SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

NOVEMBER 2005

HSC ASSESSMENT TASK #1

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 90 minutes
- Write using black or blue pen. Pencil may be used for diagrams
- Board approved calculators may be used
- All necessary working should be shown in every question

Total Marks - 70

- All questions may be attempted
- All questions are of equal value

Examiner – *R Dowdell*

Question 1: (14 marks)

Marks

(a) Write down the derivatives of

(i) $x^3 + x^2$

(ii) $x^3 - x^2$

(iii) $x^3 \times x^2$ 4

(iv) $\dfrac{x^3}{x^2}$

(b) The fifth term of an Arithmetic Series is 1 and the sum of the first 8 terms is 6. Find the 11th term. 3

(c) Find the values of A, B and C when $2x^2 - 4x + 3 \equiv A(x-1)^2 + B(x-1) + C$. 3

(d) The quadratic equation $5x^2 + 3x - 7 = 0$ has roots α and β. Find the value of:

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ 4

(iv) $\alpha^2 + \beta^2$

Question 2: (14 marks) START A NEW BOOKLET

Marks

(a) A point moves such that its distance from the point (1, 3) is always equal to its distance from the line $x = 5$. The path that point moves on is a parabola. For this parabola, determine:

(i) its Cartesian equation;

(ii) the focus; 5

(iii) the directrix;

(iv) the focal length;

(v) the axis of symmetry.

(b) (i) Find the equation of the line which is perpendicular to $3x + 2y - 7 = 0$ and which passes through the point (1, 2). 4

(ii) Find the point of intersection of these two lines

(c) Sketch the region represented by $x + 3y \geq 3$ and $x - y < 2$. 3

(d) Calculate the shortest distance from (2, 3) to the line $y = 3x + 2$. 2

Question 3: (14 marks) START A NEW BOOKLET

Marks

(a) (i) Write in simplest form $\log_a x^3 - \log_a xz + 2\log_a x$

(ii) Make y the subject of $\log_a x - \log_a y = 3\log_a (xy)$

(iii) Find $\log_6 8$, correct to 3 decimal places. 7

(iv) If $\log_a 7\cdot 5 = 3\cdot 65$, find the value of a, correct to 4 significant figures.

(b) Solve:

(i) $x^6 + 7x^3 = 8$

(ii) $(x^2 + x)^2 - 14(x^2 + x) + 24 = 0$ 7

(iii) $x^2 + 5x < 6$

Question 4: (14 marks) START A NEW BOOKLET

Marks

(a) Evaluate the discriminant of $3x^2 + 5x - 7 = 0$.
Hence, or otherwise, solve the equation. 2

(b) For what value(s) of m does $3x^2 + mx + 5 = 0$ have 2 distinct, real roots? 2

(c) 3 white marbles and 5 black marbles are placed in a container.

(i) A marble is drawn at random from the container. Its colour is noted and it is returned to the container. A second marble is then drawn and its colour noted.

(I) Draw a tree diagram which could be used to determine the probability of each possible event.

(II) Find the probability that the two marbles drawn have the same colour. 6

(ii) If the first marble drawn is not returned to the container, find the probabilty that the two marbles drawn are the same colour.

(iii) In a game, if a person draws 2 black marbles, he is allowed to roll a die. If the die roll yields a 6, the person wins \$10. What is the probability of winning \$10 in the game if the first marble drawn is not replaced.

(d) Bill borrows \$10 000 (at 9% per annum reducible interest) to set up a Home Theatre. He repays the money under the following conditions:

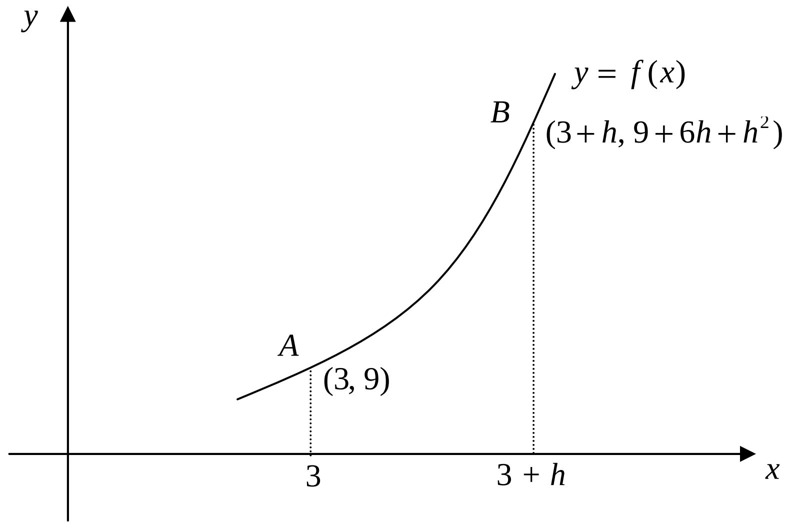
- no repayment is required until 2 months after the purchase – that is, no repayment is due after 1 month
- the loan is to be repaid by making equal monthly repayments of \$*M* 4
- the loan is to be completely repaid 24 months after the purchase date.

Calculate the size of the monthly repayment \$*M*.

Question 5: (14 marks) START A NEW BOOKLET

Marks

(a)



B represents a general point on the curve $y = f(x)$.

(i) Write down, in simplest form, the gradient of the secant AB.

(ii) Find the gradient of the tangent to $y = f(x)$ at A.

(iii) Find the equation of the tangent to $y = f(x)$ at the point (3, 9).

3

(b) (i) **Copy and complete** the following table in your answer booklet (giving answers to 3 decimal places).

h	$0\cdot 1$	$0\cdot 01$	$0\cdot 001$
$\dfrac{2^h - 1}{h}$			

4

(ii) If $A = \lim\limits_{h \to 0} \dfrac{2^h - 1}{h}$, write down an approximation to A correct to 2 decimal places.

(iii) Show, using first principles, that $\dfrac{d}{dx}\left(2^x\right) = A.2^x$

Question 5 continues on the next page

Question 5 (continued)

(c) (i) Given that $n! = 1 \times 2 \times 3 \times ..... \times (n-1) \times n$ for $n \geq 1$

and that $0! = 1$ show that $\dfrac{n}{n!} = \dfrac{1}{(n-1)!}$

(ii) Consider the function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

(I) Is $E(x)$ an infinite geometric series? Justify your answer.

(II) Find an approximation for $E(1)$ by using the first five terms of the appropriate series.
Write your answer correct to 4 significant figures. 7

(III) Find the derivative of $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

(IV) By considering your answer to part (iii), write down a simple relationship between $E(x)$ and $E'(x)$.

(V) Write down an approximation to the gradient of the tangent to the curve $y = E(x)$ at the point where x = 1.
Write your answer correct to 1 decimal place.

End of Paper

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2005

Assessment Task #1

Mathematics

Sample Solutions

Question	Marker
1	AW
2	DM
3	FN
4	AF
5	EC

Question 1

a) i) $\frac{d}{dx}(x^3+x^2)$

$= 3x^2+2x$.

ii) $\frac{d}{dx}(x^3-x^2)$

$= 3x^2-2x$.

iii) $\frac{d}{dx}(x^3 \times x^2)$

$= \frac{d}{dx}(x^5)$

$= 5x^4$

iv) $\frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x)$

$=1$

b) i). $T_n = a+(n-1)d$.

when $n=5$

$1 = a+4d$.

$\therefore a = 1-4d$ —— ①

$S_n = \frac{n}{2}(2a+(n-1)d)$

when $n=8$

$6 = 8a+28d$ —— ②

b) cont).

subs ① into ②.

$6 = 8(1-4d)+28d$.

$6 = 8-32d+28d$.

$d = \frac{1}{2}$.

$\therefore$ from ①.

$a = 1-4d$.

$= -1$

$\therefore T_{11} = a+(n-1)d$

$= -1+10(\frac{1}{2})$

$= 4$

c) $2x^2-4x+3$

$= A(x-1)^2+B(x-1)+C$

$= A(x^2-2x+1)+B(x-1)+C$.

$= Ax^2-2Ax+A+Bx-B+C$.

$= Ax^2+(B-2A)x+A-B+C$

Equating coefficients

$A=2$ —— ①

$B-2A=-4$ —— ②

$A-B+C=3$ —— ③.

subs ① into 2.

$B=0$ $\therefore$ $C=1$

d) $5x^2+3x-7.$

roots α & β.

i) $\alpha+\beta = -\frac{3}{5}$

ii) $\alpha\beta = -\frac{7}{5}.$

iii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{3}{7}.$

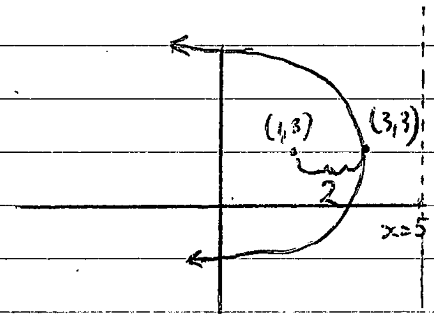
iv) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2(\alpha\beta)$

$= \left(-\frac{3}{5}\right)^2 - 2\left(\frac{-7}{5}\right)$

$= 3\frac{4}{25}$

Question 2

a)



i) $(y-3)^2 = 4(-2)(x-3)$.

$(y-3)^2 = -8(x-3)$

or

$8x + y^2 - 6y - 15 = 0$.

ii) focus $(1, 3)$

iii) directrix $x = 5$

iv) focal length 2 (-2 is acceptable)

v) axis of symmetry $y = 3$.

b) i) $y = -\frac{3}{2}x + \frac{7}{2}$

So $m_\perp = \frac{2}{3}$.

$y - 2 = \frac{2}{3}(x-1)$.

$y = \frac{2}{3}x + \frac{4}{3}$

or

$2x - 3y + 4 = 0$.

ii) $2x - 3y + 4 = 0$ (A)
$3x + 2y - 7 = 0$ (B)

$3 \times (A) - 2 \times (B)$

$6x - 9y + 12 = 0$

$6x - 4y - 14 = 0.$

$-13y + 26 = 0$

$y = 2.$

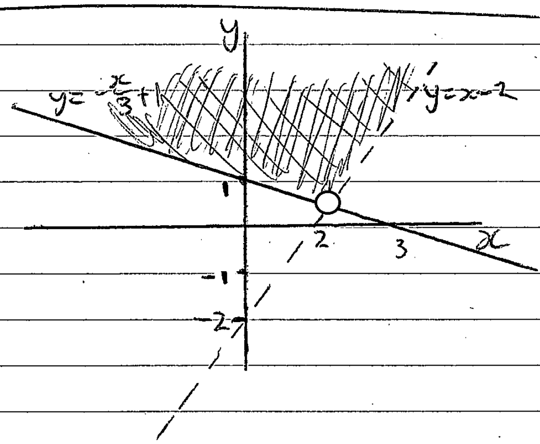
Sub into (A)

$2x - 6 + 4 = 0$

$x = 1.$

Point of intersection $(1, 2)$.

c)



$y > x - 2$ (A)
$y \geq -\frac{x}{3} + 1$ (B)

Sub $(0,0)$ into (A)
$0 > -2$ true

Sub $(0,0)$ into (B)

$0 \geq +1.$ false

d) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$3x - y + 2 = 0. \qquad (2,3)$

$d = \dfrac{|3(2) + -1(3) + 2|}{\sqrt{9+1}}$

$= \dfrac{5}{\sqrt{10}}$

$= \dfrac{5\sqrt{10}}{10}$

$= \dfrac{\sqrt{10}}{2}.$

Question 3

(a) **(i)**

$$\log_a x^3 - \log_a xz + 2\log_a x$$
$$3\log_a x - (\log_a x + \log_a z) + 2\log_a x$$
$$4\log_a x - \log_a z$$
$$\log_a\left(\frac{x^4}{z}\right)$$

(ii)

$$\log_a x - \log_a y = 3\log_a (xy)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a (xy)^3$$
$$\frac{x}{y} = x^3 y^3$$
$$y^4 = x^{-2}$$
$$y = \frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x} \quad (x, y > 0)$$

(iii) $\log_6 8 = \dfrac{\ln 8}{\ln 6} = 1\cdot 161$ (3 dec.pl.)

(iv)

$$\log_a 7\cdot 5 = 3\cdot 65$$
$$a^{3.65} = 7.5$$
$$3\cdot 65 \ln a = \ln 7\cdot 5$$
$$\ln a = 0.552$$
$$a = e^{0\cdot 552} = 1\cdot 737 \text{ (4 sig.fig.)}$$

(b) **(i)** $x^6 + 7x^3 - 8 = 0 \quad \left(\text{Let } a = x^3\right)$

$a^2 + 7a - 8 = 0$

$(a+8)(a-1) = 0$

$a = -8, a = 1 = x^3$

$\therefore x = -2, x = 1$

(ii) $\left(x^2 + x\right)^2 - 14\left(x^2 + x\right) + 24 = 0 \quad a = \left(x^2 + x\right)$

$a^2 - 14a + 24 = 0$

$(a-12)(a-2) = 0$

$a = 12, \quad a = 2$

$\left(x^2 + x\right) = 12, \quad \left(x^2 + x\right) = 2$

$(x = 4)(x-3) = 0, \quad (x+2)(x-1) = 0$

$x = -4, 3, -2, 1$

(iii) $x^2 + 5x - 6 < 0$

$(x+6)(x-1) < 0$

$\therefore -6 < x < 1$

Question 4

(a) $$\Delta = b^2 - 4ac$$
$$= (5)^2 - 4(3)(-7)$$
$$= 25 + 84$$
$$= 109$$

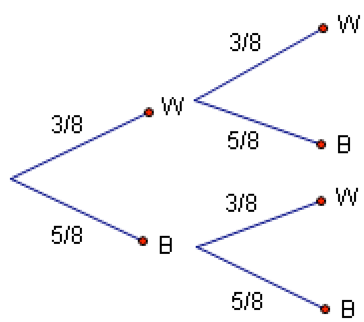
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-5 \pm \sqrt{109}}{6}$$

(b) $$\Delta = b^2 - 4ac$$
$$= m^2 - 4(3)(5)$$
$$= m^2 - 60$$

For 2 distinct real roots $\Delta > 0$.

$$m^2 - 60 > 0$$
$$\left(m - \sqrt{60}\right)\left(m + \sqrt{60}\right) > 0$$
$$m < -\sqrt{60}, m > \sqrt{60}$$
$$m < -2\sqrt{15}, m > 2\sqrt{15}$$

(c) (i) (I)



(II) P(same colour) = P(WW) + P(BB)

$$= \frac{3}{8} \times \frac{3}{8} + \frac{5}{8} \times \frac{5}{8}$$
$$= \frac{9}{64} + \frac{25}{64}$$
$$= \frac{17}{32}$$

(ii) P(same colour) = P(WW) + P(BB)

$$= \frac{3}{8} \times \frac{2}{7} + \frac{5}{8} \times \frac{4}{7}$$

$$= \frac{6}{56} + \frac{20}{56}$$

$$= \frac{13}{28}$$

(iii) $P(BB) \times P(6) = \frac{5}{8} \times \frac{4}{7} \times \frac{1}{6}$

$$= \frac{5}{84}$$

(d) Let A_n be the amount owing after n months.

$A_1 = 10000 \times 1 \cdot 0075$

$A_2 = A_1 \times 1 \cdot 0075 - M$

$A_2 = \left(10000 \times 1 \cdot 0075\right) \times 1 \cdot 0075 - M$

$A_2 = 10000 \times 1 \cdot 0075^2 - M$

$A_3 = A_2 \times 1 \cdot 0075 - M$

$A_3 = \left(10000 \times 1 \cdot 0075^2 - M\right) \times 1 \cdot 0075 - M$

$A_3 = 10000 \times 1 \cdot 0075^3 - M\left(1 + 1 \cdot 0075\right)$

:

:

$A_{24} = 10000 \times 1 \cdot 0075^{24} - M\left(1 + 1 \cdot 0075 + ... + 1 \cdot 0075^{22}\right)$

But $A_{24} = 0$

$\therefore M\left(1 + 1 \cdot 0075 + ... + 1 \cdot 0075^{22}\right) = 10000 \times 1 \cdot 0075^{24}$

$1 + 1 \cdot 0075 + ... + 1 \cdot 0075^{22}$ is a geometric series.

$a = 1, r = 1 \cdot 0075, n = 23$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$S_{23} = \frac{1\left(1 - 1 \cdot 0075^{23}\right)}{1 - 1 \cdot 0075}$$

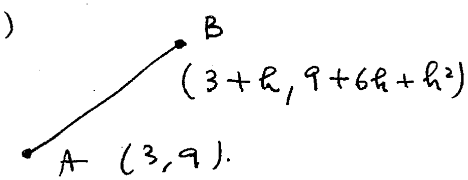
≈ 25

$\therefore 25M = 10000 \times 1 \cdot 0075^{24}$

$M = \$478 \cdot 57$

Question 5

(a)



(i) $m_{AB} = \dfrac{h^2+6h}{h}$

$= h+6.$

(ii) $y = x^2$

$\dfrac{dy}{dx} = 2x$

$\left.\dfrac{dy}{dx}\right|_{x=3} = 6$

(iii) $y - 9 = 6(x-3)$

$6x - y - 9 = 0$

is the equation of tgt to $y=f(x)$ at $A(3,9)$.

(b)

h	0·1	0·01	0·001
$\dfrac{2^h-1}{h}$	0·72	0·70	0·69

(ii) 0·69.

(iii) $A = \lim\limits_{h\to 0} \dfrac{2^h-1}{h}.$

$\dfrac{d}{dx}(2^x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2^{x+h}-2^x}{h}$

$= \lim\limits_{h\to 0} 2^x\left(\dfrac{2^h-1}{h}\right)$

$= A\cdot 2^x.$

(c)

(i) $\dfrac{n}{n!} = \dfrac{\cancel{n}\ 1}{\cancel{n}(n-1)(n-2)\cdots 3\times 2\times 1}$

$= \dfrac{1}{(n-1)!}$

(ii) $E(x)$ is not G.P.

(I) $\because \left(\dfrac{x^2}{2!}\right)\Big/(x) \neq \left(\dfrac{x^3}{3!}\right)\Big/\left(\dfrac{x^2}{2!}\right)$

(II) $E(1) = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}$

$= 2\cdot 708$ (4 s.f.)

(III) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

(IV) $E'(x) = E(x).$

(V) $E'(1) \doteqdot 2\cdot 7.$