FINAL MARK

GIRRAWEEN HIGH SCHOOL
Mathematics Extension 1
HSC ASSESSMENT Task 1
ANSWERS COVER SHEET

Name: ____

QUESTION	MARK	HE2	HE3	HE4	HE5	HE6	HE7
Q1 - Q5	/5						✓
Q6	/10						✓
Q7	/18						✓
Q8	/13	✓					✓
Q9	/13						✓
Q10	/13						✓
TOTAL							
	/72	/13					/72

HSC Outcomes **Mathematics Extension 1**

HE2 uses inductive reasoning in the construction of proofs.

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion and exponential growth and decay.

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE6 determines integrals by reduction to a standard form through a given substitution.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

GIRRAWEEN HIGH SCHOOL

HSC TASK 1

YEAR 11

2015

MATHEMATICS EXTENSION 1

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions. Write using ***Blue*** or ***Black*** pen only.
- Board-approved calculators may be used.
- All necessary working should be shown in Questions 6 - 10.
  Marks may be deducted for careless or badly arranged work.
- For Questions 1 - 5, circle the letter corresponding to the correct answer in your answer booklet.
  For Questions 6 – 10, each question is to be returned on a *separate* piece of paper clearly marked Question 6, Question 7, etc.
- You may ask for extra pieces of paper if you need them.

Multiple Choice (5 marks)

Write the letter corresponding to the correct answer in your answer booklet.

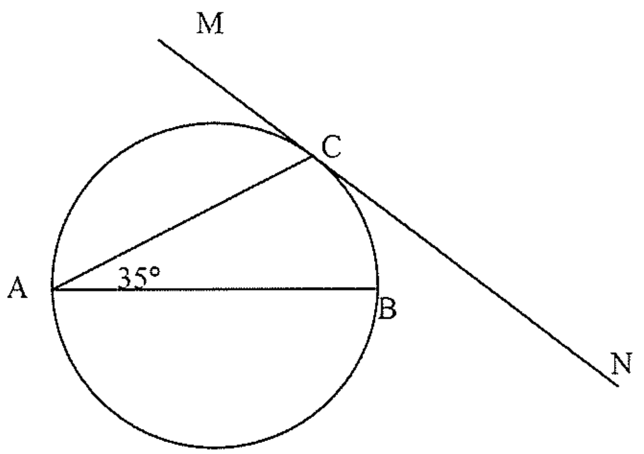
1. What is the sixth term in the expansion of $(2x - 3y)^9$?

   A. ${}^9C_3 \times 2^6 \times (-3)^3 x^6 y^3$
   B. ${}^9C_4 \times 2^5 \times (-3)^4 x^5 y^4$
   C. ${}^9C_5 \times 2^4 \times (-3)^5 x^4 y^5$
   D. ${}^9C_6 \times 2^3 \times (-3)^6 x^3 y^6$

2. What is the coefficient of x^5 in the expansion of $(1-3x+2x^3)(1-2x)^6$?

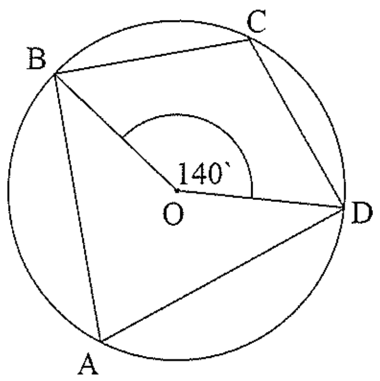
   A. -792 B. -720
   C. 120 D. 312

3. In the diagram, *AB* is a diameter of the circle and *MCN* is the tangent to the circle at *C*. $\angle CAB = 35°$. What is the size of $\angle MCA$?



   A. 35° B. 45° C. 55° D. 65°

4.



ABCD is a cyclic quadrilateral inscribed in a circle with centre O such that $\angle BOD = 140°$. What is the size of $\angle BCD$?

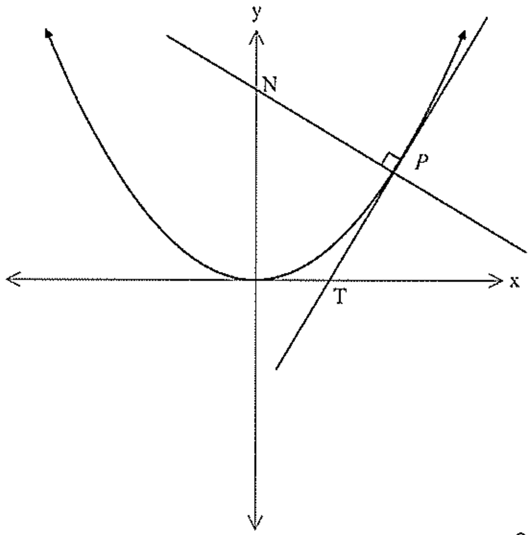
A. 100°   B. 110°   C. 120°   D. 130°

5. Which of the following is an expression for $\frac{1}{(n-1)!} + \frac{n^3+1}{(n+1)!}$

A. $\frac{n+1}{n!}$ B. $\frac{n^2+1}{n!}$ D. $\frac{n^2+n+1}{n!}$ D. $\frac{n^3+n^2+1}{n!}$

Question 6 (10 marks)

a. The diagram shows the parabola $x^2 = 4ay$

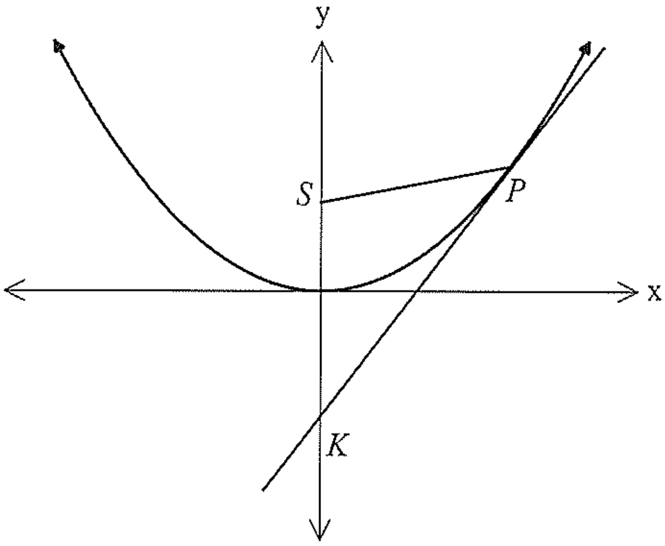


The tangent to the parabola at $P(2ap, ap^2)$ cuts the x - axis at T and the normal at P cuts the y - axis at N.
The equation of the tangent is given by $y = px - ap^2$ and the equation of the normal is given by $x + py = 2ap + ap^3$.

(i) Show that the coordinates of N are $\left(0, a(p^2+2)\right)$. [2]

(ii) Find the locus of M, the midpoint of NT. [4]

b.

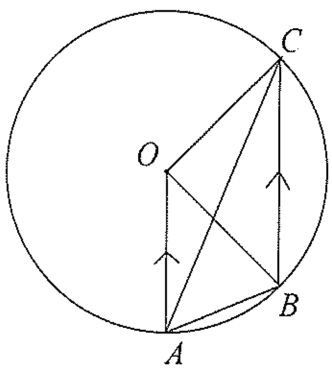


The equation of the tangent to the parabola
$x^2 = 4ay$ at $P\left(2ap, ap^2\right)$ is given by $y = px - ap^2$.

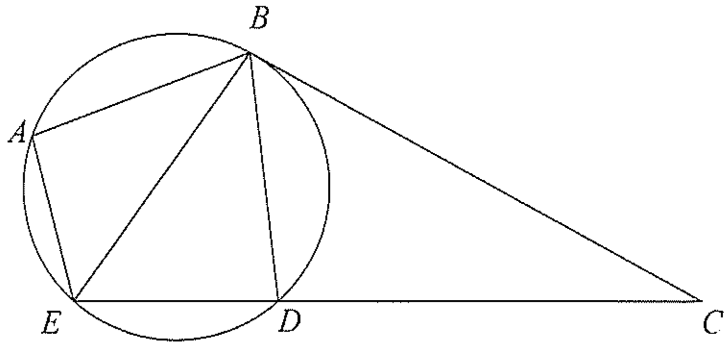
Prove that the tangent at P is equally inclined to the focal chord through P and the axis of the parabola i.e. prove that $\angle SPK = \angle SKP$ [4]

Question 7 (18 marks)

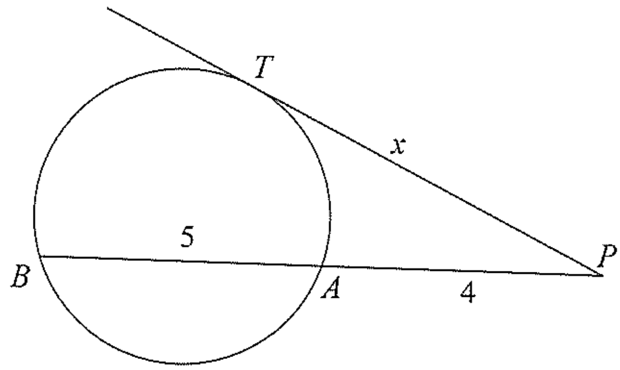
a. In the diagram O is the centre of the circle. A, B and C are points on the circle such that $AO//BC$ and $\angle OAC = 25°$.
Find the size of $\angle BOA$, giving reasons. [2]



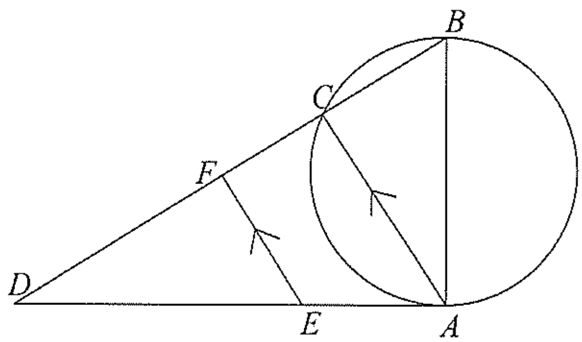
b. In the diagram, the tangent to the circle at *B* meets *ED* produced at *C*.
$\angle BAE = 105°$ and $\angle CBD = 50°$.
Find the value of $\angle EBD$, giving reasons. [3]



c. In the diagram, PT is a tangent and PB is a secant.
Given that $AB = 5$ cm and $AP = 4$ cm, find the value of x. [2]



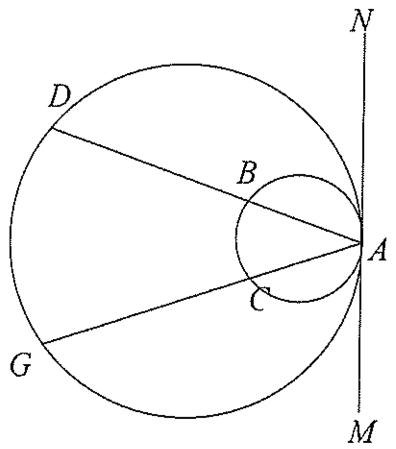
d.



AB is a diameter of the circle and *C* is a point on the circle.
The tangent to the circle at *A* meets *BC* produced at *D*.
E is a point on *AD* and *F* is a point on *CD* such that *EF* is parallel to *AC*.

(i) Copy the diagram.
(ii) Explain why $\angle EAC = \angle ABC$. [1]
(iii) Hence, show that *EABF* is a cyclic quadrilateral. [3]
(iv) Show that BE is a diameter of the circle through *E*, *A*, *B* and *F*. [2]

e.



Two circles touch each other internally at A.
MAN is the common tangent to the circles at A.
ABD and ACG are two straight lines which cut the smaller circle at B and C and the larger circle at D and G.

(i) Copy the diagram.

(ii) Show that $\Delta\, ABC$ is similar to $\Delta\, ADG$. [3]

(iii) Hence, show that $\dfrac{AB}{AD} \times \dfrac{AC}{AG} = \dfrac{BC^2}{DG^2}$ [2]

Question 8 (13 marks)

a. Use Mathematical Induction to prove that:

$1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$ for all $n \geq 1$ [5]

b. Use Mathematical Induction to prove that:

$5^{2n} - 2^{3n}$ is divisible by 17 for all $n \geq 1$ [4]

c. Prove by Mathematical Induction that for all positive integers

$5^n \geq 1 - 4n + 8n^2$ [4]

Question 9 (13 marks)

a. Expand and simplify: $(2y^2 - x)^5$. [3]

b. Find the term in x^{-2} in the expansion of $\left(2x + \frac{1}{x^2}\right)^{10}$. [3]

c. Find the coefficient of x^4 in the expansion of the product $(2 + 3x^2)\left(x - \frac{2}{x}\right)^6$. [3]

d. One of the terms in the expansion of $\left(4x^3 + \frac{1}{2x^2}\right)^{15}$ is 40 040. [4]

Find the next term.

Question 10 (13 marks)

a. For the expansion of $(1 + 2x)^{10}$, find:

(i) the ratio $\frac{T_{k+1}}{T_k}$, showing all necessary working. [4]

(ii) the greatest coefficient. [2]

(iii) the greatest term when $x = \frac{1}{2}$. [3]

b. In the expansion of $(2 + 3x)^n$, when $x = \frac{1}{2}$, the ratio of the 5th term to the 4th term is 9:8. Find the value of n. [4]

End of Examination

YEAR 11 HSC TASK 1 EXTENSION 1 MATHEMATICS

SOLUTIONS

MC

1. $T_6 = {}^9C_5 (2x)^4 (-3y)^5$ [C]

2. $(1-3x+2x^3)(1-2x)^6$

Terms in x^5

$= {}^6C_5(-2x)^5 + (-3x)\,{}^6C_4(-2x)^4 + 2x^3\,{}^6C_2(-2x)^2$

coefficient $= -792$ [A]

3. $\angle ACB = 90°$ ($\angle$ in a semi-circle)

$\angle ABC = 55°$ ($\angle$ sum of Δ)

$\therefore \angle MCA = 55°$ ($\angle$ in the alternate segment) [C]

4. Reflex $\angle BOD = 220°$ ($\angle$s at a point)

$\therefore \angle BCD = 110°$ ($\angle$ at the centre is twice the $\angle$ at the circumference on the same arc) [B]

5. $\frac{1}{(n-1)!} + \frac{n^3+1}{(n+1)!}$

$= \frac{n(n+1)+n^3+1}{(n+1)!}$

$= \frac{n^3+n^2+n+1}{(n+1)!}$

$= \frac{n^2(n+1)+1(n+1)}{(n+1)!}$

$= \frac{(n^2+1)(n+1)}{(n+1)!}$ [B]

$= \frac{n^2+1}{n!}$

Question 6 (10 marks)

a. (i) N is on the normal & the y-axis $\Rightarrow x=0$

$\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore N(0, a(p^2+2))$ (2)

ii) T is on the tangent & x-axis $\Rightarrow y=0$

$\therefore px - ap^2 = 0$

$x = ap$

$\therefore T(ap, 0)$

$M_{NT} = \left(\frac{ap}{2}, \frac{a(p^2+2)}{2}\right)$

$x = \frac{ap}{2}$; $y = \frac{a(p^2+2)}{2}$

$p = \frac{2x}{a}$

Substitute $p = \frac{2x}{a}$ into y

$y = \frac{a}{2}\left(\left(\frac{2x}{a}\right)^2+2\right)$

$= \frac{a}{2}\left(\frac{4x^2}{a^2}+2\right)$

Locus of M : $y = \frac{2x^2}{a} + a$ (4)

b. $y = px - ap^2$; $S(0,a)$

K : when $x=0$, $y = -ap^2$

$\therefore SK = a + ap^2$

Also, $SP = \sqrt{(2ap)^2 + (ap^2-a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= \sqrt{(a+ap^2)^2}$

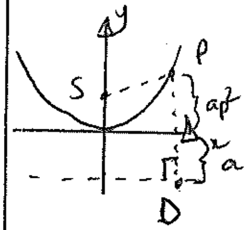
$= a + ap^2$

$\therefore SK = SP$

$\therefore \Delta SPK$ is isosceles

$\therefore \angle SPK = \angle SKP$ ($\angle$s opposite = sides of isosceles Δ) (4)

6b.



(OR)

SP = SD (by definition of the parabola)

$= ap^2 = a$

$= \ldots$

Question 7 (18 marks)

a. i) $\angle BCA = 25°$ (alternate $\angle$s, AO || BC)

$\therefore \angle BOA = 50°$ ($\angle$ at the centre is twice the $\angle$ at the circumference on arc AB) (2)

b. $\angle BED = 50°$ ($\angle$ in the alternate segment)

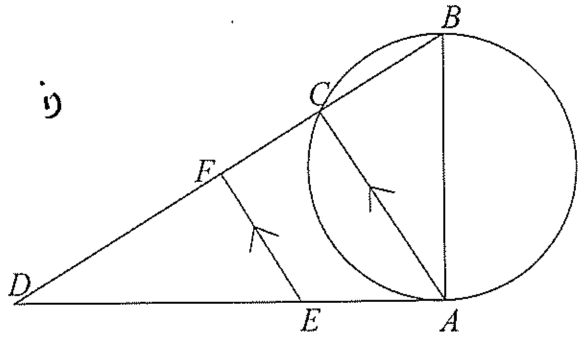
$\angle BDE = 75°$ (opposite $\angle$s of a cyclic quadrilateral)

$\therefore \angle EBD = 55°$ ($\angle$ sum of ΔBDE) (3)

c. $x^2 = 4 \times 9$ (square of tangent = product of intercepts of secants)

$\therefore x = 6$ cm (2)

d. i)



ii) $\angle EAC = \angle ABC$ ($\angle$ in the alternate segment) (1)

iii) $\angle EAC = \angle DEF$ (corresponding $\angle$s, AC || EF)

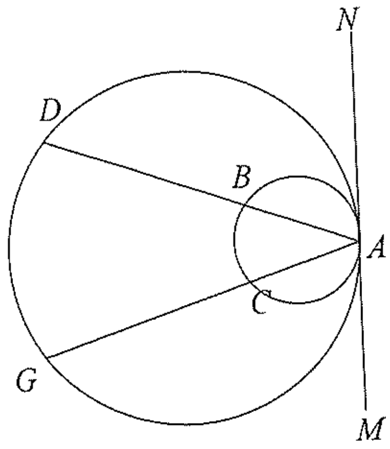
$\therefore \angle DEF = \angle ABC$ (both equal to $\angle EAC$)

$\therefore$ EABF is a cyclic quadrilateral (exterior $\angle$ is equal to interior opposite $\angle$) (3)

iv) $\angle BAE = 90°$ (tangent $\perp$ to radius at point of contact)

$\therefore$ BE is a diameter (subtends right $\angle$ at circumference of circle EABF) (2)

e. i)



ii) $\angle NAB = \angle ACB$ ($\angle$ in the alternate segment)

$\angle NAB = \angle AGD$ ($\angle$ in the alternate segment)

In Δs ABC and ADG

$\angle BAC$ is common

$\angle ACB = \angle AGD$ (both equal to $\angle NAB$)

$\therefore \Delta ABC ||| \Delta ADG$ (equiangular) (3)

iii) $\dfrac{AB}{AD} = \dfrac{AC}{AG} = \dfrac{BC}{DG}$ (ratio of matching sides of ||| Δs)

$\therefore \dfrac{AB}{AD} \times \dfrac{AC}{AG} = \dfrac{BC}{DG} \times \dfrac{BC}{DG}$

$\dfrac{AB}{AD} \times \dfrac{AC}{AG} = \dfrac{BC^2}{DG^2}$ (2)

Question 8 (13 marks)

a. $1\times3 + 2\times4 + 3\times5 + \dots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$ $n \geq 1$

Step 1 : Show true for $n=1$

$LHS = 1\times3 = 3$

$RHS = \frac{1}{6}(2)(9)$

$= 3$

$LHS = RHS$

$\therefore$ true for $n=1$

Step 2 : Assume true for $n=k$

ie. $1\times3 + 2\times4 + \dots + k(k+2) = \frac{k}{6}(k+1)(2k+7)$

Step 3 : Prove true for $n=k+1$

ie. $1\times3 + 2\times4 + \dots k(k+2) + (k+1)(k+3) = \frac{k+1}{6}(k+2)(2k+9)$

$LHS = 1\times3 + 2\times4 + \dots + k(k+2) + (k+1)(k+3)$

$= \frac{k}{6}(k+1)(2k+7) + (k+1)(k+3)$ [from step 2]

$= \frac{k+1}{6}\left[k(2k+7) + 6(k+3)\right]$

$= \frac{k+1}{6}(2k^2 + 13k + 18)$

$= \frac{k+1}{6}(k+2)(2k+9)$

$= RHS$

$\therefore$ If the result is true for $n=k$, then also true for $n=k+1$

Step 4 : By the principle of Mathematical Induction, the result is true for all $n \geq 1$

(5)

[1 mark for correct setting-out]

b. $5^{2n} - 2^{3n}$ divisible by 17 $n \geq 1$

Step 1 : Show true for $n=1$

$5^{2(1)} - 2^{3(1)}$

$= 25 - 8$

$= 17$ which is divisible by 17

$\therefore$ true for $n=1$

Step 2 : Assume true for $n=k$

ie. $5^{2k} - 2^{3k} = 17p$ for some integer p

$5^{2k} = 17p + 2^{3k}$

Step 3 : Prove true for $n=k+1$

ie. $5^{2(k+1)} - 2^{3(k+1)} = 17q$ for some integer q

$LHS = 5^{2(k+1)} - 2^{3(k+1)}$

$= 5^2 \cdot 5^{2k} - 2^3 \cdot 2^{3k}$

$= 5^2(17p + 2^{3k}) - 8 \cdot 2^{3k}$ [from step 2]

$= 25 \cdot 17p + 25 \cdot 2^{3k} - 8 \cdot 2^{3k}$

$= 25 \cdot 17p + 17 \cdot 2^{3k}$

$= 17(25p + 2^{3k})$

$= 17q$ where $q = 25p + 2^{3k}$

$= RHS$.

$\therefore$ If the result is true for $n=k$, then also true for $n=k+1$

Step 4 : By the principle of mathematical Induction, the result is true for all $n \geq 1$

(4)

Q8 (cont'd)

c) $5^n \geq 1 - 4n + 8n^2$

Step 1: Show true for $n=1$

$LHS = 5^1 = 5$

$RHS = 1 - 4(1) + 8(1)^2 = 5$

$LHS \geq RHS$

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$

ie. $5^k \geq 1 - 4k + 8k^2$

Step 3: Prove true for $n=k+1$

ie. $5^{k+1} \geq 1 - 4(k+1) + 8(k+1)^2$

$\Rightarrow 5^{k+1} \geq 5 + 8k^2 + 12k$

$LHS = 5^{k+1}$

$\geq 5(1 - 4k + 8k^2)$ [from step 2]

$= 5 - 20k + 40k^2$

$= 5 + 8k^2 + 12k - 32k + 32k^2$

$= 5 + 8k^2 + 12k + 32k(k-1)$

$\geq 5 + 8k^2 + 12k$

$\geq RHS$. (since $32k(k-1) \geq 0$ for $k \geq 1$)

$\therefore$ If the result is true for $n=k$, then also true for $n=k+1$.

Step 4: By the principle of Mathematical Induction, the result is true for all $n \geq 1$

(4)

Question 9 (13 marks)

a. $(2y^2 - x)^5 =$

${}^5C_0(2y^2)^5 + {}^5C_1(2y^2)^4(-x) + {}^5C_2(2y^2)^3(-x)^2 + {}^5C_3(2y^2)^2(-x)^3 + {}^5C_4(2y^2)(-x)^4 + (-x)^5$

$= 32y^{10} - 80y^8x + 80y^6x^2 - 40y^4x^3 + 10y^2x^4 - x^5$

(3)

b. $\left(2x + \frac{1}{x^2}\right)^{10}$; term in x^{-2}

$T_{k+1} = {}^{10}C_k(2x)^{10-k}(x^{-2})^k$

$= {}^{10}C_k\, 2^{10-k} \cdot x^{10-k} \cdot x^{-2k}$

$= {}^{10}C_k\, 2^{10-k} \cdot x^{10-3k}$

For the term in x^{-2},

$10 - 3k = -2$

$k = 4$

$\therefore T_5 = {}^{10}C_4(2x)^6(x^{-2})^4$

Term in $x^{-2} = {}^{10}C_4\, 2^6 x^{-2}$

(3)

c. $(2 + 3x^2)\left(x - \frac{2}{x}\right)^6$; x^4

$= (2+3x^2)\left[{}^6C_0x^6 + {}^6C_1x^5\left(\frac{-2}{x}\right) + {}^6C_2x^4\left(\frac{-2}{x}\right)^2 + {}^6C_3x^3\left(\frac{-2}{x}\right)^3 \ldots\right.$

$= (2+3x^2)\left[x^6 - 12x^4 + 60x^2 + \ldots\right.$

Terms in x^4:

$= -24x^4 + 180x^4$

$= 156x^4$

Coefficient of $x^4 = 156$

(3)

Question 9 (Con't)

d. $\left(4x^3+\frac{1}{2x^2}\right)^{15}$; 40040 ⇒ independent of x

$T_{k+1} = {}^{15}C_k\,(4x^3)^{15-k}\,(2x^2)^{-k}$

$= {}^{15}C_k\,2^{30-2k}\,.\,x^{45-3k}\,.\,2^{-k}\,x^{-2k}$

$= {}^{15}C_k\,2^{30-3k}\,.\,x^{45-5k}$

For the term independent of x

$45-5k=0$

$k=9$

$\therefore T_{10} = {}^{15}C_9\,2^3 = 40\,040$

$\therefore T_{11} = {}^{15}C_{10}\,2^0\,x^{-5}$

$= {}^{15}C_{10}\cdot\frac{1}{x^5}$ (4)

Question 10 (13 marks)

a. $(1+2x)^{10}$

i) $T_{k+1} = {}^{10}C_k\,(2x)^k$

$T_k = {}^{10}C_{k-1}\,(2x)^{k-1}$

$\frac{T_{k+1}}{T_k} = \frac{{}^{10}C_k\,(2x)^k}{{}^{10}C_{k-1}\,(2x)^{k-1}}$

$= \frac{{}^{10}C_k}{{}^{10}C_{k-1}}\,.\,2x$

$= \frac{10!}{k!\,(10-k)!}\times\frac{(k-1)!\,(11-k)!}{10!}\,.\,2x$

$= \frac{11-k}{k}\,.\,2x$

$\frac{T_{k+1}}{T_k} = \frac{2(11-k)}{k}\,.\,x$ (4)

ii) Greatest coefficient ⇒ $\frac{T_{k+1}}{T_k} > 1$

$22-2k > k$

$k < 7\frac{1}{3}$

$\therefore k = 7$

$\therefore$ Greatest term, $T_8 = {}^{10}C_7\,(2x)^7$

Greatest coefficient : ${}^{10}C_7\,2^7$ (2)

iii) Greatest term when $x=\frac{1}{2}$

when $x=\frac{1}{2}$, $\frac{T_{k+1}}{T_k} = \frac{2(11-k)}{k}\cdot\frac{1}{2}$

$= \frac{11-k}{k}$

Greatest term when

$\frac{11-k}{k} > 1$

$11-k > k$

$k < 5\frac{1}{2}$

$\therefore k = 5$

Greatest term,

$T_6 = {}^{10}C_5\,(2\cdot\frac{1}{2})^5$

$= {}^{10}C_5$ (3)

Question 10 (Con't)

b. $(2+3x)^n$

When $x=\frac{1}{2}$, $T_5 : T_4 = 9:8$

$T_5 = {}^nC_4 (2)^{n-4} (3x)^4$; $T_4 = {}^nC_3 (2)^{n-3} (3x)^3$

When $x = \frac{1}{2}$

$T_5 = {}^nC_4\, 2^{n-4} \left(\frac{3}{2}\right)^4$; $T_4 = {}^nC_3\, 2^{n-3} \left(\frac{3}{2}\right)^3$

$$\frac{T_5}{T_4} = \frac{{}^nC_4\, 2^{n-4} \left(\frac{3}{2}\right)^4}{{}^nC_3\, 2^{n-3} \left(\frac{3}{2}\right)^3} = \frac{9}{8}$$

$$\frac{{}^nC_4}{{}^nC_3} \cdot \frac{3/2}{2} = \frac{9}{8}$$

$$\frac{n!}{4!(n-4)!} \times \frac{3!(n-3)!}{n!} \cdot \frac{3}{4} = \frac{9}{8}$$

$$\frac{n-3}{4} \times \frac{3}{4} = \frac{9}{8}$$

$$n-3 = 6$$

$$\therefore n = 9$$

(4)