2016 Year 12
HSC Assessment Task 1
December

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 50 minutes
- Write using non erasable black or blue pen
- Board-approved calculators may be used
- Show all necessary working
- Marks may be deducted for careless or badly arranged work

Total marks – 33

Questions 1-3 (pages 2-4)

Answer each question on the appropriate pages of your answer booklet.

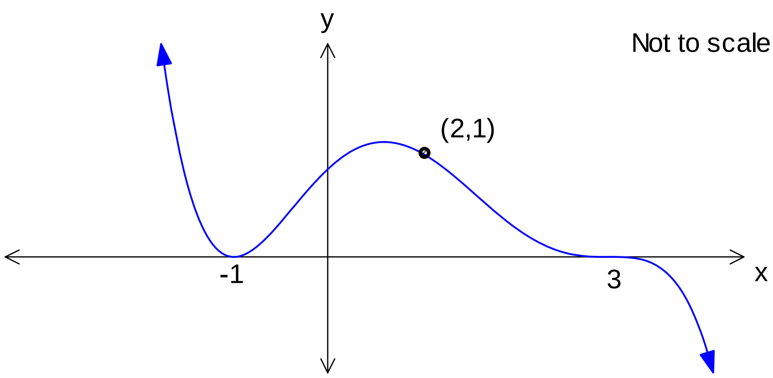
Question 1 (11 marks) - Use the Question 1 section of the writing booklet.		Marks
a)	Find the value of $\boldsymbol{k}$ if $(\boldsymbol{x-2})$ is a factor of $\boldsymbol{P(x) = x^3 - kx + 6}$	**1**
b)	If α, β and γ are the roots of the equation $2x^3 - 3x - 4 = 0$ Find the value of	
	(i) $\alpha + \beta + \gamma$	**1**
	(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$	**1**
	(iii) $\alpha\beta\gamma$	**1**
	(iv) $\alpha^2 + \beta^2 + \gamma^2$	**2**
c)	Given the polynomials $f(x) = 2x^4 - 10x^3 + 12x^2 + 2x - 3$ and $g(x) = x^2 - 2x + 1$. Find expressions for $A(x)$ and $R(x)$ if $f(x) = g(x)A(x) + R(x)$.	**2**
d)	For all positive integers of n, use mathematical induction to prove $$\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$$	**3**

Question 2 **(10 marks) - Use the Question 2 section of the writing booklet.** **Marks**

a) When the polynomial $P(x)$ is divided by $x^2 - 4$ the remainder is $3x - 1$. **2**

What is the remainder when $P(x)$ is divided by $x + 2$

b) Write an equation for the polynomial function below. **2**



c) One of the roots of the equation $x^3 + kx^2 + 1 = 0$ is equal to the sum of the other two roots.

(i) Show that $x = -\frac{k}{2}$ is a root of the equation. **2**

(ii) Find the value of k. **2**

d) Show $(x - 1)(x - n)$ is a factor of **2**

$$P(x) = x^{2n} - x^n + n^n - (xn)^n$$

where n is a positive integer.

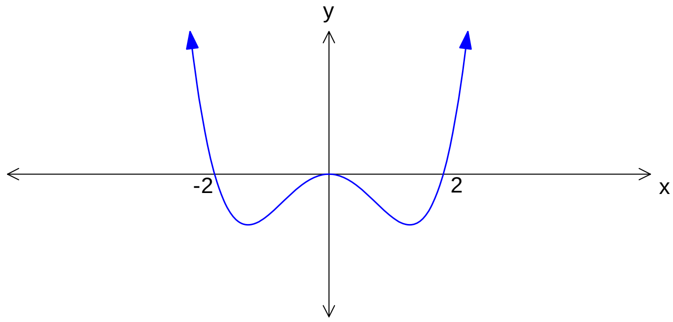
Question 3 (12 marks) - Use the Question 3 section of the writing booklet.		Marks
a)	An odd polynomial $P(x)$ of smallest degree has double root at $x = 2$ and the gradient of the tangent at $x = 1$ is -6. Find an equation for $P(x)$.	**3**
b)	(i) Sketch the polynomial $P(x) = x^4 - 4x^2$	**2**
	(ii) Show that $P(x)$has stationary points at (0,0), $\left(-\sqrt{2}, -4\right)$ and $\left(\sqrt{2}, -4\right)$.	**2**
	(iii) Hence, or otherwise, find all the values of k such that $P(x) + k = 0$ will have exactly two solutions.	**2**
c)	Use mathematical induction for prove $n^3 - n$ is divisible by 6 for all positive integers of n.	**3**
	End of the Exam	

BAULKHAM HILLS HIGH SCHOOL - EXTENSION 1
2016 HSC Task 1 SOLUTIONS

	Solution	Mks	Comments
	QUESTION 1		
1a.	$2^3 - k(2) + 6 = 0$ $k = 7$	**1**	**1 mark** • Correct substitution.
1b.(i)	$\alpha + \beta + \gamma = 0$	**1**	**1 mark** • Correct answer.
1b.(ii)	$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{2}$	**1**	**1 mark** • Correct answer.
1b.(iii)	$\alpha\beta\gamma = 2$	**1**	**1 mark** • Correct answer.
1b.(iv)	$(\alpha+\beta+\gamma)^2 = (\alpha+\beta)^2 + 2\gamma(\alpha+\beta) + \gamma^2$ $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma+\gamma^2$ $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $= (0)^2 - 2\left(-\frac{3}{2}\right)$ $= 3$	**2**	**2 marks** • Correct solution **1 mark** • Significant progress to finding an expression for $\alpha^2+\beta^2+\gamma^2$
1 c.	$2x^4 - 10x^3 + 12x^2 + 2x - 3 = (x^2 - 2x + 1)(2x^2 - 6x - 2) + 4x - 1$ $A(x) = 2x^2 - 6x - 2$ $R(x) = 4x - 1$	**2**	**2 marks** • Correct expressions **1 mark** • Significant progress towards solution.
1d	Test for $n = 1$ $LHS = \frac{1}{(1+1)(1+2)} = \frac{1}{6}$ $RHS = \frac{1}{2(1+2)} = \frac{1}{6}$ $\therefore$ true for $n = 1$ Assume true for $n = k$ $ie. \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$ Prove true for $n = k+1$ Aim: $\frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$ $LHS = \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)}$ $= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} \times \frac{2}{2}$ if the assumption is true $= \frac{1}{2(k+2)}\left[\frac{k(k+3)}{(k+3)} + \frac{2}{(k+3)}\right]$ $= \frac{1}{2(k+2)}\left[\frac{k(k+3)+2}{(k+3)}\right]$ $= \frac{1}{2(k+2)}\left[\frac{k^2+3k+2}{(k+3)}\right]$ $= \frac{1}{2(k+2)}\left[\frac{(k+2)(k+1)}{(k+3)}\right]$ $= \frac{k+1}{2(k+3)}$ If it is true for $n = k$, then it is also true for $n = k+1$. By Induction, since it is true for $n = 1$, then it is also true for $n = 2, 3, 4\ldots$ all positive integer of n	**3**	There are 4 key parts of the induction; 1. Proving the result true for $n = 1$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement **3 marks** • Successfully does all of the 4 key parts **2 marks** • Successfully does 3 of the 4 key parts **1 marks** • Successfully does 2 of the 4 key parts

Solution		Mks	Comments
QUESTION 2			
2a.	$P(x) = (x^2 - 4)Q(x) + 3x - 1$ $P(x) = (x+2)(x-2)Q(x) + 3x - 1$ $P(-2) = (-2+2)(-2-2)Q(x) + 3(-2) - 1$ $P(-2) = -7$ $ie.\ R(-2) = 3(-2) - 1 = -7$	**2**	**2 mark** • Correct solution. **1 mark** • Significant attempts to use the remainder theorem
2b.	$P(x) = a(x+1)^2(x-3)^3$ At (2,1) $1 = a \times 2^2(2-3)^3$ $a = -\frac{1}{9}$ $P(x) = -\frac{1}{9}x^2(x-3)^3$	**2**	**2 marks** • Correct solution **1 mark** • Correctly identifies $P(x) = (x+1)^2(x-3)^3$
2c (i)	Let the roots be α, β and$\alpha + \beta$ Sum of roots: $\alpha + \beta + (\alpha + \beta) = -k$ $2(\alpha + \beta) = -k$ $(\alpha + \beta) = -\frac{k}{2}$ Since $\alpha + \beta$ is a root, then $-\frac{k}{2}$ is a root	**2**	**2 marks** • Correct solution **1 mark** • Correctly identifies sum of roots $= -a$ • Identify the third root as $\alpha + \beta$
2c.(ii)	Since $-\frac{k}{2}$ is a root, then $\left(-\frac{k}{2}\right)^3 + k\left(-\frac{k}{2}\right)^2 + 1 = 0$ $-\frac{k^3}{8} + \frac{k^3}{4} = -1$ $k^3 = -8$ $k = -2$	**2**	**2 marks** • Correct solution **1 mark** • Correctly identifies $-\frac{k}{2}$ is a root
2d	$P(1) = 1^{2n} - 1^n + n^n - (1n)^n$ $= 1 - 1 + n^n - n^n$ $= 0$ $\therefore (x-1)$ is a factor $P(n) = n^{2n} - n^n + n^n + (n \times n)^n$ $= n^{2n} - n^n + n^n + (n^2)^n$ $= 0$ $\therefore (x-n)$ is a factor $\therefore (x-1)(x-n)$ is a factor of $P(x)$	**2**	**2 marks** • Correct solution **1 mark** • Correctly proves one factor.

QUESTION 3

	Solution	Mks	Comments
3a.	If $P(x)$ is odd, it must pass through the origin. Since $P(x)$ has a double root at $x = 2$, then $(x-2)^2$ is a factor. Since $P(x)$ is odd, $(x+2)^2$ is also a factor. $\therefore P(x) = ax(x-2)^2(x+2)^2$ If the gradient of the tangent -6 at $x = 1$ then $P'(1) = -6$ $P(x) = ax(x^2-4)^2$ $P(x) = ax(x^4 - 8x^2 + 16)$ $P(x) = ax^5 - 8ax^3 + 16ax$ $P'(x) = 5ax^4 - 24ax^2 + 16a$ $P'(1) = 5a - 24a + 16a$ $-6 = -3a$ $a = 2$ $\therefore P(x) = 2x(x-2)^2(x+2)^2$	**3**	1. -Recognises it passes the origin. ie. a factor of x. -Recognises $(x-2)^2(x+2)^2$ is a factor. -Recognises a factor of a. 2. Finds the derivative 3. Uses derivative to solve for a **3 marks** • Correct Solution. **2 mark** • Successfully does 2 parts. **1 mark** • Successfully does 1 parts.
3b.(i)		**2**	**2 marks** • Correct answer. **1 mark** • Correct graph without any intercepts.
3b (ii)	$P'(x) = 4x^3 - 8x$ Stationary point when $P'(x) = 0$ $0 = 4x(x^2-2)$ $x = 0, \sqrt{2}, -\sqrt{2}$ $P(0) = 0$ $P(\sqrt{2}) = \sqrt{2}^4 - 4\sqrt{2}^2$ $= -4$ $P(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2$ $= -4$	**2**	**2 marks** • Correct solution **1 mark** • Finds all stationary points without *y*-values
3b.(iii)	$k < 0$ or $k = 4$	**2**	**2 marks** • Correct solution **1 mark** • One region correct.

Solution		Mks	Comments
3c	<u>Test for $n = 1$</u> $LHS = 1^3 - 1$ $= 0$ Since 0 is divisible by 6 $\therefore$ true for $n = 1$ <u>Assume true for $n = k$</u> $ie. k^3 - k = 6P$ where P is a positive integer Prove true for $n = k + 1$ Aim: $(k+1)^3 - (k+1) = 6Q$ where P is a positive integer $LHS = (k+1)^3 - (k+1)$ $= k^3 + 3k^2 + 3k + 1 - k - 1$ $= k^3 + 3k^2 + 2(k^3 - 6P)$ if the assumption is true $= 3k^3 + 3k^2 - 12P$ $= 3k^2(k+1) - 12P$ $\therefore k(k+1) = 2M$ since k and $(k+1)$are consecutive number, one of them must be even. $LHS = 3k \times 2M - 12P$ $= 6(km - 2P)$ where $(km - 2P)$ is an integer <u>$\therefore (k+1)^3 - (k+1)$ is divisible by 6.</u> If it is true for $n = k$, then it is also true for $n = k + 1$. By Induction, since it is true for $n = 1$, then it is also true for $n = 2, 3, 4 \ldots$ all positive integer of n	**3**	There are 4 key parts of the induction; 1. Proving the result true for n = 1 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement (must explain why $k(k+1) = 2M$) **3 marks** • Successfully does all of the 4 key parts **2 marks** • Successfully does 3 of the 4 key parts **1 marks** • Successfully does 2 of the 4 key parts Note: Without using the assumption, it is not the process of mathematical induction