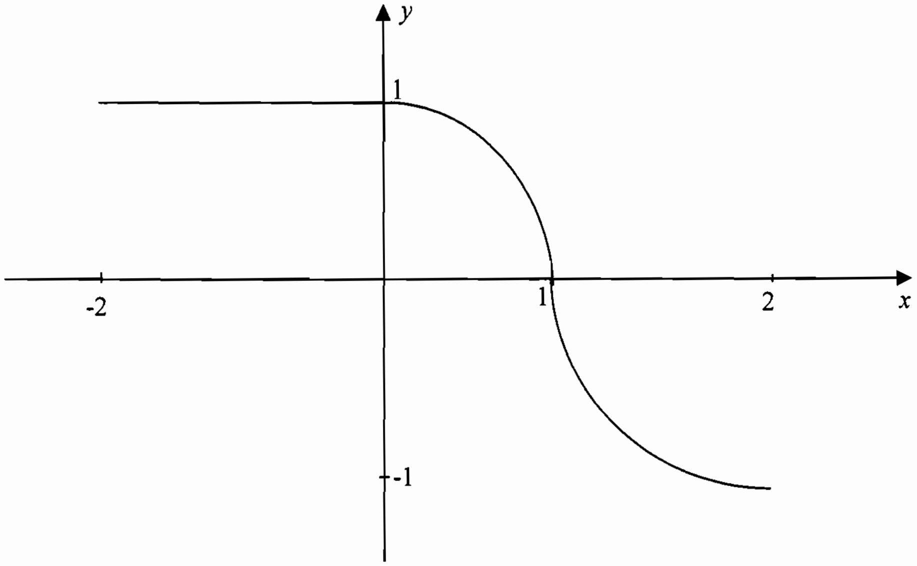
3 (d) By rationalising the numerator, find $\lim_{x\to\infty}\frac{\sqrt{2+h}-\sqrt{2}}{h}$

3 (e) Solve $\sin 2x=\cos x$ for $0°\le x\le 360°$

QUESTION 2: (14 MARKS)

Marks

(a) The curve $y = P(x)$ is shown at below for $-2 \leq x \leq 2$



6 (i) Using a different set of axes for each, neatly sketch the following showing all important features

(I) $y = |P(x)|$

(II) $y = \dfrac{1}{P(x)}$

(III) $y = 1 - P(x)$

2 (ii) What is the Domain and Range of $y = \dfrac{1}{P(x)}$

2 (b) (i) Show that for all values of x, $2x^2 - 5x + 4 > 0$

3 (ii) For what values of x is $\dfrac{x^2 - 7x + 12}{2x^2 - 5x + 4} > 1$

1 (iii) Explain clearly why the result of part (b) (i) above is vital to the solution to part (b) (ii)

QUESTION 3: (14 MARKS)

Marks

3 (a) The roots of the polynomial equation $2x^3 - 3x^2 + 1 = 0$ are α, β and γ

Find (i) $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(b) Consider the interval joining A(2, -5) to B(5, 10)

2 (i) Find the point P which divides AB externally in the ratio 2:3

2 (ii) In what ratio does the point Q(4,5) divide AB?

3 (c) The angle between the lines $y - 2x + 5 = 0$ and $y = mx + 3$ is 45°

Find all possible values of m.

3 (d) (i) Show that for the function $f(x) = \frac{2 - 3x}{5 + x^2}$ the derivative is zero for two values of x

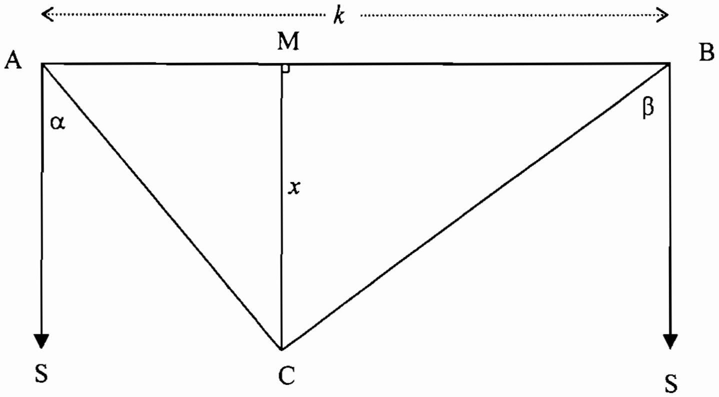
1 (ii) What does this result imply about the tangent at points with these x values?

QUESTION 4: (13 MARKS)

Marks

3 (a) Find the equation, in simplest expanded form, of the locus of a point P(x,y) which moves so that its distance from the point S(3,0) is twice its distance from the line x-5 = 0

(b) A straight road runs k km from a town A due east to a town B.
A radio mast at C has a bearing of Sα°E from A and Sβ°W from B
C is x km from the road AB



(i) Copy the diagram above onto your answer sheet

1 (ii) Explain why $\angle ACB = \alpha + \beta$

2 (iii) Show that $\dfrac{k}{\sin(\alpha+\beta)} = \dfrac{BC}{\cos\alpha}$

2 (iv) Show that $x = \dfrac{k\cos\alpha\cos\beta}{\sin(\alpha+\beta)}$

1 (c) (i) Explain why the remainder upon dividing a polynomial P(x) by x-α has a degree of zero

1 (ii) If $P(x) = 2x^4 - x^3 + 3x^2 - 1$ is divided by x-3 find the remainder.

3 (iii) One root of $ax^2 + mx + n = 0$ is three times the other.
Show that $3m^2 - 16an = 0$.

QUESTION 5: (14 MARKS)

Marks

2 (a) Prove that, for all values of m, the line $x = \frac{y}{m} + Am$ is a tangent to the parabola $x^2 = 4Ay$

3 (b) (i) Write the expression $2\cos x - 3\sin x$ in the form $A\cos(x + \alpha)$, showing the values of A and α

2 (ii) Hence, or otherwise, solve, for $0° \leq x \leq 360°$, the equation

$$2\cos x - 3\sin x = \sqrt{13}$$

Give your answer(s) to the nearest minute.

(c) The point $P(2Ap, Ap^2)$ lies on the parabola $x^2 = 4Ay$

2 (i) Derive the equation of the tangent to the parabola at the point P.

1 (ii) The point Q is another point on the parabola and has an x-value of $\frac{2A}{p}$

Show that it has a y-value of $\frac{A}{p^2}$

2 (iii) Derive the equation of the tangent to the parabola at the point Q above

2 (iv) The tangents at P and Q intersect at the point ***R***.
Accurately describe the locus of ***R***.

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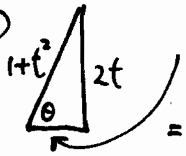
QUESTION 1

a) $2/3$ ①

b)



$$\sqrt{(1+t^2)^2-4t^2} = 1-t^2$$

$$\therefore \cos\theta = \frac{1-t^2}{1+t^2}$$ ②

c) i) $$\frac{dy}{dx} = \frac{1}{2}(-6x)(1-3x^2)^{-\frac{1}{2}}$$

$$= \frac{-3x}{\sqrt{1-3x^2}}$$

ii) $$\frac{dy}{dx} = -2x(1+x^2)^{-2}$$

$$= \frac{-2x}{(1+x^2)^2}$$

iii) $$\frac{dy}{dx} = (1+x)^{\frac{1}{2}} + x.\frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$= \frac{2+2x+x}{2(1+x)^{\frac{1}{2}}}$$

$$= \frac{2+3x}{2\sqrt{1+x}}$$

d) $$\frac{\sqrt{2+h}-\sqrt{2}}{h} \times \frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$$

$$= \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{2+h}+\sqrt{2}}$$

$$\therefore \lim_{h\to\infty}\left(\frac{\sqrt{2+h}-\sqrt{2}}{h}\right) = 0$$

e) $2\sin x\cos x = \cos x$

ie. $\cos x(2\sin x - 1) = 0$

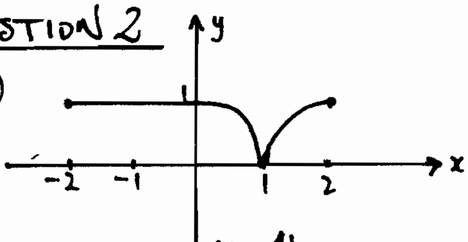
$\therefore \cos x = 0$ OR $\sin x = \frac{1}{2}$

$\therefore x = 90°, 270°$ $\therefore x = 30°, 150°$

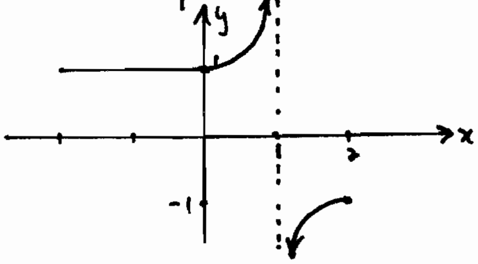
QUESTION 2

a) i)

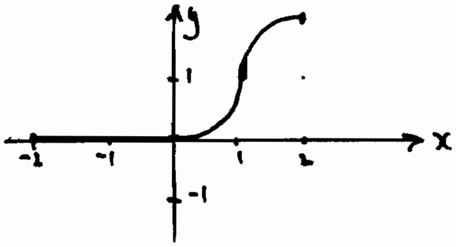
I)



II)



III)



ii) $\mathcal{D}: \{x: -2 \le x \le 2,\ x \ne 1\}$

OR $\mathcal{D}: \{x: -2 \le x < 1\} \cup \{x: 1 < x \le 2\}$

$\mathcal{R}: \{y: -\infty < y \le -1\} \cup \{y: 1 \le y < \infty\}$

b) i) For $2x^2 - 5x + 4$,

$\Delta = 25 - 4.2.4$

$= -7$

Since $\Delta < 0$, $2x^2 - 5x + 4$ is positive definite, ie. > 0

OR $2x^2 - 5x + 4 = 2(x - \frac{5}{4})^2 + \frac{7}{8}$

which has a minimum value of $\frac{7}{8}$

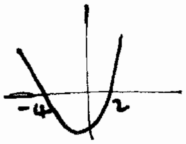
$\therefore 2x^2 - 5x + 4 > 0$

QUESTION 2 (cont)

b) ii) $\dfrac{x^2-7x+12}{2x^2-5x+4} > 1$

ie. $x^2-7x+12 > 2x^2-5x+4$ — Ⓐ

ie. $x^2+2x-8<0$



$(x+4)(x-2)<0$

$\therefore \quad -4<x<2$

iii) Inequality sign at Ⓐ above does not have to be reversed because we know we've multiplied both sides by a positive quantity.

QUESTION 3

a) i) $\alpha+\beta+\gamma = \frac{3}{2}, \ \left(-\frac{b}{a}\right)$

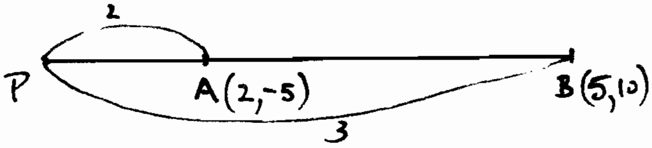
$\alpha\beta\gamma = -\frac{1}{2} \ \left(-\frac{d}{a}\right)$

ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{0}{-\frac{1}{2}}$

$= 0$

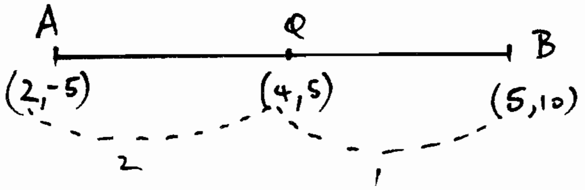
b)



OR

i) At P $x=-4$, $y=-35$

ii)



Q divides AB in the ratio 2:1.

(For real numbers, better not

c) $m_1 = m, \ m_2 = 2$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

ie $1 = \left|\frac{m-2}{1+2m}\right|$

ie $m-2 = 1+2m \Rightarrow m=-3$

or $2-m = 1+2m \Rightarrow m = \frac{1}{3}$

d) i)

$f'(x) = \frac{(5+x^2)(-3)-(2-3x)(2x)}{(5+x^2)^2}$

$= \frac{3x^2-4x-15}{(5+x^2)^2}$

$= 0$ when $3x^2-4x-15=0$

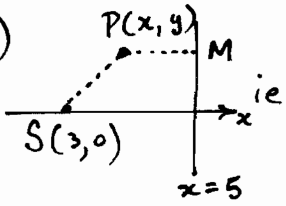
ie. $(3x+5)(x-3)=0$

$\therefore \ x = -\frac{5}{3}, 3$

ii) When $x=-\frac{5}{3}$ or 3 the tangent is horizontal ($\parallel$ the x axis)

QUESTION 4

a)



$PS = 2PM$

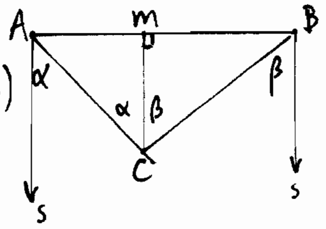
ie $PS^2 = 4PM^2$

$(x-3)^2+(y-0)^2 = 4(5-x)^2$

$x^2-6x+9+y^2 = 4(25-10x+x^2)$

ie $3x^2-34x+91-y^2=0$

b) i)



ii) $\angle ACB = \angle ACM + \angle BCM$

But $\angle ACM = \alpha$ (alternate ∠s, parallel lines)

QUESTION 4 (cont.)

b) iii) In ΔABC, $\frac{AB}{\sin A\hat{C}B} = \frac{BC}{\sin C\hat{A}B}$

ie. $\frac{k}{\sin(\alpha+\beta)} = \frac{BC}{\sin(90-\alpha)}$

But $\sin(90-\alpha) = \cos\alpha$

$\therefore \frac{k}{\sin(\alpha+\beta)} = \frac{BC}{\cos\alpha}$

iv) In ΔBMC, $\frac{x}{BC} = \cos\beta$

ie. $x = BC\cos\beta$

From iii) above, $BC = \frac{k\cos\alpha}{\sin(\alpha+\beta)}$

$\therefore x = \frac{k\cos\alpha}{\sin(\alpha+\beta)}\cos\beta$ as required

c) i) Remainder must have a degree less than the divisor, $(x-\alpha)$, which is of degree one.

ii) Remainder $= P(3)$

$= 2\times(3)^4 - 3^3 + 3(3)^2 - 1$

$= 161$

iii) Let the roots of $ax^2 + mx + n = 0$ be α and 3α.

Then $4\alpha = -\frac{m}{a}$ and $3\alpha^2 = \frac{n}{a}$

$\therefore \alpha = -\frac{m}{4a}$ and $3\left(-\frac{m}{4a}\right)^2 = \frac{n}{a}$

ie. $3m^2 = 16an$

QUESTION 5

(a) $x = \frac{y}{m} + Am$ and $x^2 = 4Ay$

intersect when $\left(\frac{y}{m} + Am\right)^2 = 4Ay$.

ie. $\underline{y^2} + 2Ay + A^2m^2 - 4Ay = 0$

ie. $y^2 - 2Aym^2 + A^2m^4 = 0$

Now $\Delta = 4A^2m^4 - 4.A^2m^4$

$= 0$

$\therefore$ Only 1 point of intersection

$\therefore x = \frac{y}{m} + Am$ is a tangent to the parabola.

(b) i) $2\cos x - 3\sin x = A\cos(x+\alpha)$

$\therefore A = \sqrt{2^2+3^2} = \sqrt{13}$

$\therefore \sqrt{13}\cos(x+\alpha) = 2\cos x - 3\sin x$.

From this we get

$\cos\alpha = 2/\sqrt{13}$ and $\sin\alpha = 3/\sqrt{13}$

$\therefore \alpha = 56°19'$

$\therefore 2\cos x - 3\sin x = \sqrt{13}\cos(x + 56°19')$

ii) $2\cos x - 3\sin x = \sqrt{13}$

$\therefore \cos(x + 56°19') = 1$

ie $x + 56°19' = 0°$ or $360°$

$\therefore x = -56°19'$ or $303°41'$.

But $-56°19'$ is outside the domain

$\therefore x = 303°41'$

(c) i) $y = \frac{x^2}{4A}$

$\therefore \frac{dy}{dx} = \frac{2x}{4A} = \frac{x}{2A}$

At $P(2Ap, Ap^2)$ $m_T = \frac{2Ap}{2A} = p$

and the equation of the tangent is $y - Ap^2 = p(x - 2Ap)$

ie. $y = px - Ap^2$

QUESTION 5 (cont.)

c) ii) $x^2 = 4Ay$ and $x = \frac{2A}{p}$

$\therefore \quad \frac{4A^2}{p^2} = 4Ay$

ie $y = \frac{4A^2}{4Ap^2} = \frac{A}{p^2}$

iii) At Q, $m_T = \frac{2A/p}{2A}$

$= \frac{1}{p}$

$\therefore$ Equation of tangent is

$y - \frac{A}{p^2} = \frac{1}{p}\left(x - \frac{2A}{p}\right)$

$y - \frac{A}{p^2} = \frac{x}{p} - \frac{2A}{p^2}$

ie. $p^2y - A = px - 2A$

ie. $p^2y = px - A$

or

$y = \frac{x}{p} - \frac{A}{p^2}$

iv) Now $y = px - Ap^2$

and $p^2y = px - A$

$\therefore \quad y(1-p^2) = -A(p^2-1)$

$\therefore \quad y = A$

The locus of R lies on the extensions of the latus rectum.