**SYDNEY BOYS HIGH SCHOOL**
**MOORE PARK, SURRY HILLS**

# 2006

YEAR 11
HSC Task 1
Term 4

# Mathematics

**General Instructions**

- Working time – 90 Minutes
- Reading time – 5 Minutes
- Write using black or blue pen
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work.
- Hand in your answer booklets in 5 sections. Section A (Question 1), Section B (Question 2), Section C (Question 3), Section D (Question 4) and Section E (Question 5).

**Total Marks – 76**

- Attempt Questions 1 – 5.
- All questions are NOT of equal value.

Examiner: *A. Ward*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

## SECTION A

**Question 1 (16 Marks)** **Marks**

**a)** Differentiate with respect to $x$ and simplify:

(i) $y = x^5 - 1$ 1

(ii) $y = \left(3x^4 - 5\right)^7$ 1

(iii) $y = \dfrac{x+1}{3-x}$ 2

**b)** Write down the third and fourth terms of the series $12 + 6 + ....$ if it is:

(i) an arithmetic series 1

(ii) a geometric series 1

**c)** (i) Find, to 2 decimal places, the roots of:

$$2x^2 - 3x - 4 = 0$$ 2

(ii) Show that $2x^2 - 3x + 4 = 0$ has no real roots. 1

**d)** Determine each of the following:

(i) $\int x^6 dx$ 1

(ii) $\int (x-1)(x+2)dx$ 2

(iii) $\int \dfrac{\sqrt{t}+1}{\sqrt{t}} dt$ 2

**e)** Three terms of an arithmetic series have the sum 21 and a product of 315.

Find the 3 numbers. 2

## End of Section A

**SECTION B – Start a new booklet** **Marks**

**Question 2. (17 marks)**

**a)** Given that $ab^c = d$ :

(i) Find $b$ in terms of $a$, $c$ and $d$. 1

(ii) Find $c$ in terms of $a$, $b$ and $d$ 2

(iii) Calculate $b$, correct to 4 significant figures, when $a = 75.12$, $c = 1.142$ and $d = 61.94$ . 1

**b)** Given that, $f(x) = a - 2x - x^2$ where a is a constant. Find:

(i) the value for $a$ for which the roots of the equation differ by 3. 2

(ii) the set of values of $a$ for which $f(x) < 0$ for all values of $x$. 2

**c)** At what points does the tangent to $f(x) = 2x^3 - 3x^2 + 1$, have slope 0. 2

**d)** Evaluate: $\int_{-1}^{2} \left(3x^2 - 2x\right) dx$ 2

**e)** For the function, $f(x) = 5x^3 - 7x^2 + 3x + 2$

(i) Show that $f(x)$ passes through the point (1,3) 1

At this point, find:

(ii) the gradient. 1

(iii) the equation of the tangent in gradient-intercept form. 1

(iv) the equation of the normal in general form. 2

## End of Section B

**SECTION C – Start a new booklet** **Marks**

**Question 3. (15 marks)**

**a)** Find the values of y which satisfy the equation:

$\left(8^y\right)^y \times \dfrac{1}{32^y} = 4$ 2

**b)** A point $P$ is equidistant from the x-axis and the point $F(0,2)$. Find the locus of the point $P$. 3

**c)** Using first principles, find the derivative of the function $f(x) = x^2 + x$, (all working must be shown). 2

**d)** Find the area bounded by $y = \sqrt{x} + 3$ and the x-axis for $1 \le x \le 4$. 2

**e)** The equation $3x^2 - 6x + 8 = 0$ has roots $\alpha$ and $\beta$. Find an equation which has roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ 2

**f)** A wall vase has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \dfrac{y^3}{64} + 1$, the $y$-axis and $y = 8$, is rotated through two right angles about the $y$-axis, the units being centimetres. Calculate its volume. 4

**End of Section C**

**SECTION D – Start a new booklet** **Marks**

**Question 4. (12 marks)**

**a)** (i) Tabulate, to 2 decimal places, the values of the function $f(x)=\sqrt{1+x^2}$, at **unit** intervals from $x=2$ to $x=5$ inclusive. 2

(ii) Use these values to find an estimate, by the trapezoidal rule, of the area between $y=f(x)$ and the $x$-axis for $2\le x\le 5$ to 3 decimal places.. 2

**b)** A point $P$ has $x$-coordinate $a$ which is taken to be on the line $y=3x-9$.

(i) If $Q$ is the point (1,4), show that $PQ^2=10a^2-80a+170$ 2

(ii) Find the value of $a$ which will make $PQ$ a minimum. 2

(iii) N is a point on the line such that QN is perpendicular to the line. Find the co-ordinates of N. 2

(iv) Find the equation of $QN$ in general form. 2

## End of Section D

**SECTION E – Start a new booklet** **Marks**

**Question 5. (16 marks)**

**a)** Solve for *x*:

(i) $\log_5 x + \log_2 8 = 0$ 1

(ii) $\log_3 x + 3\log_x 3 = 4$ 2

**b)** Prove that, if the sum of the radii of two circles remains constant, the sum of the areas of the circles is least when the circles are equal. 3

**c)** A prize fund is set up with an investment of $2000, to provide a prize of $150 each year. The fund accrues compound interest at 5% p.a. paid six monthly. The first prize is awarded 1 year after the initial investment, after interest is received.

(i) Find the value of the fund immediately after the first years prize is drawn from the fund. 1

(ii) Find the value of the fund immediately after the third prize is drawn from the fund. 2

(iii) Find the number of prizes of the full $150 which can be drawn from the fund. 3

**d)** Three real, distinct and non-zero numbers *a*, *b* and *c* are such that *a*, *b*, *c* are in arithmetic series and *a, c, b* are in geometric series.

(i) Find the numerical value of the common ratio of the geometric series. 2

(ii) Hence, find an expression in terms of *a* for the sum to infinity of the geometric series whose first terms are *a*, *c*, *b*. 2

**End of Section E**

**End of Examination Paper**

Question 1

a) i) $y = x^5 - 1$

$y' = 5x^4$

ii) $y = (3x^4 - 5)^7$

$y' = 7(3x^4 - 5)^6 12x^3$

$= 84x^3(3x^4 - 5)^6$

iii) $y = \frac{x+1}{3-x}$ $u = x+1$ $v = 3-x$ $y' = \frac{vu' - uv'}{v^2}$

$u' = 1$ $v' = -1$

$y' = \frac{(3-x)+(x+1)}{(3-x)^2}$

$= \frac{4}{(3-x)^2}$

b) i) $a = 12$, $d = -6$

$\therefore T_3 = 0$, $T_4 = -6$

ii) $a = 12$, $r = \frac{1}{2}$

$\therefore T_3 = 3$, $T_4 = \frac{3}{2}$

c) i) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $(2x^2 - 3x - 4 = 0)$

$x = \frac{3 \pm \sqrt{9 - 4(2)(-4)}}{2(2)}$

$x = \frac{3 \pm \sqrt{41}}{4}$

$x = -0.85, 2.35$ to 2 decimal places

ii) $\Delta = b^2 - 4ac$ $(2x^2 - 3x + 4 = 0)$

$\Delta = (-3)^2 - 4(2)(4)$

$\Delta = -23$

since $\Delta < 0$, $2x^2 - 3x + 4 = 0$ has no real roots.

d) i) $\int x^6 dx = \frac{x^7}{7} + C$

ii) $\int (x-1)(x+2)\,dx = \int (x^2 + x - 2)\,dx$

$= \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$

iii) $\int \frac{\sqrt{t}+1}{\sqrt{t}}\,dt = \int (1 + t^{-\frac{1}{2}})\,dt$

$= t + \frac{t^{\frac{1}{2}}}{(\frac{1}{2})} + C$

$= t + 2\sqrt{t} + C$

e) let the three terms be $a-d, a, a+d$

$(a-d) + (a) + (a+d) = 21$

$3a = 21$

$a = 7$

$(a-d)(a)(a+d) = 315$

$a(a^2 - d^2) = 315$

sub in $a = 7$

$7(49 - d^2) = 315$

$49 - d^2 = 45$

$-d^2 = -4$

$d^2 = 4$

$d = \pm 2$

$\therefore$ the three terms are 5, 7 & 9.

Q2.

(a) (i) $ab^c = d$

$b^c = \frac{d}{a}$

$b = \left(\frac{d}{a}\right)^{\frac{1}{c}}$ OR $\sqrt[c]{d/a}$.

(ii) $b^c = \frac{d}{a}$

$c = \log_b \frac{d}{a}$

(iii) $b = \left(\frac{61.94}{75.12}\right)^{\frac{1}{1.142}}$

$\doteqdot 0.8446$ (4. S. F's)

(b) (i) Let the roots be $\alpha, \alpha - 3$.

now $\alpha + \alpha - 3 = -2$ $\quad (S_1 = -\frac{b}{a})$.

$2\alpha = 1$

$\alpha = \frac{1}{2}$.

now $S_2 = \frac{c}{a} = -a$

$\therefore a = -\left(\frac{1}{2} \times \frac{-5}{2}\right)$

$a = \frac{5}{4}$.

(ii) If $f(x) < 0$ the quadratic would be negative definite.

$\therefore \Delta < 0$

ie $(-2)^2 - 4 \times -1 \times a < 0$

$4 + 4a < 0$

$4a < -4$

$a < -1$

(c) $f'(x) = 6x^2 - 6x.$

let $f'(x) = 0$

$6x(x-1) = 0$

$x = 0, 1.$ $\therefore$ **Points are (0, 1) and (1, 0)**

<u>NOTE</u> If $x = 0$

$f(0) = 0 - 0 + 1 = 1$

If $x = 1$

$f(1) = 2 - 3 + 1 = 0.$

(d) $\int_{-1}^{2} (3x^2 - 2x)\,dx = \left[x^3 - x^2\right]_{-1}^{2}$

$= (8-4) - (1 - -1)$

$= 4 - -2$

$= 6.$ (NB Answer is <u>not</u> $6u^2$)

(e)(i) $f(1) = 5 - 7 + 3 + 2 = 3.$

$\therefore$ (1,3) lies on $f(x) = 5x^3 - 7x^2 + 3x + 2.$

(ii) $f(x) = 5x^3 - 7x^2 + 3x + 2.$

$f'(x) = 15x^2 - 14x + 3$

$\therefore f'(1) = 15 - 14 + 3$

$= 4$ $\therefore$ **gradient at (1,3) is 4**

(iii) $y - 3 = 4(x-1)$

$y - 3 = 4x - 4$

**$y = 4x - 1$**

in "$y = mx + b$" form.

(iii) $y - 3 = -\frac{1}{4}(x-1)$

$4y - 12 = x + 1$

**$x - 4y + 13 = 0$**

in "general form"

2006 Mathematics Assessment 1: **Solutions Section C**

## Question 3 (15 Marks)

(a) Find the values of $y$ which satisfy the equation: **[2]**

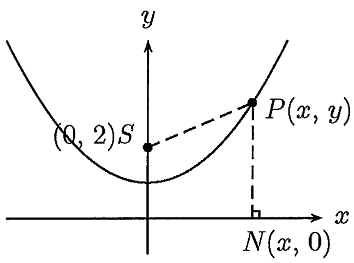
$(8^y)^y \times \dfrac{1}{32^y} = 4$

**Solution:**

$$\begin{aligned} 2^{3y^2} \times 2^{-5y} &= 2^2, \\ 3y^2 - 5y - 2 &= 0, \\ (3y+1)(y-2) &= 0, \\ \therefore\ y &= 2,\ -\tfrac{1}{3}. \end{aligned}$$

(b) A point $P$ is equidistant from the $x$-axis and the point $F(0,.2)$. Find the locus of the point $P$. **[3]**

**Solution:**




$$\begin{aligned} PN^2 &= PS^2, \\ y^2 &= x^2 + (y-2)^2, \\ y^2 &= x^2 + y^2 - 4y + 4, \\ \therefore\ x^2 &= 4(y-1). \end{aligned}$$

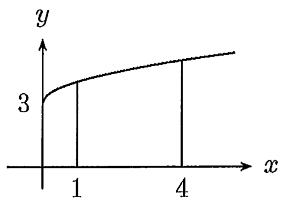
(c) Using first principles, find the derivative of the function $f(x) = x^2 + x$ (all working must be shown). **[2]**

**Solution:**

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left\{\frac{(x+h)^2 + (x+h) - x^2 - x}{h}\right\}, \\ &= \lim_{h\to 0}\left\{\frac{x^2 + 2xh + h^2 + x + h_x^2 - x}{h}\right\}, \\ &= \lim_{h\to 0}\left\{\frac{2xh + h^2 + h}{h}\right\}, \\ &= \lim_{h\to 0}\{2x + h + 1\}, \\ &= 2x + 1. \end{aligned}$$

(d) Find the area bounded by $y = \sqrt{x} + 3$ and the $x$-axis for $1 \leq x \leq 4$. [2]

**Solution:**




$$\text{Area} = \int_1^4 (x^{\frac{1}{2}} + 3)\, dx,$$
$$= \left[ \frac{2x^{\frac{3}{2}}}{3} + 3x \right]_1^4,$$
$$= \frac{16}{3} + 12 - \left( \frac{2}{3} + 3 \right),$$
$$= \frac{41}{3} \text{ or } 13\tfrac{2}{3}.$$

(e) The equation $3x^2 - 6x + 8 = 0$ has roots $\alpha$ and $\beta$. Find an equation which has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. [2]

**Solution:** $\alpha + \beta = 2,$

$$\alpha\beta = \frac{8}{3}.$$

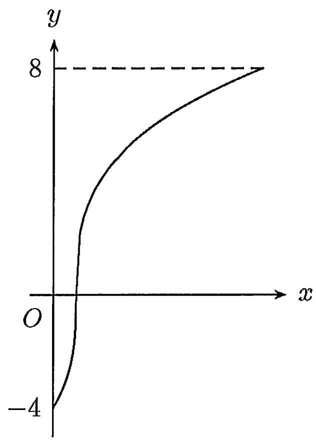
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta},$$
$$= \frac{3}{4},$$
$$\frac{1}{\alpha\beta} = \frac{3}{8}.$$

$\therefore$ New equation: $x^2 - \frac{3x}{4} + \frac{3}{8} = 0,$

*i.e.* $8x^2 - 6x + 3 = 0.$

(f) A wall vase has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \dfrac{y^3}{64} + 1$, the $y$-axis, and $y = 8$, is rotated through two right angles about the $y$-axis, the units being centimetres. Calculate its volume. [4]

**Solution:** When $x = 0$, $y = -4$.




$$
\begin{aligned}
V &= \tfrac{1}{2}\pi \int_{-4}^{8} x^2\,dy, \\
&= \frac{\pi}{2}\int_{-4}^{8}\left(\frac{y^6}{4096} + \frac{y^3}{32} + 1\right) dy, \\
&= \frac{\pi}{2}\left[\frac{y^7}{7 \times 4096} + \frac{y^4}{4 \times 32} + y\right]_{-4}^{8}, \\
&= \frac{\pi}{2}\left\{\frac{512}{7} + 32 + 8 - \left(-\frac{4}{7} + 2 - 4\right)\right\}, \\
&= \frac{405\pi}{7}, \\
&\approx 181.76\,\text{cm}^3 \text{ (2 dec. pl.)}
\end{aligned}
$$

QUESTION 4

a (i) $f(x) = \sqrt{1+x^2}$

| $x$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $f(x)$ | $\sqrt{5}$ | $\sqrt{10}$ | $\sqrt{17}$ | $\sqrt{26}$ |
| | 2.24 | 3.16 | 4.12 | 5.10 |

(ii)

$A \simeq \frac{h}{2}\{f(2) + f(5) + 2[f(3) + f(4)]\}$

$h = \frac{b-a}{n} = \frac{5-2}{3} = 1$

$A = \frac{1}{2}[2.24 + 5.10 + 2(3.16 + 4.12)]$

$= 10.950$ (3 d.p.)

b (i) $P(a, 3a-9)$ $Q(1,4)$

$PQ = \sqrt{(a-1)^2 + (3a-9-4)^2}$

$PQ^2 = a^2 - 2a + 1 + 9a^2 - 78a + 169$

$PQ^2 = 10a^2 - 80a + 170$

(ii) $(PQ^2)' = 20a - 80$

turn pt. $20a = 80$, $a = 4$

$(PQ^2)'' = 20$

$(PQ^2)'' > 0$

$\therefore$ min value of $PQ = 4$

(iii) min value when line $\perp$ to PQ

$\therefore$ N is $(4, 3)$

(iv) $y - 3 = -\frac{1}{3}(x - 4)$

$3y - 9 = -x + 4$

$x + 3y - 13 = 0$

# SECTION E QUESTION 5

a) (i) $\log_5 x + \log_2 8 = 0$

$\log_5 x = -3$

$x = 5^{-3} = \frac{1}{125}$

(ii) $\log_3 x + \frac{\log_3 27}{\log_3 x} = 4$

$(\log_3 x)^2 - 4\log_3 x + 3 = 0$

$\log_3 x = 3$ or $\log_3 x = 1$

$x = 27$ or $x = 3$

b) $r_1 + r_2 = k$ —— (*)

$S = \pi r_1^2 + \pi r_2^2$

$= \pi(r_1^2 + r_2^2)$

$= \pi[r_1^2 + (k - r_1)^2]$

$S = \pi[2r_1^2 - 2kr_1 + k^2]$

$\frac{dS}{dr_1} = \pi[4r_1 - 2k] = 0$

$\Rightarrow r_1 = k/2$

$\therefore r_2 = k/2$ from *

$\frac{d^2S}{dr_1^2} = 4\pi > 0$ MIN.

(c) $V_n$ = value at end of n years (after nth prize).

(i) $V_1 = 2000(1.025)^2 - 150$

$V_1 = \$1951.25$

(ii) $V_1 = 2000(1.025)^2 - 150$

$V_2 = [2000(1.025)^2 - 150](1.025)^2 - 150$

$V_2 = 2000(1.025)^4 - 150[1 + 1.025^2]$

$V_3 = \{2000(1.025)^4 - 150[1 + 1.025^2]\}1.025^2 - 150$

$V_3 = 2000(1.025)^6 - 150[1 + 1.025^2 + 1.025^4]$

$V_3 = \$1846.77$

(iii)

$V_n = 2000(1.025)^{2n} - 150[1 + 1.025^2 + 1.025^4 + \ldots]$

let $V_n = 0$

$\frac{2000(1.025)^{2n}}{150} = \frac{1((1.025^2)^n - 1)}{0.050625}$

$0.675(1.025)^{2n} = 1.025^{2n} - 1$

$1 = [1.025^{2n}].(0.325)$

$n = -\frac{1}{2} \cdot \frac{\log(0.325)}{\log(1.025)} \doteq 22$

$\therefore$ 22 prizes

(d))

(i) $a, b, c$ A.S. $\quad a, c, b$ G.S.

$\Rightarrow b - a = c - b$

$$r = \frac{c}{a} = \frac{b}{c}$$

$$b = \frac{a+c}{2}$$

$$c^2 = ab$$

$$\Rightarrow c^2 = a\left(\frac{a+c}{2}\right)$$

$$2c^2 = a^2 + ac$$

$$2c^2 - ac - a^2 = 0$$

$(\div a^2)$ $$2\left(\frac{c}{a}\right)^2 - \left(\frac{c}{a}\right) - 1 = 0$$

$$\Rightarrow \frac{c}{a} = \frac{1 \pm \sqrt{1-4(2)(-1)}}{4}$$

Common ratio $\frac{c}{a} = 1$ or $-\frac{1}{2}$

However only $-\frac{1}{2}$ applies $\therefore$ $r = -\frac{1}{2}$

(ii) $$S = \frac{a}{1-r}$$

$$= \frac{a}{1-(-\frac{1}{2})}$$

$$S = \frac{2}{3}a$$