Caringbah High School

2011 Year 12
Semester 1 Examination

Mathematics Extension 1

General Instructions

- All questions should be attempted
- Start each 'Question' on a new page
- Answers without appropriate working and/or diagrams may not attract full marks
- Approved silent calculators may be used

Time Allowed

1½ hours + 5 minutes reading time

Question 1 (13 marks)

MARKS

a) Convert an angle of 1.85 radians to degrees. 1

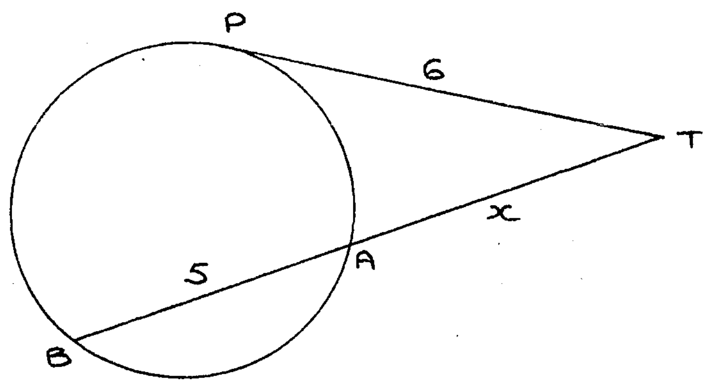
b) If α, β, γ are the roots of the equation: $x^3 - 5x^2 + 2x - 1 = 0$ find the value of:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha\beta\gamma$ 1

(iii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$ 1

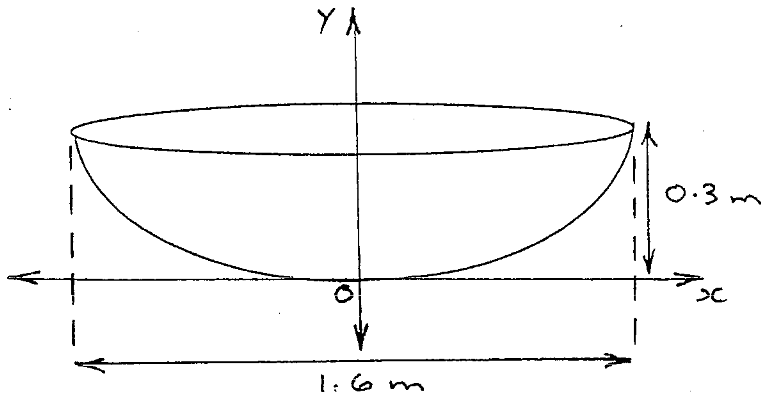
c) 2



PT is a tangent and TAB is a secant.
Find the value of x.

d) Find $\int x \,.\, e^{-x^2}\, dx$ 1

e) A satellite dish in the shape of a paraboloid has its vertex at the origin. Its diameter is 1.6 metres and its depth 0.3 metre as shown. 3



Find the equation of the parabola required, in the form $x^2 = 4ay$. Hence find the satellite dish's focal length 'a'.

f) Find $\int \frac{1+5x^2-x^3}{x^2}\,dx$ 2

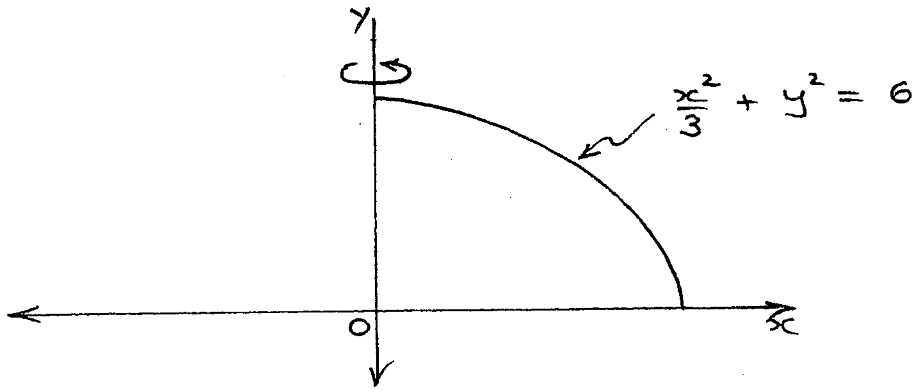
g) What is the exact value of $\sec\frac{\pi}{6}$? 1

Question 2 (13 marks)

a) Perform the long division: $(x^3 - 2) \div (x + 1)$ 2

writing your answer in the form $A(x) + \frac{R}{x+1}$

b) 4

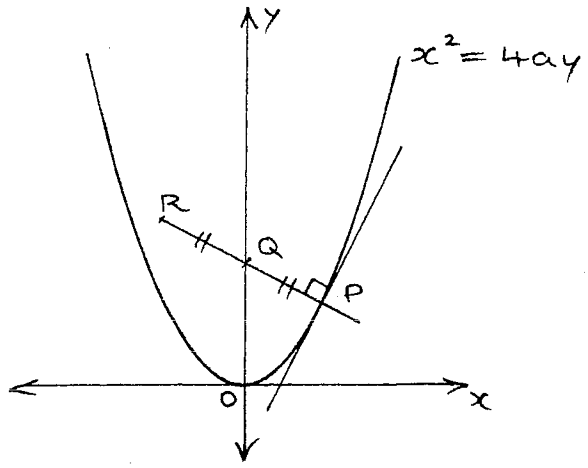


That part of the curve $\frac{x^2}{3} + y^2 = 6$ that lies in the first quadrant is rotated about the y axis. Find the exact volume of the solid of revolution.

c) Given that $y = e^{kx}$ is a solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ 3

find the values of k.

d) The normal at the point $P(2at, at^2)$ on $x^2 = 4ay$ cuts the y axis at Q, and is so produced to R such that PQ = QR.



(i) You are given that the normal at P has the equation $x + ty - at(t^2 + 2) = 0$ (you are not required to prove this). Find the co-ordinates of R in terms of t. 2

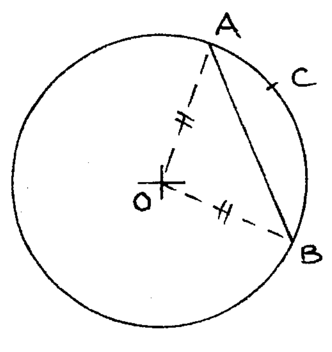
(ii) Find the cartesian locus of R. 2

Question 3 (13 marks)

a) A function $y = f(x)$ had just one stationary point, at $x = 3$. Using the table of results shown below, what is the nature of the stationary point? 1

x	2	3	4
$f'(x)$	−0.1	0	−5

b)



AB is a chord of length 10cm, in a circle of radius 6cm.

(i) Find <AOB in radians (correct to 3 significant figures). 2

(ii) Find the perimeter of segment ABC. 1

(iii) Find the area of segment ABC 1

c) Solve the equation: $x^3 - 3x^2 - 4x + 12 = 0$ given that the sum of two of its roots is zero. 2

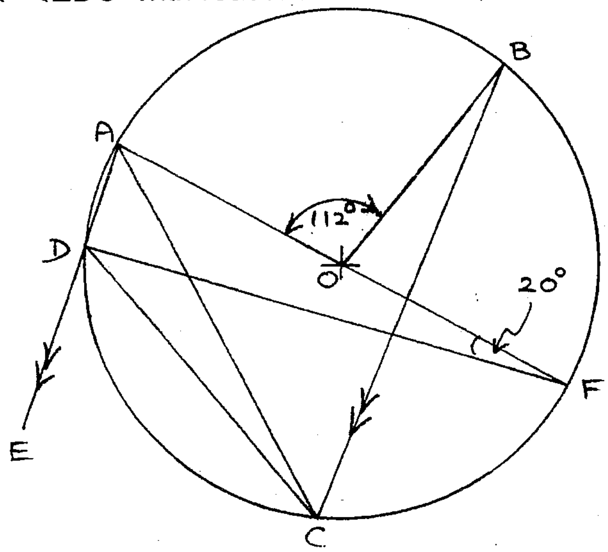
d) P is the point $(2at, at^2)$ on the parabola $x^2 = 4ay$ S is the focus. PR is drawn parallel to the parabola's axis, meeting the directrix in R. Prove that RS is parallel to the normal at P. 3

e) Starting with an initial value of $x = 1$, use one application of Newton's Method to find a better approximation to the root of the equation: $2x - \ln(x + 3) = 0$. 3

Question 4 (13 marks)

a) i) Copy the diagram below into your examination booklet.

ii) 0 is the centre of the circle and AF is a diameter. Find <EDC with reasons. 3

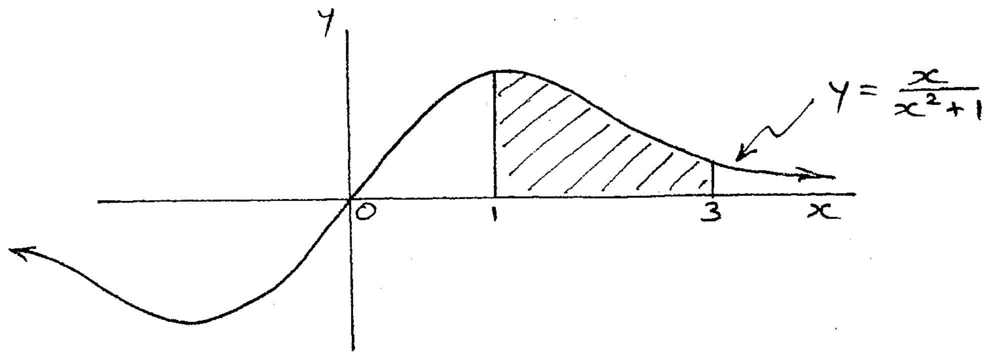


b) Solve for x correct to three significant figures: $2^x = 40$ 2

c) Find $\int \frac{dx}{\sqrt[3]{2x+1}}$ 2

d) Solve for x: $2 \ln x = \ln(x + 6)$ 3

e) Calculate the shaded area shown. 3



Question 5 (13 marks)

For the curve with equation $y = \frac{x+1}{(x-1)^2}$

(i) What is its natural domain? 1

(ii) Show that $\frac{dy}{dx} = \frac{-x-3}{(x-1)^3}$ 2

(iii) Find the one stationary point this curve has, and determine its nature. 3

(iv) You are given its second derivative as 2

$$\frac{d^2y}{dx^2} = \frac{2x+10}{(x-1)^4}$$

Prove that there is a point of inflexion at $x = -5$.

(v) What happens to the value of y as x approaches infinity? 1

(vi) Draw a neat half page sketch of the curve showing all relevant information. 3

(vii) Determine the range of the above function. 1

------- END OF PAPER -------

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1 a) $106°$

b) i) $\frac{-(-5)}{(1)} = 5$

ii) $\frac{-(-1)}{(1)} = 1$

iii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}$

$= \frac{5}{1}$

$= 5$

c) $x(x+5) = 6^2$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0$

$x = -9, 4$

$\therefore x = 4$ only since $x > 0$

d) $I = -\frac{1}{2}\int -2x . e^{-x^2} dx$

$= -\frac{1}{2} e^{-x^2} + C$

e) Curve passes thru' $(0.8, 0.3)$

$\therefore (0.8)^2 = 4a(0.3)$

$0.64 = 1.2a$

$a = 0.5\dot{3}$ is focal length

and the curve's equation is

$x^2 = 2.1\dot{3}y$

f) $I = \int (x^{-2} + 5 - x)\, dx$

$= \frac{x^{-1}}{-1} + 5x - \frac{x^2}{2} + C$

$= -\frac{1}{x} + 5x - \frac{x^2}{2} + C$

g) $\frac{2}{\sqrt{3}}$ (or $\frac{2\sqrt{3}}{3}$)

Q2 a)

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x+1 & x^3 \qquad\qquad -2 \\ & x^3 + x^2 \\ \hline & -x^2 \qquad -2 \\ & -x^2 - x \\ \hline & x - 2 \\ & x + 1 \\ \hline & -3 \end{array}$$

$\therefore \frac{x^3 - 2}{x+1} = x^2 - x + 1 - \frac{3}{x+1}$

b) $x^2 = 18 - 3y^2$

Also, when $x = 0$, $y = \sqrt{6}$

$\therefore V = \pi\int_0^{\sqrt{6}} (18 - 3y^2)\, dy$

$= \pi\left[18y - y^3\right]_0^{\sqrt{6}}$

$= \pi\{[18\sqrt{6} - 6\sqrt{6}] - [0]\}$

$= 12\pi\sqrt{6}$ (u^3)

c) $y = e^{kx}$ $y' = ke^{kx}$ $y'' = k^2e^{kx}$

$\therefore k^2e^{kx} - 4ke^{kx} + 3e^{kx} = 0$

$e^{kx}(k^2 - 4k + 3) = 0$

$e^{kx} \neq 0$ $\therefore (k-3)(k-1) = 0$

$k = 1, 3$

d) i) Let $x = 0$

$\therefore ty - at(t^2+2) = 0$

$y = a(t^2+2)$

$\therefore Q$ is $(0, at^2 + 2a)$

Let R be (x, y)

$\therefore \frac{x + 2at}{2} = 0 \rightarrow x = -2at$

$\frac{y + at^2}{2} = at^2 + 2a$

$y + at^2 = 2at^2 + 4a$

$y = at^2 + 4a$

$\therefore R$ is $(-2at, at^2 + 4a)$

ii) $\begin{cases} x = -2at & \text{--- (1)} \\ y = at^2 + 4a & \text{--- (2)} \end{cases}$

From ① $t = \frac{-x}{2a}$

Sub. into ② $y = a\left(\frac{-x}{2a}\right)^2 + 4a$

ie. $y = \frac{x^2}{4a} + 4a$ is the cartesian locus of R.

Q3. a) ✗ → ✗

Hence a stationary inflexion is present at $x = 3$.

b) i) $\cos\theta = \frac{6^2+6^2-10^2}{2\times6\times6}$

$= \frac{-7}{18}$

$\theta = \cos^{-1}\left(\frac{-7}{18}\right)$ (in radian mode)

$\doteq 1.97$ (3sf)

ii) $P = 6\times1.97 + 10$

$= 21.8$ cm

iii) $A = \frac{1}{2}\times6^2(1.97 - \sin 1.97)$

$\doteq 18.875$ cm^2

c) $\alpha+\beta+\gamma = \frac{-(-3)}{(1)}$

$\therefore 0 + \gamma = 3$

$\gamma = 3$

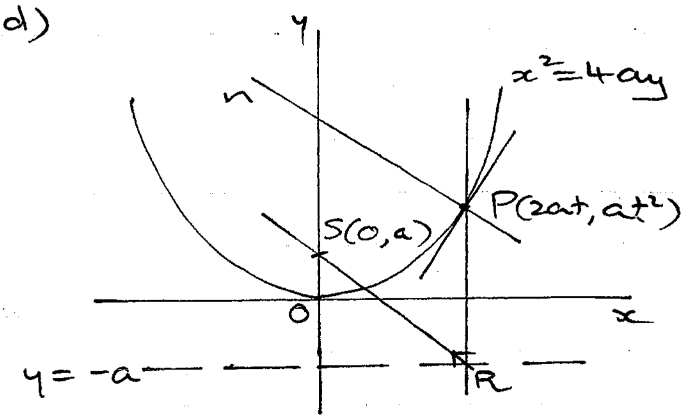
Also, $\alpha\beta\gamma = \frac{-(12)}{(1)}$

$3\alpha\beta = -12$

$\alpha\beta = -4$

$\therefore \alpha = 2, \beta = -2, \gamma = 3$

d)



$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = 2at \times \frac{1}{2a} = t$

$\therefore m_n = -\frac{1}{t}$

R has coord's $(2at, -a)$

$\therefore m_{RS} = \frac{a - -a}{0 - 2at}$

$= \frac{2a}{-2at}$

$= -\frac{1}{t}$

$= m_n$

$\therefore$ RS || normal at P.

e) Let $f(x) = 2x - \ln(x+3)$

$\therefore f'(x) = 2 - \frac{1}{x+3}$

$a_1 = 1 - \frac{[2(1) - \ln(1+3)]}{[2 - \frac{1}{1+3}]}$

$= 1 - \frac{2 - \ln 4}{1\frac{3}{4}}$

$\doteq 0.649$

Q4

a) i) —

ii) $\angle ACD = 20°$

($\angle$'s on common arc AD)

$\angle ACB = \frac{1}{2}\times112°$

$= 56°$

($\angle$ at circ. $\frac{1}{2}\times\angle$ at centre, on common arc AB)

$\angle DCB = 20° + 56°$
$= 76°$
(addition of adj. ∠'s)
$\therefore \angle EDC = 76°$
(alt. ∠'s in // lines)

b) $x = \log_2 40$
$= \frac{\ln 40}{\ln 2}$
$= 5.32192...$
$\doteqdot 5.32$

c) $I = \int (2x+1)^{-\frac{1}{3}} dx$
$= \frac{(2x+1)^{2/3}}{2 \times \frac{2}{3}} + C$
$= \frac{3}{4} \cdot \sqrt[3]{(2x+1)^2} + C$

d) $\ln x^2 = \ln(x+6)$
$\therefore x^2 = x+6$
$x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$
$\therefore x = 3$ only
($x > 0$ in the expression $2\ln x$)

e) $A = \int_1^3 \frac{x}{x^2+1} dx$
$= \frac{1}{2}\int_1^3 \frac{2x}{x^2+1} dx$
$= \left[\frac{1}{2}\ln(x^2+1)\right]_1^3$
$= \frac{1}{2}[\ln 10 - \ln 2]$
$= \frac{1}{2}\ln\left(\frac{10}{2}\right)$
$= \frac{1}{2}\ln 5 \quad (u^2)$.

Q5. $y = \frac{x+1}{(x-1)^2}$

i) $\mathbb{D} = \{\text{all } x \neq 1\}$

ii) $u = x+1 \qquad v = (x-1)^2$
$u' = 1 \qquad v' = 2(x-1)^1.1$
$= 2(x-1)$

$y' = \frac{(x-1)^2.1 - (x+1).2(x-1)}{[(x-1)^2]^2}$
$= \frac{(x-1)[(x-1) - 2(x+1)]}{(x-1)^4}$
$= \frac{x-1-2x-2}{(x-1)^3}$
$= \frac{-x-3}{(x-1)^3}$ as req'd

iii) let $y' = 0$
$\therefore -x-3 = 0$
$x = -3$
$\rightarrow y = \frac{(-3)+1}{((-3)-1)^2} = \frac{-1}{8}$

x	-4	-3	-2
y'	$-$	0	$+$

$\therefore (-3, -\frac{1}{8})$ is a local min.

iv) let $y'' = 0$ for a possible inflex^n.
$\therefore 2x + 10 = 0$
$x = -5$

x	-6	-5	-4
y''	-0.0008	0	0.0032
	⌢	.	⌣

$\therefore$ there is a change in concavity
$\therefore$ an inflexion exists at $x = -5$

(also, $y = \frac{-4}{(-6)^2} = \frac{-1}{9}$)

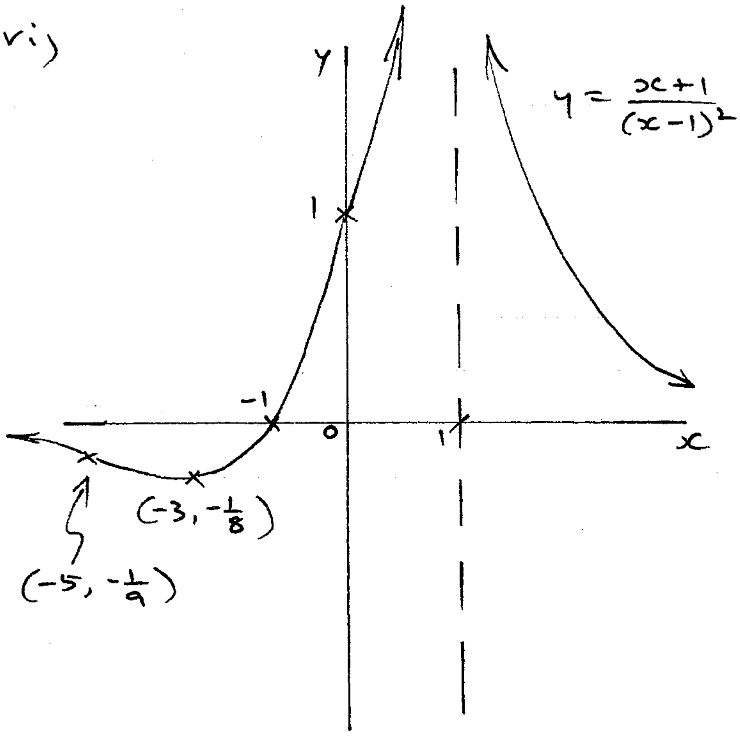
v) As $x \to \infty$, $y \to 0$

(note that $\frac{x+1}{(x-1)^2}$ becomes

$$\frac{x+1}{x^2-2x+1} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = \frac{0}{1} = 0$$

as $x \to \infty$)

vi)



vii) $R = \{ y \geq -\frac{1}{8} \}$