BAULKHAM HILLS HIGH SCHOOL

2014

YEAR 12 HSC ASSESSMENT TASK 1

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 55 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions

Total marks – 36

This paper consists of 3 questions

Questions 1-3 – Extended Response
36 marks

Start a new page for each question.

Question 1 (17 marks) - Start a new page		**Marks**
a)	A polynomial $P(x) = ax^n(x-1)^2 + 5x + 7$ is monic of degree 6.	
	(i) Find the value of a.	**1**
	(ii) Find the value of n.	**1**
	(iii) What is the remainder when $P(x)$ is divided by $(x-1)$?	**1**
b)	Sketch the graph of $y = (2-x)^3(x+1)$ showing the x and y intercepts.	**2**
c)	Solve $x^3 + 3x^2 - 9x - 27 = 0$	**3**
d)	If α, β and γ are the roots of $2x^3 - 3x + 4x + 2 = 0$, find:-	
	(i) $\alpha + \beta + \gamma$	**1**
	(ii) $\alpha\beta\gamma$	**1**
	(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$	**2**
	(iv) $(\alpha - 1)(\beta - 1)(\gamma - 1)$	**2**
e)	The polynomial $P(x) = x^3 + ax^2 + bx + 4$ has $x + 2$ as a factor. When $P(x)$ is divided by $x - 1$ the remainder is 9. Find the value of a and b.	**3**

Question 2 (9 marks) - Start a new page		
a)	Use mathematical induction to prove $\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ where n is a positive integer.	**3**
b)	Two roots of the polynomial $2x^3+x^2-kx+6=0$ are equal in absolute value but opposite in sign. Find the value of k.	**3**
c)	What is the remainder when $x^{99}-99$ is divided by x^2-1 ?	**3**

Question 3 (10 marks) - Start a new page		**Marks**
a)	Use mathematical induction to prove n^3+4n is divisible by 8 if n is an even integer , where $n \geq 2$.	**4**
b)	Two of the roots of the equation $x^3+ax^2+b=0$ are reciprocals of each other.	
	(i) Show that the third root is equal to $-b$	**1**
	(ii) Show that $a=b-\frac{1}{b}$	**2**
	(iii) Show that the 2 roots which are reciprocals will be real if $-\frac{1}{2} \leq b \leq \frac{1}{2}$	**3**

End of Examination

17

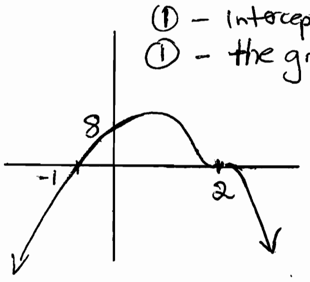
2014 DEC. EXT. 1. ASSESSMENT SOLUTIONS

1a) (i) $a = 1$ (1)

(ii) $n = 4$ (1)

(iii) $P(1) = 12$ (1)

b) (1) – Intercepts
(1) – the graph.



c) $P(x) = x^3 + 3x^2 - 9x - 27$

$P(3) = 27 + 27 - 27 - 27$

$\therefore (x-3)$ is a factor. (1)

$$\begin{array}{r} x^2 + 6x + 9 \\ x-3 \,\big)\, x^3 + 3x^2 - 9x - 27 \\ x^3 - 3x^2 \\ \hline 6x^2 - 9x \\ 6x^2 - 18x \\ \hline 9x - 27 \\ 9x - 27 \end{array}$$

$\therefore P(x) = (x-3)(x+3)^2$ (1)

$\therefore$ Solution $x = 3, -3$ (1)

d) $2x^3 - 3x^2 + 4x + 2 = 0$

(i) $\alpha + \beta + \gamma = \frac{3}{2}$ (1)

(ii) $\alpha\beta\gamma = -1$ (1)

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{2}{-1}$ (1)

$= -2$ (1)

(iv) $(\alpha-1)(\beta-1)(\gamma-1)$

$= (\alpha-1)(\beta\gamma - \beta - \gamma + 1)$

$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + \alpha + \beta + \gamma - 1$ (1)

$= -1 - 2 + \frac{3}{2} - 1$

$= -\frac{5}{2}$ (1)

e) $P(x) = x^3 + ax^2 + bx + 4$

$P(-2) = 0 \quad \therefore -8 + 4a - 2b + 4 = 0$

$4a - 2b = 4$

$2a - b = 2$ -- (1)

$P(1) = 9 \quad 1 + a + b + 4 = 9$

$a + b = 4$ - (2)

$2a - b = 2$ - (1)

$a + b = 4$ - (2)

(1)+(2) $3a = 6 \rightarrow a = 2$

$b = 2$

(1) mark for both eq'ns

(2) marks for solving for "a" and "b". (1 mark off for each error)

Question 2 [9]

a) Prove

$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1 Prove true for $n=1$

LHS $= \frac{1}{2}$ RHS $= \frac{1}{1+1} = \frac{1}{2}$

LHS = RHS $\therefore$ true for $n=1$.

(accept $\frac{1}{1.2} = \frac{1}{1+1}$)

Step 2 Assume true for $n=k$

$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ - (1)

Step 3. Prove true for $n = k+1$

ie

$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

from Assumption (1)

$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

LHS $= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

= RHS

$\therefore$ Proven true for $n=k+1$ if true for $n=k$ $\therefore$ true for $n=1, n=2$ & for all n by mathematical induction.

(1) mark for steps 1, 2 and 4 correct

(2) marks for successfully proving step 3.

2b) $2x^3 + x^2 - kx + 6 = 0$

$\alpha, -\alpha, \beta$

$\therefore \alpha + -\alpha + \beta = -\frac{1}{2}$

$\beta = -\frac{1}{2}$ -- (1)

$\alpha\beta - \alpha\beta - \alpha^2 = -\frac{k}{2}$

$\alpha^2 = \frac{k}{2}$ -- (2)

$-\alpha^2\beta = -\frac{6}{2}$

$\alpha^2\beta = 3$ -- (3)

sub (1) into (3)

$\alpha^2(-\frac{1}{2}) = 3$

$\alpha^2 = -6$

sub into (2)

$-6 = \frac{k}{2}$

$k = -12$

Alternatively sub $\beta = -\frac{1}{2}$ back into equation & solve.

Marking Scheme

(1) for the 3 equations

(2) marks for using eq'ns to solve for k.

c) $P(x) = x^{99} - 99$

$P(x) = (x^2-1)Q(x) + ax + b$

$\therefore x^{99} - 99 = (x^2-1)(Q(x)) + ax + b$

Sub in $x = 1$

$(1)^{99} - 99 = a(1) + b$

$a + b = -98$ - (1)

Sub in $x = -1$

$(-1)^{99} - 99 = -a + b$

$-a + b = -100$ - (2)

(1)+(2) $2b = -198$

$b = -99$

$a = 1$

$\therefore$ remainder is $x - 99$

(1) for $(x^2-1)Q(x) + ax + b$

(1) for equations (1) + (2)

(1) for solving successfully, but must write remainder $x - 99$.

3a) Prove n^3+4n is $\div$ by 8 if n is even

Step 1. Prove true for $n=2$

$2^3+4(2) = 16$

$\frac{16}{8} = 2$ $\therefore$ true for $n=2$.

Step 2. Assume true for $n=k$

ie $\frac{k^3+4k}{8} = M$ (where M is an integer)

ie $k^3 = 8M-4k$. —(1)

Step 3. Prove true for $n=k+2$.

ie $(k+2)^3 + 4(k+2)$ is $\div$ by 8, k is even.

$k^3+6k^2+16k+16$ is $\div$ by 8

sub in (1)

$8M-4k+6k^2+16k+16$

$= 8M+6k^2+12k+16$ —(2)

but k is even $\therefore$

let $k=2a$ where a is a positive integer

$= 8M+6(2a)^2+12(2a)+16$

$= 8M+24a^2+24a+16$

$= 8(M+3a^2+3a+2)$

and since 8 is a factor and M and a integers

$(k+2)^3+4(k+2)$ is $\div$ by 8

step 4 $\therefore$ proven true for $k+2$ if true for k $\therefore$ true for 2, 4, 6, ... & for all even n by Mathematical Induction.

(1) mark for 1, 2 and 4

Step 3 – 3 marks

(1) for substituting $k+2$.

(1) for expanding $(k+2)^3$ correctly & getting to expression (2)

(1) for substituting $k=2a$.

NB If don't state M is an integer lose 1 mark

NB If make same error in step 4 as in earlier M. I. question 2a) don't deduct marks twice.

b) $x^3+ax^2+b=0$

roots $\alpha, \frac{1}{\alpha}, \beta$

(i) $\alpha \times \frac{1}{\alpha} \times \beta = -b$

$\therefore \beta = -b$ (1) mark

(ii) $\alpha+\frac{1}{\alpha} - b = -a$

$\alpha+\frac{1}{\alpha} = b-a$ —(1)

$\alpha(\frac{1}{\alpha}) + \alpha(\beta) + \frac{\beta}{\alpha} = 0$

ie $1 - \alpha b - \frac{b}{\alpha} = 0$

$1 - b(\alpha+\frac{1}{\alpha}) = 0$ —(2) (1 mark)

sub (1) into (2)

$1 - b(b-a) = 0$

$-\frac{1}{b} = (b-a)$

$a = b - \frac{1}{b}$ (3) → 1 mark

(iii) from (1) $\alpha+\frac{1}{\alpha} = b-a$

sub in (3) $\alpha+\frac{1}{\alpha} = b-(\frac{1}{b}-b)$

$\alpha+\frac{1}{\alpha} = b-(b-\frac{1}{b})$

$\alpha+\frac{1}{\alpha} = \frac{1}{b}$

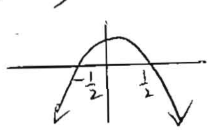
$\alpha^2 b - \alpha + b = 0$ (4)

$\therefore \alpha = \frac{1 \pm \sqrt{1-4b^2}}{2b}$

roots are real if

$1-4b^2 \geq 0$

$(1+2b)(1-2b) \geq 0$



$\therefore$ roots are real if $-\frac{1}{2} \leq b \leq \frac{1}{2}$

1 mark for sub. in (3) into (1) & getting eq'n (4)

1 mark for using quad. form. correctly.

1 mark for correct conclusion using Δ.