Name: ____________________

Teacher: ____________________

SYDNEY TECHNICAL HIGH SCHOOL

Mathematics Extension 1

Preliminary Yearly Examination
September 2009

Time allowed — 90 minutes

Instructions

- Reading time — 5 minutes
- Approved calculators may be used.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks awarded are shown on each question.
- Total marks — 60
- Attempt all questions.
- Start each question on a new page.

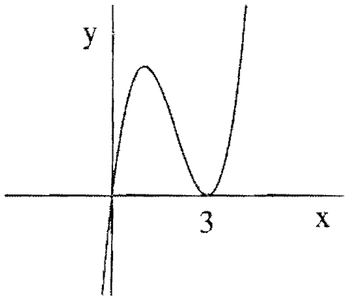
Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total

Question 1 **Marks 10**

a) Use the Remainder Theorem to find the remainder when $P(x) = 2x^5 - 3x^2 + 7$ is divided by $(x - 3)$. *(1)*

b) Find the coordinates of the point which divides the interval $A(-2,1)$ to $B(3,2)$ in the ratio $4:3$. *(2)*

c) The sketch shows the cubic $y = P(x)$.



(i) What is the equation of $P(x)$ if it is monic? *(2)*

(ii) If it is not monic, find the equation of $P(x)$ if $P(2) = 4$? *(2)*

d) Find the equation of the normal to the curve $y = x^{\frac{3}{2}}$ at the point $(4, 8)$. *(3)*

Question 2 **Marks 10**

a) Find the acute angle between the lines $2x + y - 3 = 0$ and $5x - 2y + 4 = 0$ to the nearest minute. *(2)*

b) The cubic $3x^3 - 8x^2 + 6x - 1$ has roots α, β and γ.

Evaluate: *(i)* $\alpha + \beta + \gamma$ *(1)*

(ii) $\alpha\beta\gamma$ *(1)*

(iii) $\alpha\beta + \alpha\gamma + \beta\gamma$ *(1)*

(iv) $\left(\alpha\beta + \frac{1}{\alpha\beta}\right) + \left(\alpha\gamma + \frac{1}{\alpha\gamma}\right) + \left(\beta\gamma + \frac{1}{\beta\gamma}\right)$ *(3)*

c) Describe, with the aid of diagrams and mathematical language, the meaning of:

(i) *continuity* at a point on a curve. (1)

(ii) *differentiability* at a point on a curve. (1)

Question 3 **Marks 10**

a) Find the derivative of $y = x^2\sqrt{1-x}$, expressing your answer as a single fraction. *(3)*

b) Solve $\frac{x-2}{x+2} \geq 1$ (3)

c) (i) Find the distance of *R(1,-2)* from the line $x - 2y + 3 = 0$. *(2)*

(ii) Show, without graphing, that *R(1,-2)* and *S(-2,3)* are on opposite sides of the line $x - 2y + 3 = 0$. *(1)*

d) Find the Cartesian equation represented by $x = \frac{t}{2}$ and $y = 2t^2$. *(1)*

Question 4 **Marks 10**

a) $f(x) = \sqrt{x^2 - 9}$

What is: (i) the domain? *(2)*

(ii) the range? *(1)*

b) (i) Show that $\frac{1+\cos 2A}{\sin 2A} = \cot A$ *(2)*

(ii) Hence find the exact value of *cot* $22°30'$. *(1)*

c) *P(x,y)* is constrained such that $\angle APB = 90^\circ$ where *A* is *(1,-1)* and *B* is *(7,3)*.

(i) Find an expression for the gradient of *AP*. *(1)*

(ii) Show that the locus of *P(x,y)* is a circle. *(2)*

(iii) Find its centre and radius. *(1)*

Question 5 **Marks 10**

a) A parabola has the equation $6y = x^2 - 4x - 14$.

(i) Express the equation of the parabola in the form $(x-h)^2 = 4a(y-k)$. (2)

Find:

(ii) the coordinates of the vertex. (1)

(iii) the coordinates of the focus. (1)

(iv) the equation of the directrix. (1)

b) Solve $3\sin x - 2\cos x = 2$, for $0° \leq x° \leq 360°$. (3)

c) Show that $\sqrt{x} + \dfrac{5}{\sqrt{x}} = 3$ has no real solutions. (2)

Question 6 **Marks 10**

a) Solve $\cos^2\theta - \sin 2\theta = 0$, for $0° \leq \theta° \leq 360°$. (3)

b) $P(8p,4p^2)$ and $Q(8q,4q^2)$ are variable points on the parabola $x^2=16y$.
The chord PQ produced passes through the fixed point $(4,0)$.
The tangents at P and Q meet at R.

(i) Find the equation of the chord PQ. (1)

(ii) Show that $p + q = 2pq$. (2)

(iii) Find the coordinates of R in terms of p and q. (2)

(iv) Find the equation of the locus of R. (2)

2009 Preliminary Extension 1 Mathematics Yearly – Solutions

Q1 a) $P(x) = 2x^5 - 3x^2 + 7$

$P(3) = 2\times 3^5 - 3\times 3^2 + 7$

$= 486 - 27 + 7$

$= 466$

$\therefore P(x) = (x-3).Q(x) + 466$

b) $A(-2,1)$ $B(3,2)$ $4:3 = m:n$

$\therefore x = \frac{nx_1 + mx_2}{m+n} = \frac{3\times -2 + 4\times 3}{7} = \frac{6}{7}$

$y = \frac{ny_1 + my_2}{m+n} = \frac{3\times 1 + 4\times 2}{7} = \frac{11}{7}$

coordinates $\Rightarrow \left(\frac{6}{7}, \frac{11}{7}\right)$

c) (i) equation $\Rightarrow y = x(x-3)^2$

$= x(x^2 - 6x + 9)$

$= x^3 - 6x^2 + 9x$

(ii) $y = a(x^3 - 6x^2 + 9x)$

$\therefore 4 = a(8 - 24 + 18)$

$= 2a$

$\therefore a = 2$

$\therefore y = 2x^3 - 12x^2 + 18x$

d) $y = x^{\frac{3}{2}}$

$\therefore y' = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

$= 3$ when $x = 4$ (positive root as $y = 8$)

$\therefore y - 8 = -\frac{1}{3}(x-4)$

$\therefore 3y - 24 = 4 - x$

$\therefore x + 3y - 28 = 0$

Q2 a) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-2 - \frac{5}{2}}{1 + -2.\frac{5}{2}}\right|$

$= \left|\frac{-4\frac{1}{2}}{-4}\right|$

$= \left|\frac{9}{8}\right|$

$\therefore \theta = \tan^{-1}\left(\frac{9}{8}\right)$

$= 48^\circ 22'$

$2x + y - 3 = 0$

$m_1 = -2$

$5x - 2y + 4 = 0$

$m_2 = \frac{5}{2}$

b) $3x^3 - 8x^2 + 6x - 1$

(i) $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{8}{3}$

(ii) $\alpha\beta\gamma = \frac{-d}{a} = \frac{1}{3}$

(iii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 2$

(iv) $\left(\alpha\beta + \frac{1}{\alpha\beta}\right) + \left(\alpha\gamma + \frac{1}{\alpha\gamma}\right) + \left(\beta\gamma + \frac{1}{\beta\gamma}\right)$

$= \alpha\beta + \alpha\gamma + \beta\gamma + \left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right)$

$= 2 + \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}$

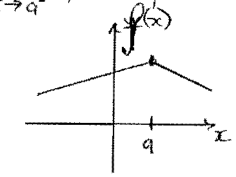
$= 2 + \frac{8}{3} \times 3$

$= 10$

c) (i) continuity at $x = a$...

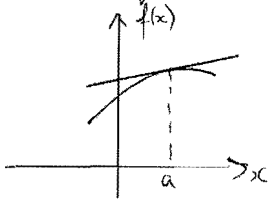
the limit from the left = value of fn = limit from the right.

or $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$



(ii) differentiability at $x = a$...

It is possible to find a derivative at $x = a$ or it is possible to draw a tangent to the curve at $x = a$.



example above is continuous but NOT differentiable at $x = a$.

Q3 a) $y = x^2\sqrt{1-x}$
$= x^2(1-x)^{\frac{1}{2}}$

let $u = x^2$
$\therefore u' = 2x$
$v = (1-x)^{\frac{1}{2}}$
$\therefore v' = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$
$= \frac{-1}{2\sqrt{1-x}}$

$\therefore y' = vu' + uv'$
$= \sqrt{1-x}\,.\,2x + x^2 \,.\, \frac{-1}{2\sqrt{1-x}}$
$= 2x\sqrt{1-x} - \frac{x^2}{2\sqrt{1-x}}$
$= \frac{4x(1-x) - x^2}{2\sqrt{1-x}}$
$= \frac{4x - 4x^2 - x^2}{2\sqrt{1-x}}$
$= \frac{4x - 5x^2}{2\sqrt{1-x}}$
$= \frac{x(4-5x)}{2\sqrt{1-x}}$

b) $\frac{x-2}{x+2} \geq 1$ NB $x \neq -2$
$\therefore (x-2)(x+2) \geq (x+2)^2$
$\therefore x^2 - 4 \geq x^2 + 4x + 4$
$\therefore 4x + 8 \leq 0$
$\therefore x + 2 \leq 0$
$\therefore x \leq -2$
but $x \neq -2$
$\therefore x < -2$

c) (i) $R(1,-2)$ $x - 2y + 3 = 0$

$d_1 = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \right|$
$= \left| \frac{1\times1 - 2\times-2 + 3}{\sqrt{1+4}} \right|$
$= \left| \frac{8}{\sqrt{5}} \right|$
$= \frac{8}{\sqrt{5}}$ units

(ii) $S(-2,3)$

$d_2 = \left| \frac{1\times-2 - 2\times3 + 3}{\sqrt{5}} \right|$
$= \left| \frac{-5}{\sqrt{5}} \right|$

$\therefore$ S is on opposite side of line $\because$ values inside absolute value signs are opposite in sign.

d) $x = \frac{t}{2}$ $y = 2t^2$
$\therefore t = 2x$ $\therefore y = 2(2x)^2$
$= 8x^2$
or $x^2 = \frac{y}{8}$

Q4 a) $f(x) = \sqrt{x^2 - 9}$

(i) $f(x)$ is undefined when $x^2 - 9 < 0$
$x^2 < 9$
$-3 < x < 3$
$\therefore$ domain $\Rightarrow \{x : x \leq -3, x \geq 3\}$

(ii) $\sqrt{x^2-9} \geq 0$
$\therefore$ range $\Rightarrow \{y : y \geq 0\}$

b) (i) RTS $\frac{1+\cos 2A}{\sin 2A} = \cot A$

LHS $= \frac{1+\cos 2A}{\sin 2A}$
$= \frac{1 + \cos^2 A - \sin^2 A}{2\sin A\cos A}$
$= \frac{\cos^2 A + \cos^2 A}{2\sin A\cos A}$
$= \frac{2\cos^2 A}{2\sin A\cos A}$
$= \frac{\cos A}{\sin A}$
$= \cot A$
$=$ RHS QED

NB Use $\Rightarrow$ $\cos 2A = 2\cos^2 A - 1$ instead. It is better.

(ii) $\cot 22°30' = \frac{1+\cos 45°}{\sin 45°}$
$= \frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$
$= \frac{\sqrt{2}+1}{\sqrt{2}} \times \sqrt{2}$
$= \sqrt{2}+1$

c) $A(1,-1)$ $B(7,3)$ $\angle APB = 90°$

(i) grad $AP = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{y - -1}{x - 1}$
$= \frac{y+1}{x-1}$

(ii) grad $BP = \frac{y-3}{x-7}$

Now $\frac{y+1}{x-1} \times \frac{y-3}{x-7} = -1$
$\therefore (y+1)(y-3) = -(x-1)(x-7)$
$\therefore y^2 - 2y - 3 = -x^2 + 8x - 7$
$\therefore (x-4)^2 + (y-1)^2 = 13$ which is a circle.

(iii) centre $\Rightarrow (4,1)$
radius $\Rightarrow \sqrt{13}$ units

Q5

a) $6y = x^2 - 4x - 14$

(i) $x^2 - 4x - 14 = 6y$

$\therefore (x-2)^2 - 18 = 6y$

$\therefore (x-2)^2 = 6y + 18$

$= 6(y+3)$

$\therefore (x-2)^2 = 4 \times \frac{3}{2}(y+3)$

(ii) vertex $\Rightarrow (2, -3)$

(iii) focus $\Rightarrow (2, -\frac{3}{2})$

(iv) directrix $\Rightarrow y = -4\frac{1}{2}$

b) $3\sin x - 2\cos x = 2$

$\therefore 3 \times \frac{2t}{1+t^2} - 2\frac{(1-t^2)}{1+t^2} = 2$

$\therefore 6t - 2 + 2t^2 = 2 + 2t^2$

$\therefore 6t = 4$

$\therefore t = \frac{2}{3}$

$\therefore \tan\frac{\theta}{2} = \frac{2}{3}$

$\therefore \frac{\theta}{2} = 33^\circ 41' 24''$

$\therefore \theta = 67^\circ 23'$

c) $\sqrt{x} + \frac{5}{\sqrt{x}} = 3$

$\therefore x + 5 = 3\sqrt{x}$

$\therefore x^2 + 10x + 25 = 9x$

$\therefore x^2 + x + 25 = 0$

$\therefore x = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 25}}{2A}$

$= \frac{-1 \pm \sqrt{-99}}{2}$

$\therefore \Delta = -99$

< 0

$\therefore \nexists$ real solutions.

Q6

a) $\cos^2\theta - \sin 2\theta = 0$

$\therefore \cos^2\theta - 2\sin\theta\cos\theta = 0$

$\therefore \cos\theta(\cos\theta - 2\sin\theta) = 0$

$\therefore \cos\theta = 0$ or $\cos\theta = 2\sin\theta$

$\therefore \theta = 90^\circ, 270^\circ$ $\quad \therefore \frac{1}{2} = \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

$\therefore \theta = 26^\circ 34', 206^\circ 34'$

$\therefore \theta = 26^\circ 34', 90^\circ, 206^\circ 34', 270^\circ$

b) $P(8p, 4p^2)$ $\quad Q(8q, 4q^2)$ on $x^2 = 16y$

$(4, 0)$

$\therefore a = 4$

a) chord PQ $\Rightarrow y = \left(\frac{p+q}{2}\right)x - apq$

$\therefore y = \left(\frac{p+q}{2}\right)x - 4pq$

b) $(4,0)$ satisfies chord

$\therefore 0 = 4\left(\frac{p+q}{2}\right) - 4pq$

$\therefore 4pq = 4\left(\frac{p+q}{2}\right)$

$\therefore pq = \frac{p+q}{2}$

$\therefore 2pq = p + q$

c) Tangents meet at $(a(p+q), apq) = (a.2pq, apq)$

$= (2apq, apq)$

d) $x = 2apq$

$y = apq$

$\therefore pq = \frac{y}{a}$

$\therefore x = 2a\frac{y}{a}$

$\therefore x = 2y \Rightarrow$ locus of R (with $x \leq 0$)