Sydney Technical High School

Name: ....

Maths Class: ....

Year 11
Mathematics Extension 1

Preliminary Course

Assessment 3

September, 2017

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1
Multiple Choice
Questions 1-10
10 Marks

Section II
Questions 11-16
60 Marks

Section 1 Multiple Choice (10 marks)
Use the multiple choice answer sheet for Question 1-10

1. If the exact value of $\cos x$ is $\frac{1}{\sqrt{5}}$, what is the exact value of $\cos 2x$?

(A) $-\frac{3}{5}$

(B) $-\frac{2}{\sqrt{5}}$

(C) $\frac{3}{5}$

(D) $\frac{2}{\sqrt{5}}$

2. The coordinates of the focus of the parabola $x^2 = 4ay$ are

(A) (0, -a)

(B) (0, a)

(C) (0, 1)

(D) (0, 4a)

3. Given $f(x) = 3x^2 - 5x + 2$, find $f(a+1)$

(A) $3a^2 - 5a + 3$

(B) $3a^2 + 11a$

(C) $3a^2 + a + 1$

(D) $3a^2 + a$

4. If $x = \frac{1}{2}at$ and $y = 2at^2$ which of the following is an expression for $\frac{dy}{dx}$?

(A) $8t$

(B) $4at$

(C) $2t$

(D) t

5. Which statement is true of the quadratic expression $2x^2 + 6x + 9$?

(A) It is positive definite

(B) It has two unreal roots

(C) It is a perfect square

(D) The zeros add to 3

6. If $\sin 25^0 = \cos(x - 45^o)$ find x if $45^o < x < 135^o$

(A) 45^o

(B) 70^o

(C) 110^o

(D) 135^o

7. The correct solution of $\frac{x}{x-3} > 0$ is:

(A) $x < 0$ or $x > 3$

(B) $0 < x < 3$

(C) $x > 0$

(D) $x > 0$ or $x > 3$

8. Which is the correct condition for $y = mx + b$ to be a tangent to $x^2 = 4ay$?

(A) $am + b = 0$

(B) $am^2 + b = 0$

(C) $am - b = 0$

(D) $am^2 - b = 0$

9. Which of the following functions does NOT have a horizontal asymptote $y = 1$?

(A) $y = 1 + 2^x$

(B) $y = \frac{x^2+1}{x^2-1}$

(C) $y = 3 - \frac{2x+1}{x+1}$

(D) $y = \frac{3x^2+1}{3x+1}$

10. What is the total number of solutions of the equation $3cosx + 4sinx = 5$ for $0° \leq x \leq 360°$?

(A) 0

(B) 1

(C) 2

(D) 3

Section II Total Marks 60
Attempt Questions 11 – 16. Answer each question in your writing booklet.
In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **(10 Marks)** Use a Separate Sheet of paper

(a) A parabola has equation $x^2 = -12y + 24$.

(i) Give the coordinates of its focus. **1**

(ii) Give the equation of its directrix. **1**

(iii) Sketch the parabola, showing its main features. **2**

(b) Solve the equation $\sin 2x + \cos x = 0$ over the Domain $0^o \le x \le 360^o$ **3**

(c) Let $P(x) = 2x^3 - 3x^2 - 3x + 2$

(i) Show that $(x + 1)$ is a factor of $P(x)$ **1**

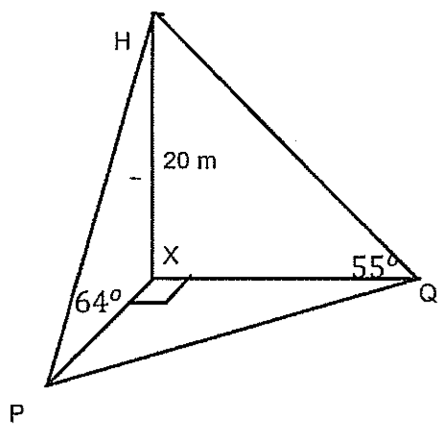
(ii) Hence express $P(x)$ as a product of three linear factors. **2**

Question 12 (10 Marks) Use a Separate Sheet of paper

(a) Solve the equation: $\frac{3x+1}{x-3} \leq 4$ **3**

(b) Find the co-ordinates of the point P(x,y) which divides the interval joining A($-3, -7$) To the point B($-1, -4$) <u>externally</u> in the ratio 4:3 **2**

(c) Two sailors P and Q, floating in the ocean, spot a helicopter above. From P the angle of elevation to the helicopter is 64^o, while from Q the angle of elevation is 55^o. Using a point X immediately below the helicopter, the triangle PQX is right angled at X.



(i) Show that XQ $= \frac{20}{tan55^o}$ **1**

(ii) How far apart are P and Q, to 3 significant figures? **2**

(d) The equation $x^3 + 3x^2 + 2x + 1 = 0$ has roots α, β and γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

Question 13 (10 Marks) Use a Separate Sheet of paper

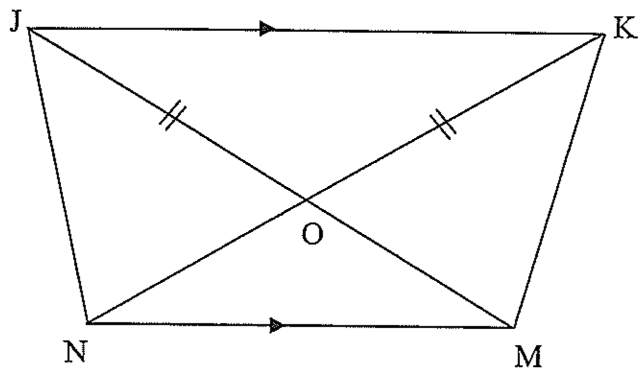
(a) Find the derivative of these expressions in simplified form :

(i) $x(2x^2-4)^5$ **2**

(ii) $\frac{x+1}{(x-1)^2}$ **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to solve the equation $2 + cosx - 2\sin x = 0$ for $0° \leq x \leq 360°$, giving answers correct to the nearest degree. **3**

(c) In the diagram below, JK || NM, JO = KO.



Redraw the diagram into your answer booklet

(i) Prove that Δ JOK ||| Δ NOM **2**

(ii) Hence prove that ΔJON ≡ ΔKOM **1**

Question 14 (10 Marks) Use a Separate Sheet of paper

(a) Find the values of A, B and C if

$$2x^2 - 3x + 5 \equiv A(x-1)(x-2) + B(x-1) + C$$ **3**

(b) (i) Find the Cartesian equation of the curve which has the parametric equations **2**

$$x = 3 + t$$

$$y = 2t^2 - 2$$

(ii) Describe Geometrically the curve found in part (i) **1**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$ with focus S.

(i) Show that the normal to the parabola at P has equation $x + py - 2ap - ap^3 = 0$ **2**

(ii) Hence find the coordinates of the three points on the parabola such that the normals to the parabola at these three points pass through the point $(0, 6a)$. **2**

Question 15 (10 Marks) Use a Separate Sheet of paper

(a) For the parabola with parametric equations $x = 10t$ and $y = 5t^2$,

(i) Find the Cartesian equation of the parabola **1**

(ii) Find the coordinates of the focus. **1**

(iii) Sketch the parabola showing the focus, vertex and directrix. **1**

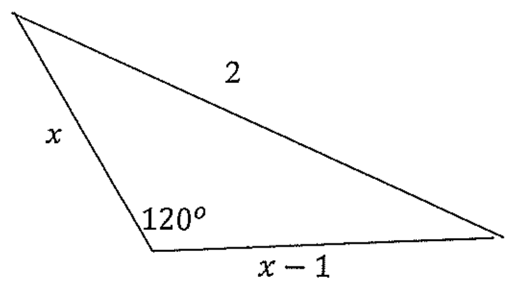
(iv) Show that the focal chord that passes through the point on the parabola where $t = 2$ has equation $3x - 4y + 20 = 0$. **2**

(b) For the polynomial $G(x) = x^2(1 - x)(x + 3)$ draw a sketch of $y = G(x)$. **3**

(c) Three tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + x + 3$ are parallel to the line $y = x$. Find the x value of the point of contact for each of these three tangents. **2**

Question 16 (10 Marks) Use a Separate Sheet of paper

(a)



In the triangle above, find the <u>exact</u> value of x. **3**

(b) (i) Find the acute angle between the lines $y = \frac{1}{\sqrt{3}}x$ and $y - \sqrt{3}x + 4\sqrt{3} = 0$. **2**

(ii) Prove that the lines and x-axis form an isosceles triangle. **2**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$ with focus S

The normal to the parabola at P has equation $x + py - 2ap - ap^3 = 0$ and cuts the y axis at N. Show that $PS = NS$. **3**

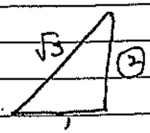
End of Examination

YEAR 11 2017 FINAL EXAMS

EXTENSION 1 SOLUTIONS

MULTIPLE CHOICE

Q1.



$\cos 2x = 2\cos^2 x - 1$

$= 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1$

$= -\frac{3}{5}$ so (A)

Q2. (B)

Q3. $f(a+1) = 3(a+1)^2 - 5(a+1) + 2$

$= 3a^2 + 6a + 3 - 5a - 5 + 2$

$= 3a^2 + a$ (D)

Q4. Either: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 4at \times \frac{2}{a}$

$= 8t$ (A)

OR $y = 2a\left(\frac{2x}{a}\right)^2$

$= \frac{8x^2}{a}$

$\therefore \frac{dy}{dx} = \frac{16x}{a}$

$= \frac{8at}{a}$

$= 8t$

Q5. $\Delta = 36 - 4(2)(9)$

$= -36$

$\therefore$ POSITIVE DEF. (A)

Q6. $\sin 25 = \cos 65$

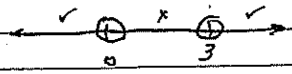
$\therefore 65 = x - 45$

$\therefore x = 110°$ (C)

Q.9 (D) as $x \to \infty$, $y \to x$

Q7. C.V. $x \neq 3$

Solution $x \neq 0$



(A)

$\therefore x < 0$ or $x > 3$

Q10 - (B)

Q 8. $x^2 = 4a(mx + b)$

$x^2 - 4amx - 4ab = 0$

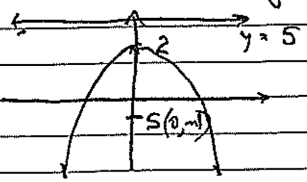
$\Delta = 16a^2m^2 + 16ab$

For tangency $\Delta = 0$

$am^2 + b = 0$ (B)

QUESTION 11:

(a) $x^2 = -12(y - 2) \Rightarrow a = 3$ (or $a = -3$)



Vertex is $(0, 2)$

(i) S is $(0, -1)$

(ii) d is $y = 5$

(iii) above.

(b) $2\sin x\cos x + \cos x = 0$

$\therefore \cos x(2\sin x + 1) = 0$

$\therefore \cos x = 0$ or $\sin x = -\frac{1}{2}$

$\therefore x = 90°, 270°$ or $x = 210°, 330°$

(c) $P(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2$

$= -2 - 3 + 3 + 2$

$= 0$

$\therefore (x+1)$ is a factor

$$\begin{array}{r} 2x^2 - 5x + 2 \\ x+1 \overline{)\, 2x^3 - 3x^2 - 3x + 2} \\ 2x^3 + 2x^2 \\ \hline -5x^2 - 3x + 2 \\ -5x^2 - 5x \\ \hline 2x + 2 \end{array}$$

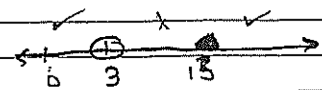
$\Rightarrow P(x) = (x+1)(2x^2 - 5x + 2)$

$= (x+1)(2x-1)(x-2)$

QUESTION 12:

(a) $3x + 1 = 4x - 12$

C.V. $x \neq 3$ $\therefore x = 13$



$\therefore x < 3$ or $x \geq 13$

(b) A$(-3, -7)$ B$(-1, -4)$

$-4 : 3$

M is $\left(\frac{-9+4}{-1}, \frac{-21+16}{-1}\right) = (5, 5)$

Q12

(c) (i) In $\triangle HXQ$, $\frac{HX}{XQ} = \tan 55^\circ$

$\therefore XQ = \frac{20}{\tan 55^\circ}$

(ii) Similarly $XP = \frac{20}{\tan 64^\circ}$

By Pythagoras, in $\triangle PQX$,

$$\left(\frac{20}{\tan 55}\right)^2 + \left(\frac{20}{\tan 64}\right)^2 = PQ^2$$

$\therefore PQ^2 = 196.116 + 95.153$

$\therefore PQ = 17.07$ m.

(d)

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= (-3)^2 - 2(2)$$
$$= 5$$

QUESTION 13:

(a)(i) $(2x^2-4)^5 . 1 + x . 20x(2x^2-4)^4$

$= (2x^2-4)^4 [2x^2 - 4 + 20x^2]$

$= (2x^2-4)^4 (22x^2-4)$

(ii) $\frac{(x-1)^2 . 1 - (x+1) . 2(x-1)}{(x-1)^4} = \frac{(x-1) - 2(x+1)}{(x-1)^3}$

$= \frac{-x-3}{(x-1)^3}$

(b)

$2 + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2} = 0$

$\therefore 2 + 2t^2 + 1 - t^2 - 4t = 0$

$t^2 - 4t + 3 = 0$ $\qquad 0 \le x \le 360$

$\therefore (t-3)(t-1) = 0$ $\qquad 0 \le x/2 \le 180$

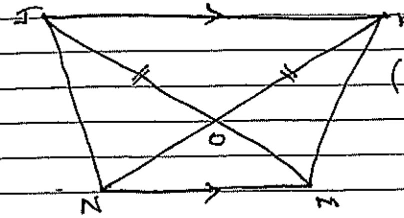
$\therefore t = 3$ OR $t = 1$

$\therefore \tan x/2 = 3$, $\tan x/2 = 1$

$\therefore x/2 = 71^\circ 34'$ OR $x/2 = 45^\circ$

$\therefore x = 143^\circ 8'$ OR $x = 90^\circ$

Q13 (c)



(i) In $\triangle JOK$ and $\triangle NOM$,

$\angle JOK = \angle NOM$ (vertically opposite angles)

$\angle KJO = \angle NMO$ (alternate angles $JK \parallel NM$)

$\therefore \triangle JOK \,|||\, \triangle NOM$ (equiangular)

(ii) Since the triangles are similar the sides are in ratio

$\therefore \frac{JO}{MO} = \frac{KO}{NO}$ OR $\frac{JO}{KO} = \frac{MO}{ON}$

$\therefore \frac{MO}{ON} = 1$ (since $JO = KO$)

$\therefore MO = NO$.

In $\triangle JON$ and $\triangle KOM$

$MO = NO$ (above)

$JO = KO$ (given)

$\angle JON = \angle KOM$ (vertically opposite)

$\therefore \triangle JON \equiv \triangle KOM$ (SAS)

QUESTION 14:

(a) By equating x^2: $A = 2$

let $x = 1$ $\quad 4 = C$

let $x = 2$ $\quad 7 = B$

(b) (i) $t = x - 3$ $\quad y = 2t^2 - 2$

$= 2(x-3)^2 - 2$

$\therefore (x-3)^2 = \frac{1}{2}(y+2)$

(ii) A parabola, vertex $(3, -2)$ focus $(3, -1\frac{7}{8})$

(c) (i) $dy/dx = x/2a$ $\quad$ At P, $m_T = p$

$\therefore m_N = -1/p$

Equation of normal:

$y - ap^2 = -1/p\,(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

Q 14 (c) (ii) Passes through $(0, 6a)$

$\therefore\ 6ap - 2ap - ap^3 = 0$

$\therefore\ 4ap - ap^3 = 0$

$\therefore\ ap(4 - p^2) = 0$

$\therefore\ p = 0$ or $p = 2$ or $p = -2$

$\therefore$ Points are

$(0,0)$, $(4a, 4a)$ and $(-4a, 4a)$

QUESTION 15:

(a) (i) $y = 5\left(\frac{x}{10}\right)^2$

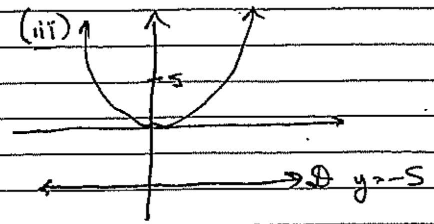
$\therefore\ y = x^2/20$

(ii) $x^2 = 20y$

$= 4(5)y$

$\therefore$ focus is $(0, 5)$

(iii)



(iv) At $t = 2$, P is $(20, 20)$

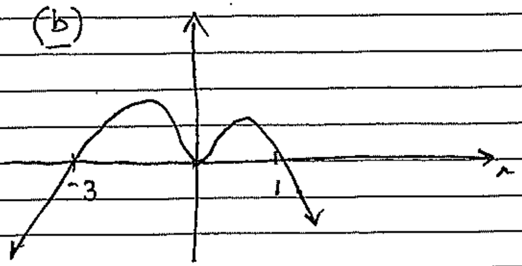
$m_{PS} = \frac{15}{20} = \frac{3}{4}$

$\therefore\ y - 20 = \frac{3}{4}(x - 20)$

$4y - 80 = 3x - 60$

$3x - 4y + 20 = 0$

(b)



(c) $12x^3 + 12x^2 - 24x + 1 = 1$ ← slope of $y = x$

$\therefore\ 12x(x^2 + x - 2) = 0$

$\therefore\ 12x(x+2)(x-1) = 0$

$\therefore\ x = 0$ or $x = 1$ or $x = -2$

QUESTION 16:

(a). By cosine rule, $2^2 = x^2 + (x-1)^2 - 2x(x-1)\cos 120°$

$\therefore\ 4 = 2x^2 - 2x + 1 - 2x(x-1)(-\frac{1}{2})$

$\therefore\ 4 = 2x^2 - 2x + 1 + x^2 - x$

$\therefore\ 3x^2 - 3x - 3 = 0$

$x^2 - x - 1 = 0$

$\therefore\ x = \frac{1 \pm \sqrt{1+4}}{2}$

Since $x > 0$, $x = \frac{1 + \sqrt{5}}{2}$

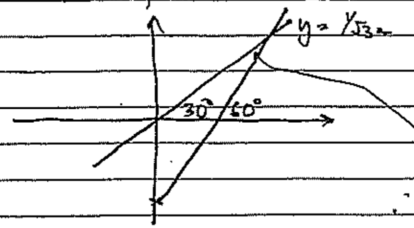
(b) (i) $m_1 = \frac{1}{\sqrt{3}}$, $m_2 = \sqrt{3}$.

$\therefore\ \tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + 1}\right|$

$= \left|\frac{1 - 3}{2\sqrt{3}}\right|$

$= \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$

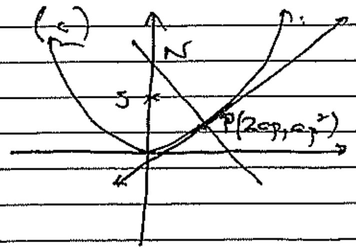
(ii) The slopes give tan of the angle made with the x-axis, so we have



By exterior angles, or by part (i) the missing angle is 30°

$\therefore$ Isosceles triangle.

(c)



Normal is $x + py - 2ap - ap^3 = 0$

$\therefore$ N is $(2a + ap^2)$

S is $(0, a)$

$\therefore\ NS = a + ap^2$

$PS^2 = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 + a^2 - 2a^2p^2$

$= (ap^2 + a)^2$

$\therefore\ PS = ap^2 + a = NS$

YEAR 11 EXTENSION 1 MARKERS' COMMENTS SEPTEMBER 2017

QUESTION 11:

a) Forgot to factorise the left hand side and got some weird vertices, focal lengths etc. Also, the directrix will NEVER cut the parabola so if this is the case your locus picture is wrong
b) Factorise out the *cos x* or you will lose 2 answers
c) Sub in value is easier for proving remainder than dividing
Factorised quadratic incorrectly after all the hard work of dividing.

QUESTION 12:

a) Must consider undefined point not in the solution. Most used critical points and handled it well. Others used multiplying by (x-3) squared, some succeeding many not.
b) Some took wrong direction. Others did not consider the ratio a negative.
c) Well done. A few did not consider 3 significant figures.
d) Some students did not know the relationship to be used. Most did.

QUESTION 13:

a) In Extension 1 it is not sufficient to just differentiate- you also have to have the algebraic skills to simplify answers, particularly factorisation. AND, PLEASE, USE THE QUOTIENT RULE when the question involves a quotient. Using the product rule does not lend itself neatly to simplification.
b) This was a bigger problem than it should have been and shows an abysmal lack of understanding of trigonometric functions. viz.,

If $tan\frac{x}{2} = 1$
$tan\, x \neq 2$

c) (ii) This mark was hard to earn. It had to be explicitly shown that NO = MO. It was not sufficient to just say "similar triangles" or "corresponding sides in similar triangles".
In the same way, if you did it by proving that ΔNOM was isosceles, you needed to really show or state the reason it was isosceles. The best solution is in the printed solutions, and this method was done by about 20 students. A difficult mark to get.

QUESTION 14:

a) This is a standard type of question and should be able to complete easily, however, mistakes occurred with the signs of the coefficients on the RHS of the expression easier not to factorise the RHS but to look at the individual terms to find the coefficients of *x*.
b) ii. When asked to describe the Cartesian equation (or locus) key words such as the NAME of the function should be used – parabola with vertex at (3, -2) – A diagram is not a description.
c) i. 'Show that' is clear – you must show each line of the proof including where the gradient of the tangent and hence the normal comes from – do not just quote it!
ii. Simply sub in the point x = 0 and y = 6a and see what to do next – Students need to realise that a represents the FOCAL length and is a constant – once the 3 values (including the zero) were found then the question still needed to be answered in terms of finding the three points.

QUESTION 15:

b) Many missed the negative sign in the leading term which reflected the shape of the graph.

c) The derivative needed to be equated to 1.

QUESTION 16:

a) Length cannot be negative; many of you missed one mark for that. In addition, you need to improve your algebra skill.
b) (ii) It is better to start with a diagram otherwise find what angles these two lines are making with x axis then prove base angles are equal
   Or
   Find the point of intersection of the given lines and find the length of each side.
c) Some tried to prove PS=PN, read the question carefully. Again, many of you demonstrated poor algebra skill to calculate the distance.