Girraween High School

Mathematics

Year 12 HSC Task 1

November 2014

General Instructions

- Working Time - 1 hour & 30 minutes
- Calculators and ruler may be used
- All necessary working out must be shown
- Write on one side of the paper only

Total Marks - 87

- Attempt all questions
- Marks may be deducted for careless or badly arranged work
- Start each question on a new sheet of paper

Part A

5 marks

Attempt Questions 1-5

Circle the letter corresponding to the correct answer on your multiple choice answer sheet.

Question 1 (1 mark)

A coin is tossed three times. What is the probability of obtaining at least one head?

A. $\frac{1}{8}$

B. $\frac{7}{8}$

C. $\frac{3}{8}$

D. $\frac{5}{8}$

Question 2 (1 mark)

Which of the following is true for the sequence: $4, 2\sqrt{2}, 2, \sqrt{2}, \ldots$?

A. there are negative terms in this sequence

B. its infinite sum does not exist

C. its infinite sum does exist

D. the common ratio is $\sqrt{2}$

Question 3 (1 mark)

Which is the value of $\sum_{n=11}^{39} (-1)^{n+1}$?

A. -28

B. -1

C. 0

D. 1

Question 4 (1 mark)

Which of the following best describes the locus of a point which moves such that it is equidistant from a point and a line?

A. the locus is a line

B. the locus is a circle

C. the locus is a parabola

D. the locus is a hyperbola

Question 5 (1 mark)

n is an integer chosen randomly from the set $\{5, 7, 9, 11\}$

p is an integer chosen randomly from the set $\{2, 6, 10, 14, 18\}$

What is the probability that $n + p = 23$?

A. 0.1

B. 0.2

C. 2.5

D. 0.3

Section B

83 marks

Attempt Questions 11-16

Write your answers on the paper provided.

Question 6 (14 marks)

(a) For the arithmetic sequence: $-57, -49, -41 \ldots$

i. Find the value of a and d [2]

ii. Find the expressions for the n^{th} term [2]

iii. Find the 20^{th} term [2]

iv. Find the first positive term in the sequence [2]

(b) The 10^{th} and 20^{th} term of an arithmetic sequence are 26 and -24 respectively.

i. Find the values of a and d [4]

ii. Find the sum of the first 20 terms [2]

Question 7 (14 marks)

(a) Consider the geometric sequence: $18,\ 9,\ 4\frac{1}{2},\ \cdots,\ \frac{9}{64}$

i. State the common ratio of this sequence [1]

ii. Determine the number of terms in this sequence [2]

iii. Find the value of $18 + 9 + 4\frac{1}{2} + \cdots + \frac{9}{64}$ [2]

(b) Using a limiting sum, express $0.\dot{5}\dot{7}$ as a fraction [2]

(c) Find the value of x such that x, $x + 1$ and $x + 10$ form a geometric sequence [2]

(d) A gardener plants a tree which was initially 80cm tall. At the end of the first year after planting, it was 130cm tall, that is it grew 50cm. Each year's growth was then 90% of the previous year's.

i. What was the growth of the tree in the second year? [1]

ii. How tall was the tree after three years? [1]

iii. How tall was the tree after ten years? Give your answer to the nearest cm. [2]

iv. Show that the tree never reaches the height of 6m. [1]

The exam continues on the next page

Question 8 (15 marks)

(a) A die is biased so that the face showing 1 is likely to come up with a probability of 0.5. However, the other five faces, numbered $2, 3, 4, 5, 6$ are all equally likely to appear.

i. Find the probability of a 6 appearing [1]

ii. Find the probability of any odd number appearing [1]

(b) An athlete knows that he has a 20% chance of winning the 100m sprint event and a 30% chance of winning the 200m sprint. If he competes in both events, what is the probability that he will:

i. win both events? [1]

ii. not win the 100m sprint and the 200m sprint [1]

iii. win one event only [2]

iv. win atleast one event [2]

(c) There are five nominees for President and Vice President of a club. Three are women and two are men. The first name, selected at random, will be the President and the second name will be the Vice President.

i. Draw a probability tree to represent all possible outcomes [2]

ii. Determine the probability that the two positions will be filled by a woman and a man in either order. [2]

(d) To win a game a player has to throw a 6 with a regular die. Find the probability that :

i. the player wins at his first throw [1]

ii. the player wins at his second throw [1]

iii. Find the expression for the probability that the player wins on the n^{th} throw [1]

The exam continues on the next page

Question 9 (15 marks)

For the parabola $(x-3)^2 = 12(y+1)$:

(a) Find the focal length [1]

(b) Find the coordinate of the vertex [1]

(c) Find the equation of the axis of symmetry [1]

(d) Find the equation of the directrix [1]

(e) Find the coordinate of the focus [1]

(f) Find the x intercept and y intercept of the parabola [3]

(g) Find the equation of the focal chord passing through the origin [3]

(h) Sketch the parabola showing the vertex, directrix, axis of symmetry, focus and the intercepts. [4]

Question 10 (10 marks)

(a) Given that a parabola has its focus at $(-1, 1)$ and its directrix is given by $y = 7$, find:

i. the focal length [1]

ii. the equation of the parabola [3]

(b) A parabola has equation $x = \frac{1}{8}y^2 - \frac{1}{2}y + \frac{3}{2}$. Find:

i. the coordinate of its vertex [4]

ii. the coordinate of its focus [1]

iii. the equation of its directrix [1]

The exam continues on the next page

Question 11 (14 marks)

(a) Out of 120 students, 80 study Biology and 60 study Physics and 20 study neither Biology nor Physics.

i. Draw a Venn diagram to illustrate this situation using B for Biology and P for Physics. [3]

ii. Find the number of students who study both Biology and Physics. [2]

iii. What is the probability that a student chosen at random does not do Physics? [2]

(b) Find all possible values of x such that [3]

$$1 + (x^2 - 5) + (x^2 - 5)^2 + \cdots$$

will have a limiting sum.

(c) Suppose we have two integers p and q such that: [4]

$$1, 3, 5, \cdots, p$$

and

$$1, 3, 5, \cdots, q$$

form arithmetic sequences.

Find p and q if:

$$(1 + 3 + 5 + \cdots + p) + (1 + 3 + 5 + \cdots + q) = (1 + 3 + 5 + \cdots + 33)$$

End of exam

Yr12 2U TASK 1 SOLUTIONS

Multiple Choice: B, C, D, C, A

Q1

$1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{8}$ $\therefore$ (B)

Q2

$r = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$

$-1 < \frac{1}{\sqrt{2}} < 1$ $\therefore$ (C)

Q3

number of terms $= 39 - 11 + 1 = 29$

So 28 pairs of "$-1+1$" with one "1" $\therefore$ (D)

Q4

(C)

Q5

$5 + 18 = 23$

$9 + 14 = 23$

$\therefore P(23) = \frac{2}{4 \times 5} = \frac{2}{20} = \frac{1}{10}$

$\therefore$ (A)

Q6

(a)

(i) $a = -57$ $d = 8$

(ii) $T_n = a + (n-1)d$.

$T_n = -57 + 8(n-1)$

$T_n = -57 + 8n - 8$

$T_n = -65 + 8n$

(iii) $T_{20} = -65 + 8 \times 20$

$T_{20} = 95$

(iv) $-65 + 8n > 0$

$8n > 65$

$n > \frac{65}{8}$

$n > 8.125$ $\therefore n = 9$.

$T_9 = -65 + 72 = 7$

(b)

(i) $26 = a + 9d$ ... ①

$-24 = a + 19d$ ... ②

② − ① gives:

$10d = -50$ $\therefore d = -5$

$a = 26 - 9d = 26 + 45 = 71$

$\therefore a = 71$

Q6/

(b)

(iii) $S_{20} = \frac{20}{2}(71 - 24)$

$S_{20} = 470$

Q7

(a)

(i) $r = \frac{1}{2}$

(ii) $T_n = ar^{n-1}$

$\frac{9}{64} = 18\left(\frac{1}{2}\right)^{n-1}$

$\frac{9}{64} = 18 \times \frac{1}{2^{n-1}}$

$2^{n-1} = \frac{18 \times 64}{9} = 128$

$\therefore n - 1 = 7 \quad \therefore n = 8$

$\therefore$ 8 terms in sequence.

(iii) $S_n = \frac{a(1 - r^n)}{1 - r}$

$S_8 = \frac{18\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$

$S_8 = \frac{2295}{64}$

(b)

$0.5\dot{7} = 0.575757\ldots$

$= 0.57 + 0.0057 + \ldots$

$= \frac{0.57}{1 - 0.01} = \frac{19}{33}$

(c) $\frac{x+1}{x} = \frac{x+10}{x+1}$

$(x+1)^2 = x(x+10)$

$x^2 + 2x + 1 = x^2 + 10x$

$1 = 8x$

$\therefore x = \frac{1}{8}$

(d)

(i) $0.9 \times 50 = 45$ cm

(ii) total growth by the end of the 3rd year is

$50 + 45 + 0.9 \times 45 = 135.5$

$\therefore$ height $= 80 + 135.5 = 215.5$ cm

(iii) the growth of the tree follows a GP:

$50 + 0.9 \times 50 + 0.9^2 \times 50 + \ldots$

$S_{10} = \frac{50(1 - 0.9^{10})}{1 - 0.9} = 325.66\ldots$

$\therefore$ height $= 80 + 325.66\ldots = 406$ cm

(nearest cm)

Q7

(d)

(iv) The eventual growth is given by the infinite sum:

$$S_\infty = \frac{50}{1-0.9} = 500$$

$\therefore$ Max height $= 80 + 500 = 580\text{cm}$

$\therefore$ Never reach 600cm.

Q8

(a)

(i) 0.1

(ii) $P(1) + P(3) + P(5)$

$= 0.5 + 0.1 + 0.1 = 0.7$

(b)

(i) $0.2 \times 0.3 = 0.06$

(ii) $1 - 0.06 = 0.94$

(iii) $P(WL) + P(LW)$

$= 0.2 \times 0.7 + 0.8 \times 0.3$

$= 0.38$

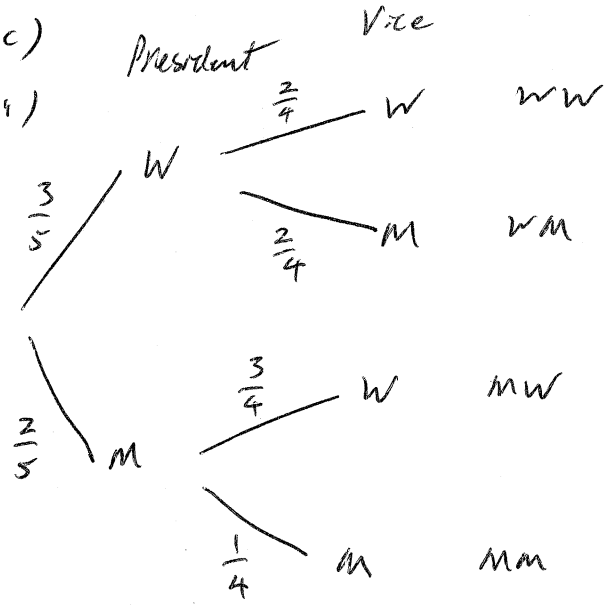
(iv) $P = 1 - P(\text{lose both})$

$= 1 - 0.8 \times 0.7$

$= 0.44$

(c)

(i)



(ii) $P(MW) + P(WM)$

$= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4}$

$= \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$

(d)

(i) $\frac{1}{6}$

(ii) $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

(iii)

Let P_n denote the probability of winning on the n^{th} throw.

$$P_n = \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$$

(iv) Probability of winning decreases with each throw if

$P_{n+1} < P_n$

i.e. $P_{n+1} - P_n < 0$.

Q8

(d)

(iv) $P_{n+1} - P_n$

$= \left(\frac{5}{6}\right)^n \times \frac{1}{6} - \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$

$= \left(\frac{5}{6}\right) \times \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6} - \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$

$= \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}\left[\frac{5}{6} - 1\right]$

$= -\frac{1}{6^2}\left(\frac{5}{6}\right)^{n-1}$

Since $\left(\frac{5}{6}\right)^{n-1} > 0$ therefore

$-\frac{1}{36}\left(\frac{5}{6}\right)^{n-1} < 0$

$\therefore$ P of winning decreases with each throw.

Q9 $(x-3)^2 = 4 \times 3 \times (y+1)$

(a) $a = 3$

(b) $V = (3, -1)$

(c) $x = 3$

(d) $y = -4$

(e) $F = (3, 2)$

(f)

when $x = 0$

$9 = 12(y+1)$

$y + 1 = \frac{3}{4}$

$y = -\frac{1}{4}$. $\therefore (0, -\frac{1}{4})$

when $y = 0$

$(x-3)^2 = 12$

$x - 3 = \pm 2\sqrt{3}$

$x = 3 \pm 2\sqrt{3}$ $\therefore (3+2\sqrt{3}, 0)$ & $(3-2\sqrt{3}, 0)$

$x = 6.46\cdots$

$x = -0.46\cdots$

(g)

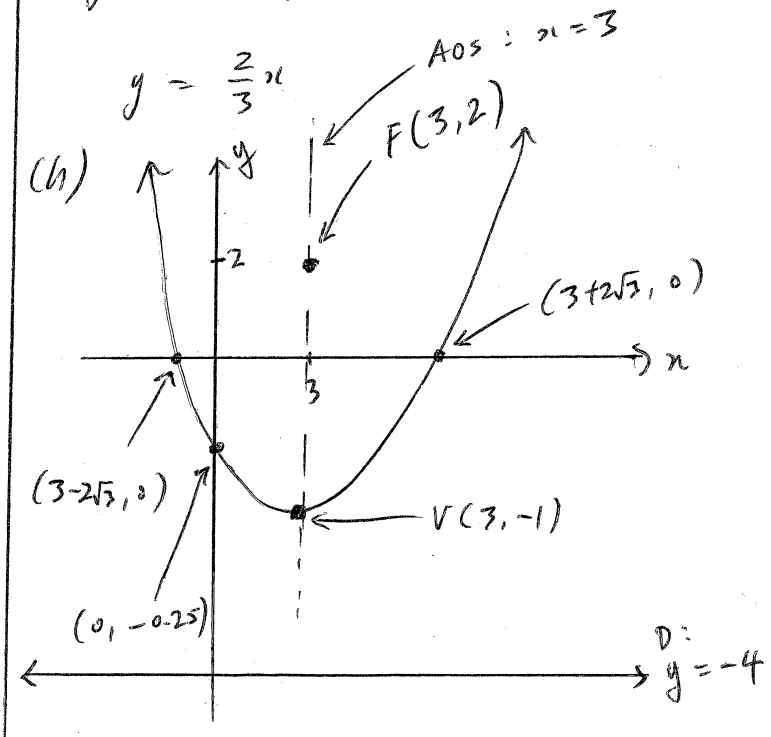
Line passing through $(3,2)$ & $(0,0)$

$m = \frac{2-0}{3-0} = \frac{2}{3}$

$y - 0 = \frac{2}{3}(x - 0)$

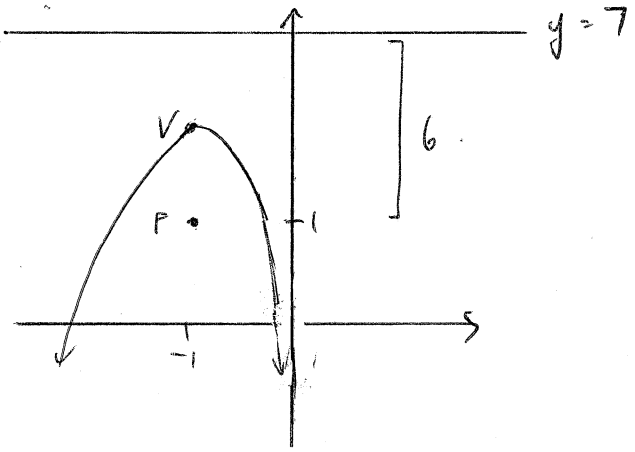
$y = \frac{2}{3}x$

(h)



Q10

(a)



(i) $a = 3$

(ii) $V = (-1, 4)$

$\therefore$ equation is

$(x+1)^2 = -4 \times 3(y-4)$

$(x+1)^2 = -12(y-4)$

(b)

(i) $x = \frac{1}{8}y^2 - \frac{1}{2}y + \frac{3}{2}$

$8x = y^2 - 4y + 12$

$8x - 12 = y^2 - 4y$

$8x - 12 + 4 = y^2 - 4y + 4$

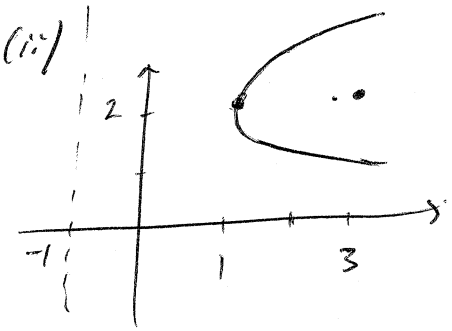
$8x - 8 = (y-2)^2$

$8(x-1) = (y-2)^2$

$4 \times 2 \times (x-1) = (y-2)^2$

$\therefore V = (1, 2)$

(ii)



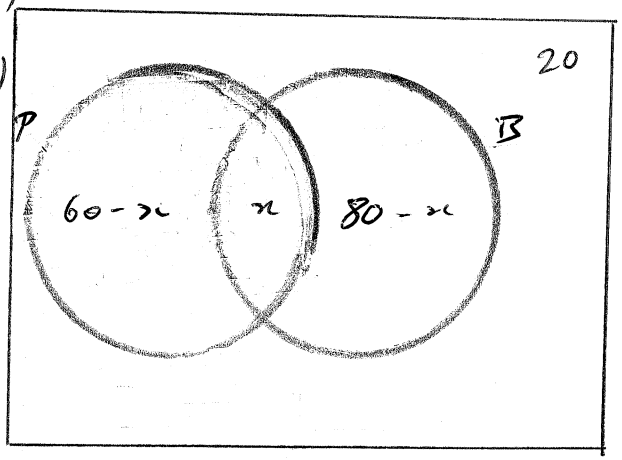
$\therefore F = (3, 2)$

(iii) D: $x = -1$

Q11

(a)

(i)



(ii) Let x be number of students who study both physics & biology.

$\therefore 60 - x + x + 80 - x = 120 - 20$

$140 - x = 100$

$\therefore x = 40$

(iii)

$P = \frac{20 + 80 - 40}{120} = \frac{1}{2}$.

a11

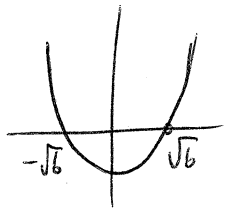
(b) $r = x^2 - 5$

For S_∞ :

$$-1 < x^2 - 5 < 1$$

for $x^2 - 5 < 1$

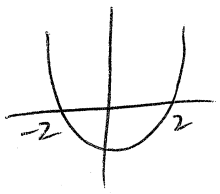
$$x^2 - 6 < 0$$



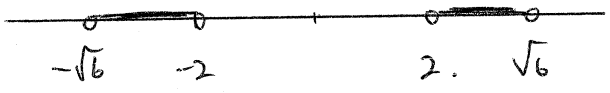
$\therefore -\sqrt{6} < x < \sqrt{6}$

for $-1 < x^2 - 5$

$$0 < x^2 - 4$$



$\therefore \quad x < -2 \quad \& \quad x > 2.$



$\therefore -\sqrt{6} < x < -2$

and $2 < x < \sqrt{6}$

(c)

Let $p = u_1^{th}$ term in the sequence

Let $q = u_2^{th}$ term in the sequence.

Now $a = 1$ & $d = 2$

So $p = 1 + (u_1 - 1)2$

$p = 2u_1 - 1 \quad (*)$

So $q = 2u_2 - 1$

Now $S_{u_1} = 1 + 3 + \ldots + p = \frac{u_1}{2}(1 + p)$

$S_{u_1} = \frac{u_1}{2}(2u_1) = u_1^2 \quad (\Delta)$

So $S_{u_2} = 1 + 3 + \ldots + q = u_2^2$

$\therefore 1 + 3 + \ldots + p + 1 + 3 + \ldots + q = u_1^2 + u_2^2$

$\therefore u_1^2 + u_2^2 = 1 + 3 + \ldots + 33$

$u_1^2 + u_2^2 = 17^2$ ←

trial & error gives

$8^2 + 15^2 = 17^2$

$\therefore u_1 = 8$ & $u_2 = 15$

$\therefore p = 15$ & $q = 29$

Note: using $(*)$ $33 = 2n - 1$

$2n = 34$

$\therefore n = 17$

So 33 is the 17th term in the sequence.

using (Δ) $S_{17} = 1 + 3 + \ldots + 33 = 17^2$