Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 11 PRELIMINARY COURSE

Mathematics Extension 1

September 2010

TIME ALLOWED: 90 minutes

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied at the time of marking

(FOR MARKERS USE ONLY)

1	2	3	4	5	6	TOTAL
/13	/13	/13	/13	/13	/13	/78

QUESTION 1: (13 Marks)

Marks

(a) Find $\frac{d}{dx}(\frac{3}{x^2})$ **1**

(b) Find $\frac{d}{dx}(1+x^2)\sqrt{1+x^2}$ leaving your answer in surd form. **2**

(c) Find the least positive integer n such that $\frac{2n}{3} > \frac{1}{n}$ **3**

(d) Find the acute angle between the lines $x=1$ and $y=2x+1$, giving your answer to the nearest minute **3**

(e) (i) By rationalizing the numerator, show that $\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$ **1**

(ii) Hence, using the method of Differentiation From First Principles, find **3**

$$\frac{d}{dx}\sqrt{x}$$

QUESTION 2: (13 Marks)

Marks

(a) Find the co-ordinates of the point, P, which divides the line joining the points A(9,4) to B(3, 2) externally in the ratio 7:5 **2**

(b) Show that the gradient of the tangent to the curve $y = \dfrac{x^3}{1+x^2}$ is always positive and only zero at the origin. **2**

(c) If $0 < x < \dfrac{\pi}{2}$ and $\dfrac{\pi}{2} < y < \pi$ and you are also given that $\sin x = \dfrac{3}{5}$ and that $\cos y = -\dfrac{5}{13}$, calculate, without using a calculator, the value of $\tan(x-y)$, leaving your answer as a fraction. **3**

(d) Show that $\dfrac{1+cosx}{sinx} + \dfrac{sinx}{1+cosx} = 2cosecx$ **2**

(e) (i) Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ **2**

(ii) Hence, or otherwise, solve $\cos\theta - \sin\theta = 1$ for $-\pi \leq \theta \leq 2\pi$ **2**

QUESTION 3: (13 Marks):

Marks

(a) Find the monic polynomial of degree 3 which has roots of -1, $2-\sqrt{2}$, $2+\sqrt{2}$ **2**
Write your answer in expanded form.

(b) (i) Simplify $\sin(x+y)-\sin(x-y)$ **1**

(ii) Use this result to evaluate $2\cos 75^0 \sin 15^0$ in exact terms. **2**

(c) Solve the equation $x-3\sqrt{x}-4=0$ **2**

(d) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$.

M is the foot of the perpendicular from P to the y-axis, while N is the foot of the perpendicular from P to the Directrix.

O is the Origin.

(i) Draw a neat diagram showing the above information. **1**

(ii) What is the equation of the Directrix. **1**

(iii) Find the co-ordinates of M and N. **1**

(iv) Find the midpoint R of MN. **1**

(v) Find the locus of the point R, in algebraic form, and describe the locus geometrically **2**

QUESTION 4: (13 Marks)

Marks

(a) Find the values of a and b if the polynomial $ax^4 - x^3 - 12x^2 + bx + 2$ is divisible by $x^2 - x - 2$ **3**

(b) Solve the equation $\sin 2\theta = \cos\theta$, for $0 \le \theta \le 2\pi$ **3**

(c) If $x^2 + x(p+2) + p^2 + 3p - 6 = 0$ has 2 real roots, show that **3**

$$-\frac{14}{3} < p < 2$$

(d) (i) If α and β are the roots of the quadratic equation $x^2 - 12x + 1 = 0$, find, without solving the equation, the values of:

(A) $\alpha^2\beta^2$ **1**

(B) $\alpha^2 + \beta^2$ **1**

(ii) Hence, or otherwise, find the quadratic equation with roots of α^2 and β^2 **2**

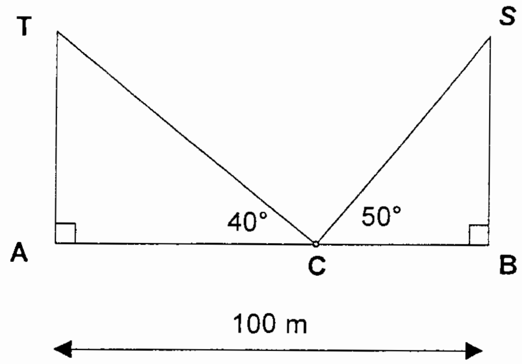
QUESTION 5: (13 Marks)

Marks

(a) Express $n^3 - 3n^2 + 2n - 1$ in the form **3**

$An(n-1)(n-2) + Bn(n-1) + Cn + D$

(b) A man stands on level ground directly between two light towers of identical height which are 100m apart. From his spot, the angles of elevation to the tops of the towers are 50^0 and 40^0 respectively.



(i) If h is the height of the towers, prove that **4**

$$h = \frac{100 \tan 50^0 \tan 40^0}{\tan 50^0 + \tan 40^0}$$

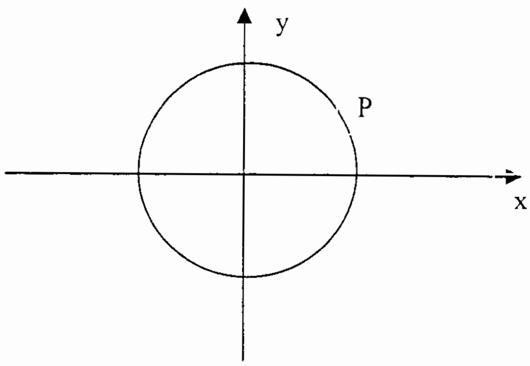
(ii) Find how far the man is from the closest tower to the nearest metre. **1**

(c) (i) Show that the the point P(acosθ, asinθ) always lies on a circle, and give the radius and centre of that circle. **2**

3

(ii) Use this form of the point P and the diagram below to prove the following geometric theorem:

"The angle in a semi-circle is a right angle."



QUESTION 6 (13 Marks)

Marks

(a) If the equation $x^3 + 2x^2 + 3x + 4 = 0$ has roots of α, β and γ, find the value of:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$ **2**

(iv) $\alpha^2 + \beta^2 + \gamma^2$ **2**

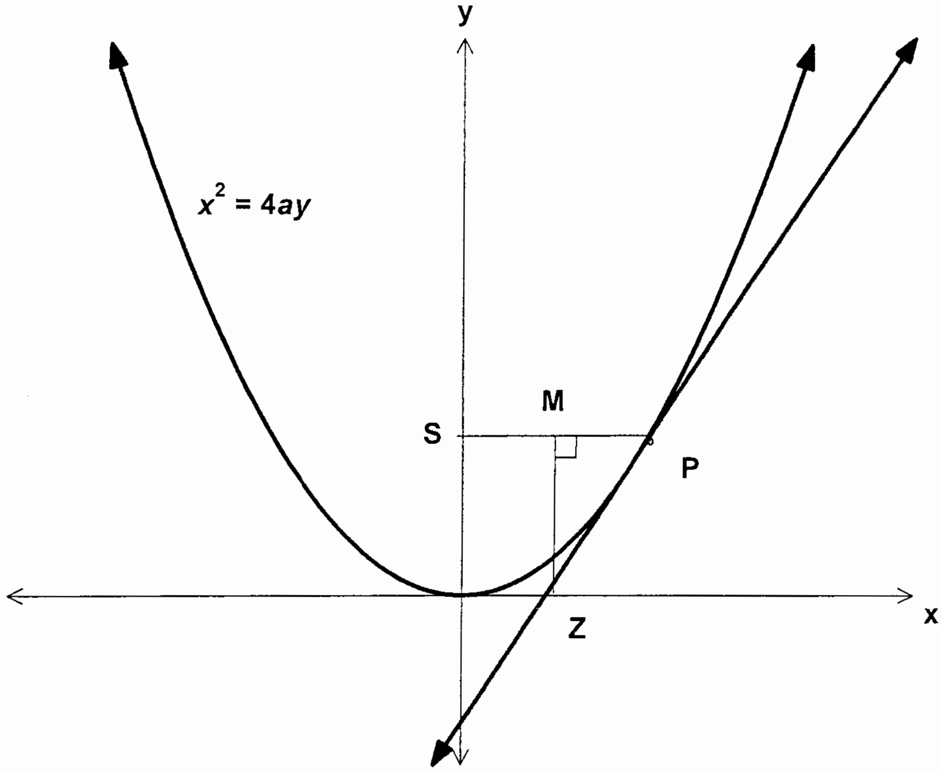
(b) Without performing any calculations, explain why the roots of $x^4 + 3x^3 - 8x^2 + 10x - 9 = 0$ cannot all be positive. **1**

QUESTION 6 CONTINUES OVERPAGE

(c) In the diagram below, the tangent to the curve $x^2 = 4y$ at the point $P(2p, p^2)$ cuts the x-axis at Z.

A line is drawn from Z at right angles to the line joining P to the focus S of the parabola, meeting it at M.

(NOTE: S and P do not necessarily have the same y-values.)



(i) Find the equation of the tangent at P. **1**

(ii) Find the co-ordinates of the point Z. **1**

(iii) Find the equation of PS **2**

(iv) Show that MZ=OZ **2**

END OF EXAMINATION PAPER

Teacher's Name: Student's Name/N°:

YEAR 11 EXTENSION 1

September 2010 SOLUTIONS and MARKING

QUESTION 1

(a) $-6x^{-3}$ or $-6/x^3$ EITHER, 1 MARK

(b) $\frac{3}{2}(2x)(1+x^2)^{1/2}$

$= 3x(1+x^2)^{1/2}$ ← 1 MARK

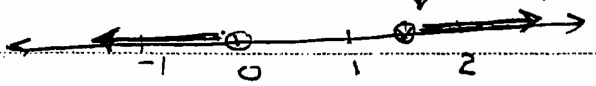
$= 3x\sqrt{1+x^2}$ ← 1 MARK

(c) $2n/3 > 1/n$ C.V's $n \neq 0$

$2n^2 = 3$

$n = \sqrt{3/2}$.

$\sqrt{3/2}$

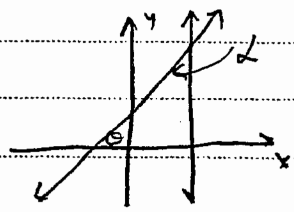


$n > \sqrt{3/2}$

$\Rightarrow n = 2$ is the smallest positive integer

ANY METHOD = 3 MARKS (1 only for $n > \sqrt{3/2}$)

(d) EITHER



$\tan\theta = 2$ OR $\tan\alpha = \left|\frac{2.1-1.0-1}{\sqrt{5}\sqrt{1}}\right|$ 3 MARKS

$\theta = 63° \ 26'$

$\therefore \alpha = 90 - \theta$

$= 26° \ 34'$

(e) $\frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$ 1 MARK

(ii) $\frac{d}{dx}\sqrt{x} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \frac{1}{\sqrt{x}+\sqrt{x}}$

$= 1/2\sqrt{x}$

LOSE (1) MARK for "$\lim_{h\to 0}$" being omitted or in the wrong spot

LOSE (1) for putting in $h\to 0$ too early

(1) MARK is awarded for seeing the connection betw part (i) and part (ii)

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QUESTION 2:

(a) $(-12, -3)$ — 2 MARKS. Accept any method.

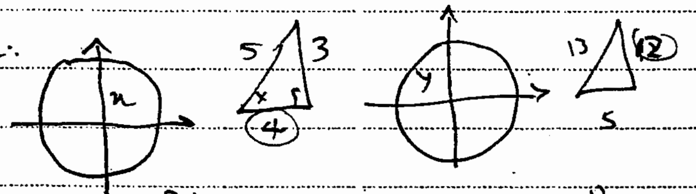
(b) $$\frac{dy}{dx} = \frac{(1+x^2)\,3x^2 - x^3\,2x}{(1+x^2)^2}$$

$$= \frac{3x^2 + x^4}{(1+x^2)^2} \geq 0$$

Since both numerator and denominator are positive for all x, and zero when $x=0$ only.

2 MARKS. (must explain why its positive for the 2^{nd} mark.)

(c) x:



$\tan x = 3/4$ $\quad\tan y = -12/5$ ← 1 mark each

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{3/4 + 12/5}{1 - 36/20} = -\frac{63}{16}$$

1 MARK.

(d) $$\frac{1+\cos x}{\sin x} + \frac{\sin x}{1+\cos x} = \frac{(1+\cos x)^2 + \sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{1 + (\cos^2 x + \sin^2 x) + 2\cos x}{\sin x(1+\cos x)}$$

$$= \frac{2(1+\cos x)}{\sin x(1+\cos x)}$$

$$= 2\operatorname{cosec} x$$

2 MARKS, 1 off/error

(e) (i) $\cos\theta - \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$

$= \sqrt{2}\cos(\theta + \alpha)$ where $\cos\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \alpha = 45°$

$= \sqrt{2}\cos(\theta + \pi/4)$

1 for R, 1 for α

(ii) $\sqrt{2}\cos(\theta + \pi/4) = 1$

$\therefore \theta + \pi/4 = \pi/4$ OR, $7\pi/4$

$\therefore \theta = 0$ OR, $\theta = 3\pi/2$

1 for each answer.

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QUESTION 3:

(a) Polynomial is $(x+1)(x-2+\sqrt{2})(x-2-\sqrt{2})$ ← 1 MARK for this

$= (x+1)((x-2)^2-2)$

$= (x+1)(x^2-4x+2)$

$= x^3-3x^2-2x+2$ ← 1 MARK

(b) (i) $\sin x\cos y + \cos x\sin y - \sin x\cos y + \cos x\sin y$

$= 2\cos x\sin y$ 1 MARK

(ii) Let $x = 75^\circ$, $y = 15^\circ$ ← 1 MARK

$\therefore 2\cos 75 \sin 15 = \sin(90^\circ) - \sin 60^\circ$

$= \begin{cases} 1-\sqrt{3}/2 \\ \dfrac{2-\sqrt{3}}{2} \end{cases}$ 1 MARK for either answer

(c) Let $u=\sqrt{x}$

$\therefore u^2-3u-4=0$

$(u-4)(u+1)=0$

$u=4$ or $u=-1$

$\therefore \sqrt{x}=4$ or $\sqrt{x}=-1$

$x=16$. NO SOL^N

(TEST: $16-3\times4-4=0$ ✓)

OR/ $(\sqrt{u}-4)(\sqrt{u}+1)=0$

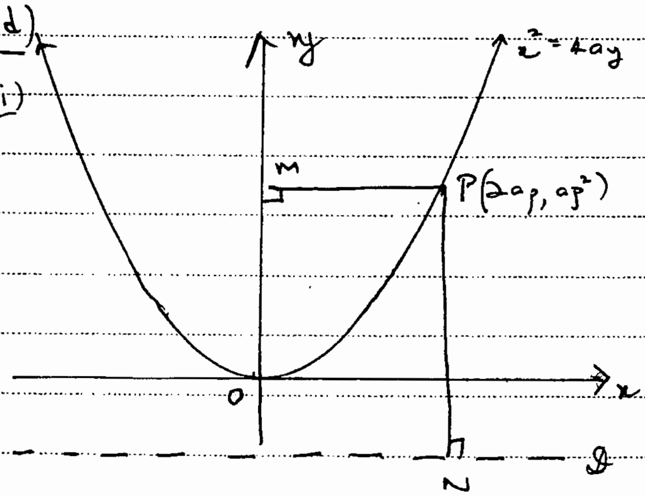
$\sqrt{u}=4$ or $\sqrt{u}=-1$

etc...

1 for solving

1 for x

(d) (i)



① for the diagram (must be neat and clear.)

(ii) directrix is $y=-a$ 1 MARK

(iii) M is $(0, ap^2)$ and N is $(2ap, -a)$ 1 MARK for BOTH

(iv) R is $\left(ap, \dfrac{ap^2-a}{2}\right)$ 1 MARK

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(v) $x = ap \Rightarrow p = x/a.$

$$y = \frac{ap^2 - a}{2}$$

$$= \frac{a\left(x^2/a^2 - 1\right)}{2}$$

$$\therefore 2y = x^2/a - a$$

$$\therefore x^2 = 2ay + a^2$$ (1) mark

A parabola (1) for parabola.

(no more detail req'd)

QUESTION 4:

(a) $x^2 - x - 2 = (x-2)(x+1)$

$P(2) = 16a + 2b - 54 = 0$

$\Rightarrow \quad 8a + b = 27 \quad (1)$ — 1 MARK

$P(-1) = a + 1 - 12 - b + 2 = 0$

$\therefore \quad a - b = 9 \quad (2)$ — 1 MARK

$(1) + (2) \quad 9a = 36$ — 1 MARK

$\begin{cases} a = 4 \\ b = -5 \end{cases}$

(b) $2\sin\theta\cos\theta = \cos\theta$ — 1 for $\sin 2\theta = 2\sin\theta\cos$

$\cos\theta(2\sin\theta - 1) = 0$

$\therefore \cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

2 MARKS for this (they lose 1 MARK for "losing" $\cos\theta = 0$)

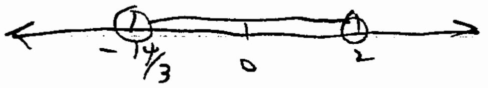
(c) $x^2 + x(p+2) + p^2 + 3p - 6 = 0$

$\Delta = (p+2)^2 - 4(p^2 + 3p - 6) > 0$ ← 1 for this line

$\therefore p^2 + 4p + 4 - 4p^2 - 12p + 24 > 0$

$\therefore 3p^2 + 8p - 28 < 0$

$(3p + 14)(p - 2) < 0$ ← 1 for getting to this



$\therefore -\frac{14}{3} < p < 2$ ← 1 for this. they must show you how (an[y] me[thod])

(d) (i) $\alpha + \beta = 12, \quad \alpha\beta = 1$

(A) $\alpha^2\beta^2 = 1$ (B) $\alpha^2 + \beta^2$ — 1 EACH

$= (\alpha+\beta)^2 - 2\alpha\beta$

$= 142$

(ii) $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

$\therefore x^2 - 142x + 1 = 0$

2 MARKS

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QUESTION 5:

(a) $n^3 - 3n^2 + 2n - 1 = An(n-1)(n-2) + B(n)(n-1) + Cn + D$

Let $n=0 \Rightarrow D = -1$ 1 MARK for D

Let $n=1 \Rightarrow C + D = -1$

$\therefore C = 0$

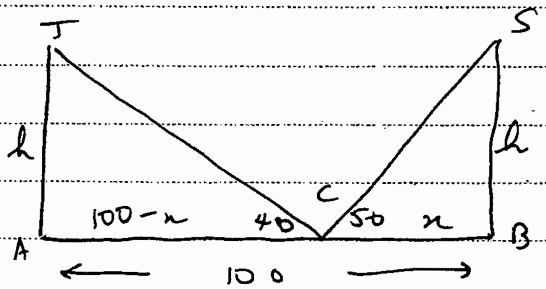
Let $n=2 \Rightarrow 2B + 2C + D = -1$

$\therefore B = 0$

and A ~~is~~ 1 (by coefficients) 1 MARK for A

$\therefore P(x) = n(n-1)(n-2) - 1$ 1 for SOLUTION

(b)



(i) $h/_{100-x} = \tan 40°$

$\therefore h = (100-x)\tan 40°$ $\underline{(1)}$

$h/_x = \tan 50°$

$\therefore h = x\tan 50°$

$\Rightarrow x = h/\tan 50°$

2 MARKS for getting 2 expressions for h

In (1): $h = 100\tan 40° - \frac{h}{\tan 50°}\tan 40°$

$\therefore h\left(1 + \frac{\tan 40°}{\tan 50°}\right) = 100\tan 40°$

$\therefore h = \frac{100\tan 40°\tan 50°}{\tan 50° + \tan 40°}$

② for making h the subject.

(ii) In the diagram, x is the closer distance

and $x = h/\tan 50°$

$= \frac{100\tan 40}{\tan 50 + \tan 40°}$

$= 41.32$ m 1 MARK

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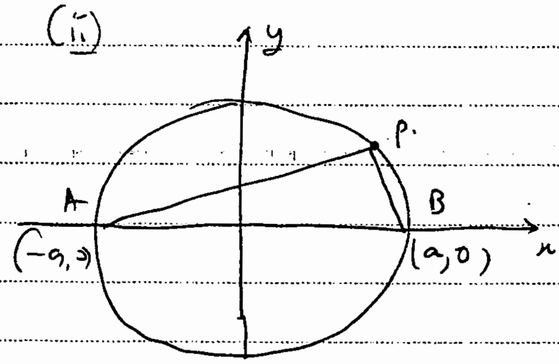
(c) (i) $x^2+y^2 = a^2\cos^2\theta + a^2\sin^2\theta$ 1 MARK for using Pythagoras

$= a^2$

which is a circle

centre $(0,0)$ radius a 1 for both centre and radius.

(ii)



$m_{PA} = \dfrac{a\sin\theta}{a\cos\theta + a}$

$m_{PB} = \dfrac{a\sin\theta}{a\cos\theta - a}$

1 MARK for each slope = 2 m

$m_{PB}\cdot m_{PA} = \dfrac{a^2\sin^2\theta}{a^2\cos^2\theta - a^2}$

$= \dfrac{\sin^2\theta}{\cos^2\theta - 1}$

$= -1$

$\therefore$ $PA \perp PB$

and result is proven

1 for this working and conclusion

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QUESTION 6:

(a) (i) $\alpha+\beta+\gamma=-2$ (b) $\alpha\beta\gamma=-4$ 1 MARK EACH = 2

(iii) $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$ ← 1. MARK

$=\frac{1}{2}$ ← 1 MARK

(iv) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ ← 1 MARK

$=4-2(3)$

$=-2$ ← 1 MARK

(b) Because if the roots are α, β, γ and δ,

$\alpha\beta\gamma\delta=-9$ 1 MARK

and there has to be one or three negatives

(c) (i) $x^2=4y \Rightarrow dy/dx = x/2$

At $P(2p, p^2)$ $m_T=p$

Tangent is: $y-p^2=p(x-2p)$ 2 MARKS

$y-p^2=px-2p^2$ {1 for getting m_T

$y=px-p^2$ {1 for answer ~~OR~~

2 for answer

(ii) Z is the point $(p, 0)$ ← 1 MARK

(iii) S is $(0,1)$ so, $m_{PS}=\frac{p^2-1}{2p}$ ← 1 MARK

Equation is:

$y-1=\frac{p^2-1}{2p}x$ 1 for equivalent

or $(p^2-1)x-2yp+2p=0$ answer or these

(iv) By perpendicular distance formula, ← 1 MARK

$$MZ=\frac{(p^2-1)p-0+2p}{\sqrt{(p^2-1)^2+4p^2}}$$

$$=\frac{p^3+p}{\sqrt{(p^2+1)^2}}$$

$$=\frac{p(p^2+1)}{p^2+1}=p$$ and $OZ=p$ ← 1 MARK to get here

$\therefore OZ=MZ$