BAULKHAM HILLS HIGH SCHOOL

2015

YEAR 12 HSC ASSESSMENT TASK 1

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions

Total marks – 36

Questions 1-3 – Extended Response
36 marks

Answer each question on the appropriate pages of your answer booklet.

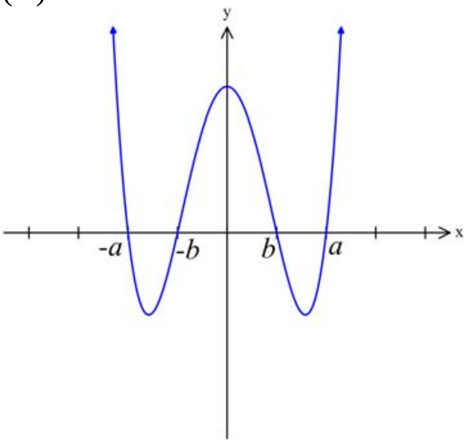
Question 1 (12 marks) - Answer on the appropriate page of your answer booklet		**Marks**
a)	If $2x - 3$ is a factor of $3x^3 + 3x + a$, find the value of a.	**1**
b)	A polynomial $P(x) = ax^n(2x-5)^2 + 3x - 4$ is monic and has degree 8.	
	(i) Find the value of a.	**1**
	(ii) Find the value of n.	**1**
	(iii) What is the remainder when $P(x)$ is divided by x?	**1**
c)	If $P(x) = 2x^3 + 11x^2 - 4$ and $A(x) = x + 1$, find the quotient and remainder when $P(x)$ is divided by $A(x)$.	**2**
d)	Without using calculus, sketch the graph of $y = (x-1)^3(3-x)$, showing the intercepts with the coordinate axes.	**3**
e)	Use mathematical induction to prove that $$\frac{2}{1\times 3} + \frac{2}{3\times 5} + \frac{2}{5\times 7} + \cdots + \frac{2}{(2n-1)(2n+1)} = 1 - \frac{1}{2n+1}$$ for all positive integers n.	**3**
	End of Question 1	

Question 2 **(12 marks) - Answer on the appropriate page of your answer booklet**		
a)	If α, β and γ are the roots of $x^3 + 2x^2 - 3x - 5 = 0$, find the value of:	
	(i) $\alpha + \beta + \gamma$	**1**
	(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$	**1**
	(iii) $\alpha\beta\gamma$	**1**
	(iv) $\frac{\gamma}{\alpha\beta} + \frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha}$	**2**
b)	Solve $2x^3 - 3x^2 - 8x + 12 = 0$.	**3**
c)	Millie is proving a statement by Mathematical Induction and has obtained the expression: $$\frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$ Which of the following expressions is equivalent to the above expression? (A) $\frac{(k+1)(k+2)(k+3)}{6}$ (B) $\frac{(k+1)(k+2)(2k+3)}{6}$ (C) $\frac{(k+1)(2k+2)(k+3)}{6}$ (D) $\frac{(2k+1)(k+2)(k+3)}{6}$	**1**
d)	Use mathematical induction to prove $3^{2n} - 1$ is divisible by 8 for all positive integers n.	**3**
	End of Question 2	

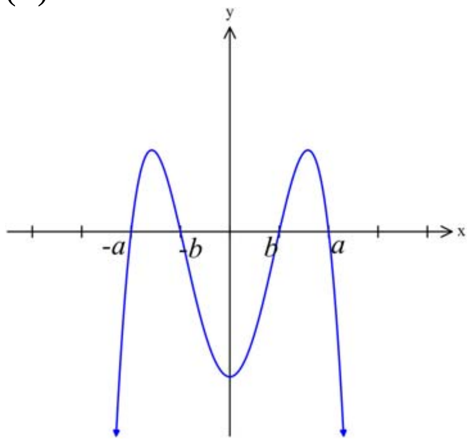
Question 3 (12 marks) - Answer on the appropriate page of your answer booklet **Marks**

a) If $a > b$, the graph of $y = (x-a)^2(b^2 - x^2)$ could be: **1**

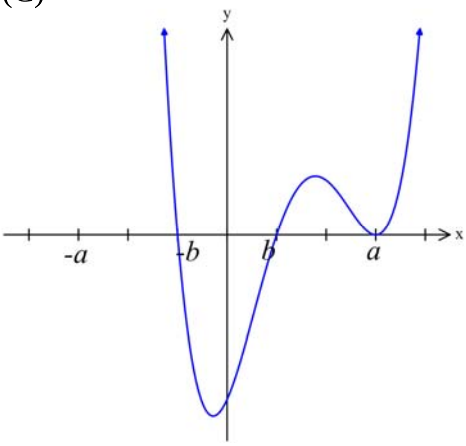
(A)



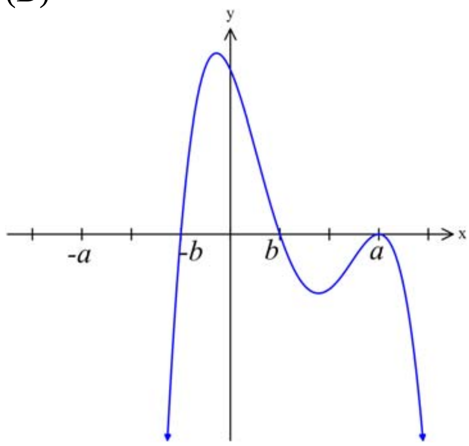
(B)



(C)



(D)



b) $P(x)$ and $Q(x)$ are two polynomials such that $P(x) = (2x^2 + x + 3)Q(x) + 4x - 1$. **3**
When $Q(x)$ is divided by $x + 2$, the remainder is 1.
Show that $x + 2$ is a factor of $P(x)$.

c) (i) Sketch the graph of $y = x^2(x-1)(x-2)$, showing the x-intercepts. **2**

(ii) Hence, or otherwise, solve the inequality: **3**

$$\frac{x^2}{(x-1)(x-2)} \leq 0$$

d) $P(x)$ is a polynomial of degree 3. **3**
Find $P(x)$, given that :

- $y = P(x)$ is an odd function.
- $(x - 2)$ is a factor of $P(x)$.
- when $P(x)$ is divided by $(x + 4)$, the remainder is 96.

End of Examination

X1 Dec 2015. Solutions.

· Induction
· Polynomials

Q1.

a) $P(\frac{3}{2}) = 0 \therefore 3\left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right) + a = 0$

$$\frac{81}{8} + \frac{9}{2} + a = 0$$

$$a = -\frac{117}{8} \quad \leftarrow (1)$$

b) (i) $a = \frac{1}{4}$ ← (1)

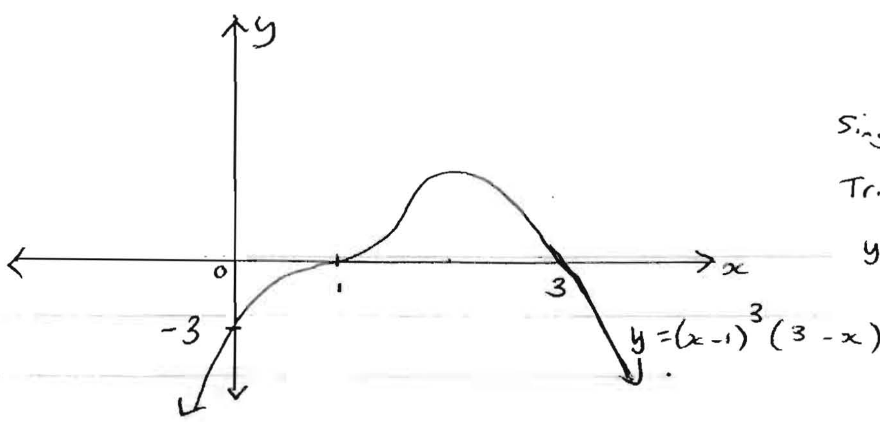
(ii) $n = 6$ ← (1)

(iii) Rem. $= P(0) = 0 + 0 - \frac{4}{\cancel{4}} = \cancel{16}\ -4$ ← (1)

c)

$$\begin{array}{r|l} & 2x^2 + 9x - 9 \quad \leftarrow \text{quotient (1)} \\ \hline x+1 & 2x^3 + 11x^2 \qquad -4 \\ & 2x^3 + 2x^2 \\ \hline & 9x^2 \\ & 9x^2 + 9x \\ \hline & -9x - 4 \\ & -9x - 9 \\ \hline & 5 \quad \leftarrow \text{remainder (1)} \end{array}$$

d)



Single root $x=3$ (1)

Triple root $x=1$ (1)

y-int (1)

[-1 if ends incorrect]

e) If $n=1$: $LHS = \frac{2}{(2-1)(2+1)} = \frac{2}{1\times3} = \frac{2}{3}$

$RHS = 1 - \frac{1}{2+1} = 1 - \frac{1}{3} = \frac{2}{3}$

$LHS = RHS \therefore$ True for $n=1$ (1)

Assume true for $n=k$

ie. Assume

$$\frac{2}{1\times3}+\frac{2}{3\times5}+\frac{2}{5\times7}+\dots+\frac{2}{(2k-1)(2k+1)}=1-\frac{1}{2k+1}$$

Now prove true for $n=k+1$

ie. Prove

$$\underbrace{\frac{2}{1\times3}+\frac{2}{3\times5}+\frac{2}{5\times7}+\dots+\frac{2}{(2k-1)(2k+1)}}+\frac{2}{(2k+1)(2k+3)}=1-\frac{1}{2k+3}$$

$$LHS = 1-\frac{1}{2k+1}+\frac{2}{(2k+1)(2k+3)}$$ by assumption (1)

$$= 1-\left(\frac{1}{2k+1}-\frac{2}{(2k+1)(2k+3)}\right)$$

$$= 1-\frac{2k+3-2}{(2k+1)(2k+3)}$$

$$= 1-\frac{\cancel{2k+1}^{\,1}}{\cancel{(2k+1)}(2k+3)}$$

$$= 1-\frac{1}{2k+3}$$

$= RHS$ ∴ If true for $n=k$, then also true for $n=k+1$ (1)

Now statement is true for $n=1$

∴ Also true for $n=2,3,4\dots$ and by induction true for all positive integers n.

Q2.

a) i) $\alpha+\beta+\gamma=-2$ — (1)

ii) $\alpha\beta+\beta\gamma+\gamma\alpha=-3$ — (1)

iii) $\alpha\beta\gamma=5$ — (1)

iv) $$\frac{\gamma}{\alpha\beta}+\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}=\frac{\gamma^2+\alpha^2+\beta^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$$ — (1)

$$=\frac{(-2)^2-2(-3)}{5}$$

$$=2.$$ — (1)

b) $2x^3-3x^2-8x+12=0$

$P(1)\neq 0$

$P(-1)\neq 0$

$P(2)=16-12-16+12=0$ $\therefore$ $x-2$ is a factor $\longleftarrow$ (1)

$P(x)=(x-2)(2x^2+x-6)$ $\longleftarrow$ by long div. (1)
or otherwise

$=(x-2)(2x-3)(x+2)$

$=0$ when $\underline{x=2,-2,\frac{3}{2}}$ $\longleftarrow$ (1)

ALT:

$P(x)=x^2(2x-3)-4(2x-3)$ $\longleftarrow$ (1)

$=(x^2-4)(2x-3)$

$=(x-2)(x+2)(2x-3)$ $\longleftarrow$ (1)

$\therefore P(x)=0$ when $\underline{x=2,-2,3/2}$ $\longleftarrow$ (1)

c) $\cancel{\frac{1}{6}k(}$ $\frac{1}{6}(k+1)\left(k(2k+1)+6(k+1)\right)$

$$= \frac{1}{6}(k+1)(2k^2+7k+6)$$

$$= \frac{1}{6}(k+1)(2k+3)(k+2) \quad \therefore \text{ B} \quad \longleftarrow (1)$$

d) If $n=1$ $\quad 3^{2n}-1 = 3^2-1 = 8$ which is divisible by 8

$\therefore$ True for $n=1$

Assume true for $n=k$

ie. Assume $3^{2k}-1 = 8m$ where m is an integer.

Now prove true for $n=k+1$

ie. Prove $3^{2(k+1)}-1 = 8p$ where p is an integer.

Now $3^{2(k+1)}-1 = 3^{2k+2}-1$

$$= 3^2 . 3^{2k} - 1$$

$$= 9(3^{2k}) - 1$$

$$= 9(8m+1) - 1 \quad \text{by assumption}$$

$$= 8(9m) + 9 - 1$$

$$= 8(9m) + 8$$

$$= 8(9m+1)$$

$$= 8p \quad \text{where } p = 9m+1 \text{ is an integer}$$

$\therefore$ If true for $n=k$, then also true for $n=k+1$.

Now statement is true for $n=1$

$\therefore$ Also true for $n=2,3,4\ldots$ and by induction true for all positive integers n.

Q3.

a) Double root at $x=a$
Single root at $x=b, -b$.
Ends: $\swarrow \quad \searrow$

D $\leftarrow$ (1)

b) $Q(x) = (x+2).S(x) + 1 \leftarrow$ (1) by the division transformation where $S(x)$ is a polynomial

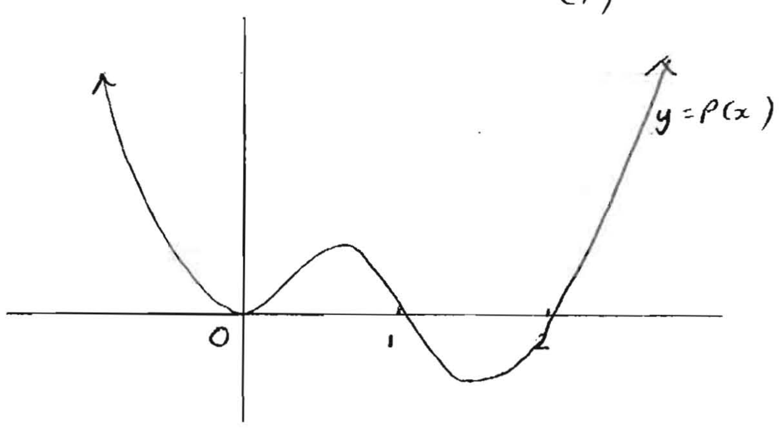
$P(x) = (2x^2+x+3)((x+2).S(x)+1) + 4x - 1 \leftarrow$ (1)

$P(-2) = (8-2+3)(0.S(-2)+1) + 4(-2) - 1$

$= 9 \times 1 - 8 - 1$

$= 0 \quad \therefore \; x+2$ is a factor of $P(x)$ $\leftarrow$ (1)

c) (i)



Single roots at $x = 1, 2$ (1)

Double root at $x = 0$ (1)

(ii) Sign of $\dfrac{x^2}{(x-1)(x-2)}$ = Sign of $P(x)$ except $x \neq 1, 2$

$\dfrac{x^2}{(x-1)(x-2)} \leq 0$ when $\underline{x = 0}$ (1), $\underline{1 < x < 2}$ (2)

[$1 \leq x \leq 2$ is only 1 mark]

d) $P(x) = ax^3 + bx$

$P(2) = 0 \quad : \quad 8a + 2b = 0$

$4a + b = 0$ —— ①

$P(-4) = 96 \quad -64a - 4b = 96$

$16a + b = -24$ —— ②

← one or both (1)

Solve simultaneously

$12a = -24 \qquad a = -2$

$-8 + b = 0 \qquad b = 8$

← one or both (1)

$\therefore P(x) = -2x^3 + 8x$ ← (1)