BAULKHAM HILLS HIGH SCHOOL

**2013**

**HSC Assessment Task 1**

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Attempt all questions

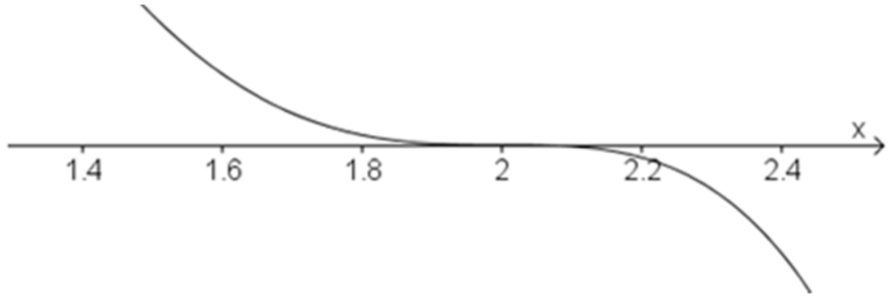
**Total marks –**

This paper consists of TWO sections.

**Section 1 – Multiple Choice**
**4 Marks:** Answer by shading the appropriate circle in your answer booklet

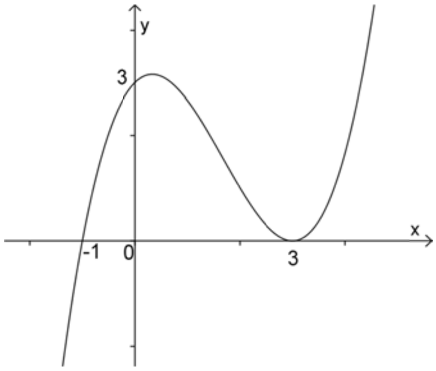
**Section 2 – Extended Response**
**33 marks**
Attempt all questions
Answer each question on the appropriate page of your answer booklet. Show all necessary working.

**Section 1 – Multiple choice (4 marks)**
**Attempt all questions.**

| | | Marks |
|---|---|---|
| 1 | If two polynomials have degrees *m* and *n* respectively, and $m<n$, what is the maximum number of points of intersection of their graphs?<br>(A) $m$ (B) $n$ (C) $m+n$ (D) $n$-$m$ | **1** |
| 2 | The polynomial $P(x) = 2x^3 - 9x^2 + 13x + k$ is divisible by $x - 2$, and $k$ is a constant. $P(x)$ is also divisible by:<br>(A) $2x - 1$ (B) $x + 1$ (C) $x - 1$ (D) $x + 2$ | **1** |
| 3 | If one root of the equation $x^3 - 5x^2 + 5x - 1 = 0$ is $2 - \sqrt{3}$ , then the sum of the other two roots is:<br>(A) $-7 + \sqrt{3}$ (B) $-1 + \sqrt{3}$ (C) $3 + \sqrt{3}$ (D) $-3 + \sqrt{3}$ | **1** |
| 4 | Part of the graph $y = P(x)$, where $P(x)$ is a polynomial of degree four is shown below.<br><br><br>Which of the following could be the polynomial $P(x)$?<br>(A) $P(x) = x(x-2)^3$ (B) $P(x) = (x-2)^3(x+3)$<br>(C) $P(x) = (2-x)^3(3-x)$ (D) $P(x) = (x-1)(x-2)^3$ | **1** |
| | **End of Section 1** | |
| | **Section 2 – Extended Response**<br>**Attempt all questions on the appropriate page of your answer booklet.**<br>**Show all necessary working.** | |
| 5 | When the polynomial $P(x) = x^3 + ax^2 - 4x + 1$ (where $a$ is a constant) is divided by $x - 1$, the remainder is 2.<br>What is the remainder with $P(x)$ is divided by $2x - 1$? | **2** |

6 Prove by Mathematical Induction that $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers $n$ **3**

7 The graph below shows all the important features of a polynomial. Find an equation for this polynomial. **3**




8 $\alpha, \beta$ and $\gamma$ are the roots of $2x^3 - 6x^2 + x - 9 = 0$. Evaluate:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ **1**

(iii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ **2**

(iv) $\alpha^2 + \beta^2 + \gamma^2$ **2**

9 Given $P(x) = 2x^3 + 7x^2 - 46x + 21$:

(i) Show that $x - 3$ is a factor of $P(x)$ **1**

(ii) Fully factorise $P(x)$ **2**

(iii) Hence or otherwise, solve: $2x^3 + 7x^2 - 46x + 21 \geq 0$ **2**

(iv) Solve the inequality:

$$\frac{2x^3+7x^2-46x+21}{x} > 0$$ **2**

10 Prove by Mathematical Induction that

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$ **4**

for all positive integers n.

11 For $\mathrm{P}(x) = x^3 + ax^2 + 2ax + b$ , where $a$ and $b$ are constants, the roots are $x = 2, -3, \gamma$. Find the value of $a, b$ and $\gamma$. **4**

12 If $P(x) = x^4 - 6x^3 + x^2 - 4x - 4$ and $A(x) = x^2 + x + 1$:

(i) Find the quotient and remainder when $P(x)$ is divided by $A(x)$ **2**

(ii) Hence find **two** possible polynomials $L(x)$ such that $x^2 + x + 1$ **i**s a factor of $P(x) + L(x)$ **2**

***End of Task***

Mult. Choice: 1 B 2 C 3 C 4 C

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**Question 5**

**BOS#:** SOLUTIONS.

$P(1) = 1 + a - 4 + 1 = 2$

$a - 2 = 2$

$a = 4$ —— 1

$P(x) = x^3 + 4x^2 - 4x + 1$

Remainder $= P\left(\frac{1}{2}\right) = \frac{1}{8} + 1 - 2 + 1$

$= \frac{1}{8}$ —— 1

**Question 6**

If $n = 1$ $\quad 3^{2n+4} - 2^{2n} = 3^6 - 2^2$

$= 729 - 4$

$= 725$ which is divisible by 5

$\therefore$ True for $n = 1$

Assume true for $n = k$.

ie Assume: $3^{2k+4} - 2^{2k} = 5m$ for some integer $m$.

Now need to prove true for $n = k + 1$:

$3^{2(k+1)+4} - 2^{2(k+1)} = 3^{2k+6} - 2^{2k+2}$

$= 3^2 \cdot 3^{2k+4} - 2^2 \cdot 2^{2k}$

$= 9(3^{2k+4}) - 4(2^{2k})$ 1

$= 9(3^{2k+4} - 2^{2k}) + 5(2^{2k})$ by assumption

$= 9(5m) + 5(2^{2k})$

$= 5(9m + 2^{2k})$

$= 5p$ where $p = 9m + 2^{2k}$ is an integer. 1

$\therefore$ If true for $n=k$, then also true for $n=k+1$

Statement is true for $n=1$

$\therefore$ Also true for $n=2,3,4 \ldots$ and by induction,

true for all positive integers $n$.

(n = 1, assump, concl.) —— 1

Mult. Choice : Working

1. The larger degree $= n$ $\therefore$ B

2. $P(2) = 16 - 36 + 26 + k = 0$ $\therefore k = -6$

$P(x) = 2x^3 - 9x^2 + 13x - 6$

$P(\frac{1}{2}) \neq 0$

$P(-1) \neq 0$

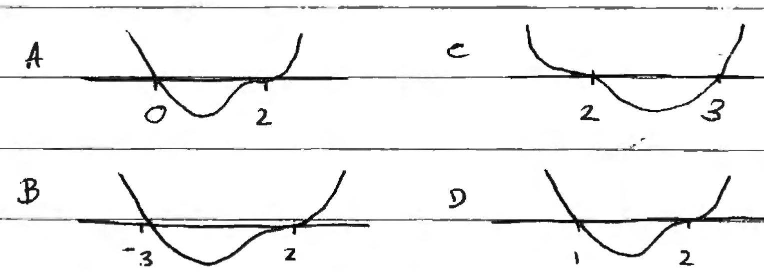
$P(1) = 0$ $\therefore$ $x-1$ is a factor

C

3. $\alpha + \beta + 2 - \sqrt{3} = 5$

$\alpha + \beta = 5 - 2 + \sqrt{3} = 3 + \sqrt{3}$ C

4.




C

You may ask for extra writing paper if you need more space to answer question 5 & 6

**Question 7** **BOS#:**______________________

$y = a(x+1)(x-3)^2$ — 1 for $(x+1)$

— 1 for $(x-3)^2$

Sub. $(0,3)$ :

$3 = a \times 1 \times (-3)^2$

$3 = 9a$

$a = 1/3$ — 1

$\therefore y = \frac{1}{3}(x+1)(x-3)^2$

**Question 8**

$2x^3 - 6x^2 + x - 9 = 0$

(i) $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{6}{2} = 3$ 1

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{1}{2}$ 1

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ 1

$= \frac{1/2}{9/2}$

$= 1/9$ 1

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 3^2 - 2\left(\frac{1}{2}\right) = 8$

You may ask for extra writing paper if you need more space to answer question 7 & 8

1 1

**Question 9** **BOS#:** ______________________

$P(x) = 2x^3 + 7x^2 - 46x + 21$

(i) $P(3) = 2(3)^3 + 7(3)^2 - 46(3) + 21$

$= 54 + 63 - 138 + 21$ — 1

$= 0 \quad \therefore \quad x - 3$ is a factor

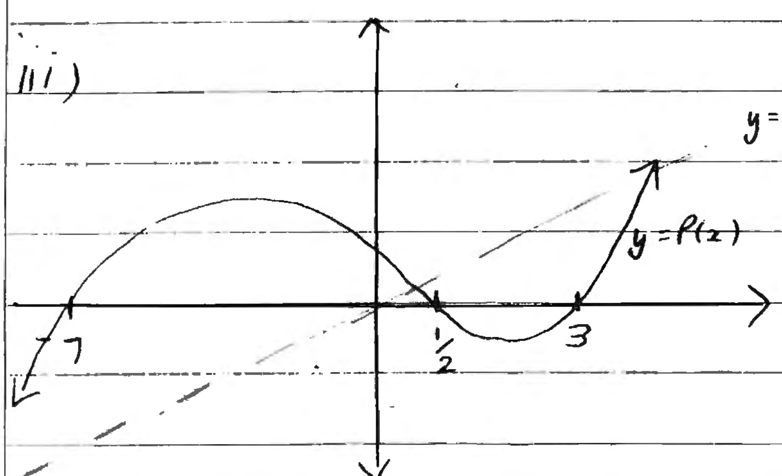
(ii) (or alternate method)

$$\begin{array}{r l} & 2x^2 + 13x - 7 \\ x-3 \,\big) & 2x^3 + 7x^2 - 46x + 21 \\ & 2x^3 - 6x^2 \\ \hline & 13x^2 - 46x \\ & 13x^2 - 39x \\ \hline & -7x + 21 \\ & -7x + 21 \\ \hline & 0 \end{array}$$

$\therefore P(x) = (x-3)(2x^2 + 13x - 7)$ — 1

$= (x-3)(2x-1)(x+7)$ — 1

(iii)




$P(x) \geq 0$ when:

$\underbrace{-7 \leq x \leq \tfrac{1}{2}}_{1}, \ \underbrace{x \geq 3}_{1}$

(iv) Fraction $> 0$ when numerator + denom. have the same sign

$\underbrace{0 < x < \tfrac{1}{2}, \ x > 3}_{\text{both pos } 1}, \ \underbrace{x < -7}_{\text{both neg } 1}$

~~1 Partly correct~~
~~2 Fully correct~~

**Question 10** **BOS#:**

If $n=1$, $LHS = \frac{1}{1.2.3} = \frac{1}{6}$

$RHS = \frac{1 \times 4}{4 \times 2 \times 3} = \frac{4}{24} = \frac{1}{6}$

$LHS = RHS$

$\therefore$ True for $n=1$

Assume true for $n=k$:

ie Assume

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

Prove true for $n=k+1$

ie Prove

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$LHS = \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \text{by assumption}$$

$$= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)}$$

Now numerator has a factor of $x+1$, since

$(-1)^3 + 6(-1)^2 + 9(-1) + 4 = -1+6-9+4 = 0$

Numerator = $(k+1)(k^2+mk+4)$ | OR EXPANSION OR LONG DIVISION

and since $mk+4k = 9k$, $m=5$

$\therefore$ Numerator $= (k+1)(k^2+5k+4)$

$= (k+1)(k+1)(k+4)$

$$LHS = \frac{(k+1)^{\cancel{2}}(k+4)}{4\cancel{(k+1)}(k+2)(k+3)} = RHS$$

[Justification of $(k+1)^2(k+4)$ needed]

$\therefore$ If true for $n=k$, then also true for $n=k$

$\therefore$ ETC

(No mark for conclusion)

**Question 11** **BOS#:**____________________

$2 - 3 + \gamma = -a$ $\qquad \gamma = 1 - a$ —— ①

$-6 - 3\gamma + 2\gamma = 2a$ $\qquad -\gamma = 2a + 6$

$\gamma = -2a - 6$ —— ②

$-6\gamma = -b$

$b = 6\gamma$ —— ③

(Any 2 eqns) 1

From ①, ② $\qquad 1 - a = -2a - 6$

$\underline{a = -7}$ —— 1

Into ① $\qquad \gamma = 1 - -7 \underline{= 8}$ —— 1

Into ③ $\qquad b = 6(8) = \underline{48}$ —— 1

**Question 12** **BOS#:**

(i)

$$\begin{array}{r} x^2 - 7x + 7 \\ x^2+x+1 \overline{) x^4 - 6x^3 + x^2 - 4x - 4} \\ \underline{x^4 + x^3 + x^2} \\ -7x^3 \quad -4x \\ \underline{-7x^3 - 7x^2 - 7x} \\ 7x^2 + 3x - 4 \\ \underline{7x^2 + 7x + 7} \\ -4x \quad -11 \end{array}$$

← quotient 1

← remainder 1

(ii) Any two expressions equivalent to

$k(x^2+x+1)+4x+11$ ($k$ = integer)

Expected answers:

$(k=0)$ $4x+11$

$(k=1)$ $x^2+5x+12$

$(k=2)$ $2x^2+6x+13$

2 (any two)

or $S(x)(x^2+x+1)+4x+11$ where $S(x)$ is a polynomial.

You may ask for extra writing paper if you need more space to answer question 12