GIRRAWEEN HIGH SCHOOL

HSC TASK 1

YEAR 11

2016

MATHEMATICS EXTENSION 1

Time Allowed: 90 minutes

Instructions:

- Attempt ALL questions. Write using **blue** or **black** pen only.
- Board-approved calculators may be used.
- For Questions 1 – 5, circle the letter corresponding to the correct answer in your answer booklet. For Questions 6 – 10, start each question on a new page in your answer booklet.
- All necessary working should be shown in Questions 6 – 10.
- Marks may be deducted for careless or badly arranged work.
- You may ask for extra pieces of paper if you need them.

Multiple Choice (5 marks)

Write the letter corresponding to the correct answer in your answer booklet.

1. What is the coefficient of x^4 in the expansion of $(1-3x+2x^3)(1-2x)^5$

(A) - 408 **(B)** 546 **(C)** 300 **(D)** -792

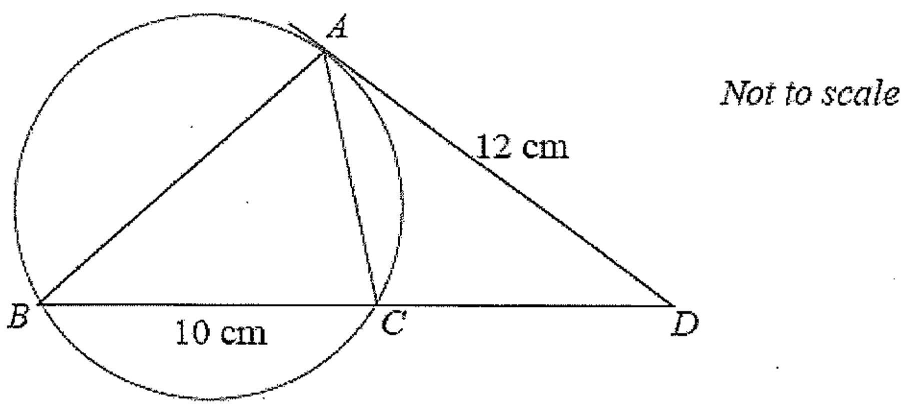
2. Write as a single fraction: $\frac{1}{(n-1)!}+\frac{1}{n!}-\frac{1}{(n+1)!}$

(A) $\frac{n}{(n-1)!(n+1)}$ **(B)** $\frac{n+2}{(n-1)!(n+1)}$ **(C)** $\frac{n+2}{(n+1)!(n+1)}$ **(D)** $\frac{n}{(n+1)!(n+1)}$

3. What is the term independent of x in the expansion of $\left(x^2-\frac{2}{x}\right)^9$?

(A) ${}^9C_6\times2^6$ **(B)** $-{}^9C_6\times2^6$ **(C)** ${}^9C_3\times2^3$ **(D)** $-{}^9C_3\times2^3$

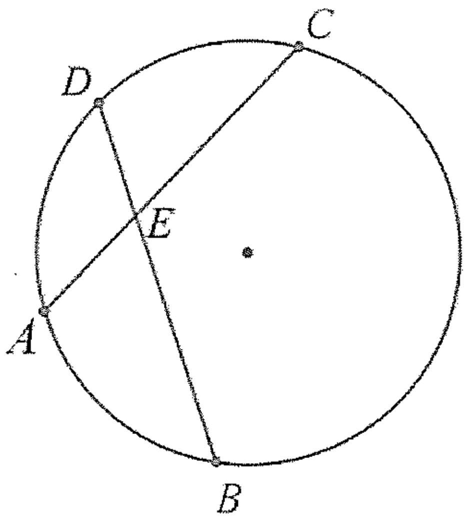
4.



ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced at D where $BC = 10$ cm and $AD = 12$ cm. What is the length of CD?

(A) 6 cm **(B)** 7 cm **(C)** 8 cm **(D)** 9 cm

5. In the diagram below, chords AC and BD of the circle intersect at E. $AC = 9$ units, $CE = m$ units, $DE = x$ units and $EB = y$ units.



NOT TO SCALE

Which of the following statements is true?

(A) $xy = 9m$ **(B)** $\dfrac{x}{y} = \dfrac{9-m}{m}$

(C) $x(x+y) = m(9-m)$ **(D)** $xy = m(9-m)$

Question 6 (15 marks) Marks

(a) Expand

(i) $\left(1+\dfrac{2x}{5}\right)^5$ **(ii)** $\left(x^2-2y^3\right)^7$ 3+4

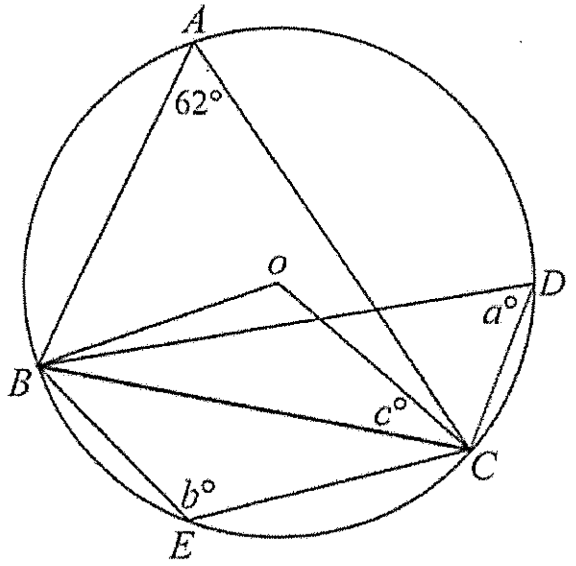
(b) Find the term independent of x^9 in the expansion of $\left(5x^2-\dfrac{1}{x}\right)^{18}$ 3

(c) Find the value of k in the expansion of $(1+kx)^7$ in ascending power of x if the second, third and fourth coefficients form an arithmetic sequence. 5

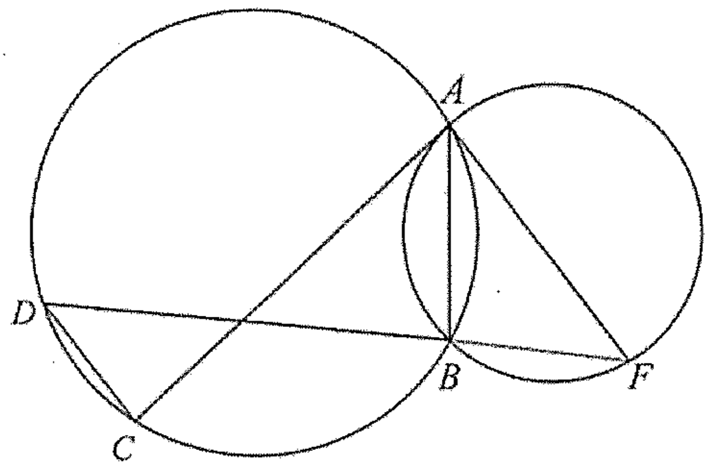
Question 7 (16 marks)

Find the value of the pronumerals. Give reasons.

(a) **6**

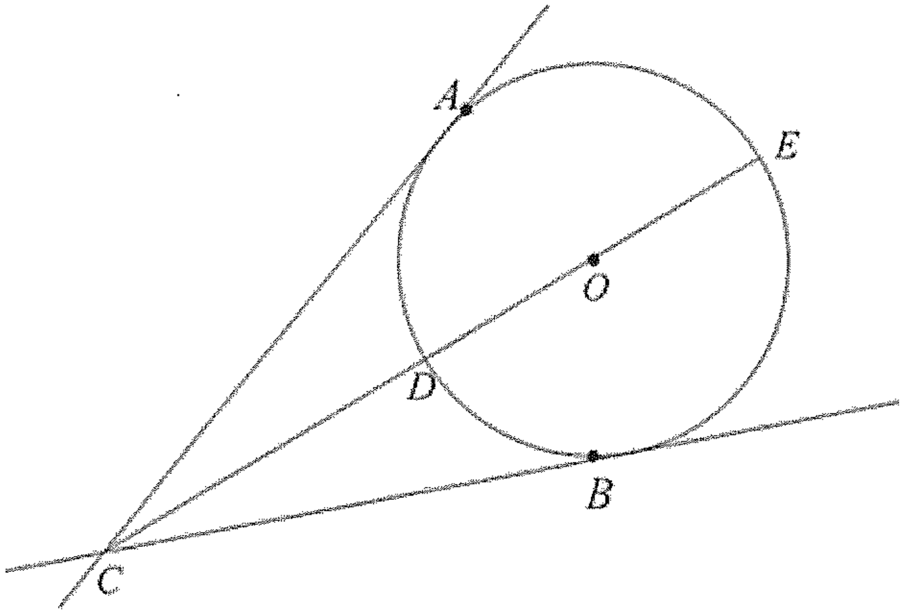


(b) AC is tangent to the circle ABF. Let $\angle CDB = \theta$. Prove that $CD \parallel FA$. 3



(c) CA and CB are tangents drawn to the circle with centre O.

CO is produced to E and intersects the circle at D.



(i) Prove that $OACB$ is a cyclic quadrilateral. 3

(ii) Show that $CD = \dfrac{CB \times CA}{CD + 2DO}$ 4

Question 8 (18 marks)

(a) Tangents are drawn from the point $T(8,6)$ to the parabola $x^2 = 8y$. P and Q are the points of contact of the tangents.

(i) Find the equation of the chord of contact. (Given that the equation of the chord of contact is $xx_1 = 2a(y + y_1)$). 3

(ii) Find the coordinates of the point P and Q. 4

(iii) Find the midpoint M of PQ. 2

(iv) Write the equation of TM. 1

(v) Hence or otherwise show that TM is parallel to the axis of the parabola. 2

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of the tangent at P is $y = px - ap^2$.

(i) The tangent at P and the line passing through Q parallel to the $y-$axis intersect at T. Show that the coordinates of T are $(2aq, 2apq - ap^2)$ 3

(ii) Find the coordinates of M, the midpoint of PT. 2

(iii) Find the Cartesian equation of the locus of M when $pq = -1$. 1

Question 9 (18 marks)

Prove the following by Mathematical Induction:

(a) $2(1!)+5(2!)+10(3!)+...+(n^2+1)(n!)=(n+1)!\times n$ for positive integers $n\geq 1$. 6

(b) $4\times 2^n+3^{3n}$ is divisible by 5 for all integers $n\geq 0$. 6

(c) $5^n\geq 4n+12$ for all integers $n>1$. 6

Question 10 (18 marks)

(a) Solve for n where n is a positive integer.

$${}^{n}C_2+{}^{n+1}C_1=7$$ 3

(b) For the expression $(2+5x)^{20}$

(i) derive an expression for $\dfrac{T_{r+1}}{T_r}$. 5

(ii) hence, find the greatest coefficient.

(You may leave your answer in the form ${}^{20}C_r2^a5^b$). 3

(iii) Find the greatest term when $x=\dfrac{1}{3}$. 3

(c) In the expansion of $(2+3x)^n$ the coefficients of x^4 and x^3 are in the ratio 15 : 8. Find

the value of n. 4

END OF EXAMINATION

Year 12 Extension 1 HSC Task 1 2016

Multiple choice

1. C 2. B 3. A 4. C 5. D

Question 6 (15 marks)

(a) (i) $\left(1+\frac{2x}{5}\right)^5$

$$= 1 + {}^5C_1\left(\frac{2x}{5}\right) + {}^5C_2\left(\frac{2x}{5}\right)^2 + {}^5C_3\left(\frac{2x}{5}\right)^3 + {}^5C_4\left(\frac{2x}{5}\right)^4 + \left(\frac{2x}{5}\right)^5$$

$$= 1 + 2x + \frac{8x^2}{5} + \frac{16x^3}{25} + \frac{16x^4}{125} + \frac{32x^5}{3125}$$ 3

(ii) $(x^2-2y^3)^7$

$$= (x^2)^7 - {}^7C_1(x^2)^6(2y^3) + {}^7C_2(x^2)^5(2y^3)^2 - {}^7C_3(x^2)^4(2y^3)^3 + {}^7C_4(x^2)^3(2y^3)^4 - {}^7C_5(x^2)^2(2y^3)^5 + {}^7C_6(x^2)(2y^3)^6 - (2y^3)^7$$

$$= x^{14} - 7\times x^{12}\times 2y^3 + 21x^{10}\times 4y^6 - 35x^8\times 8y^9 + 35x^6\times 16y^{12} - 21x^4\times 32y^{15} + 7x^2\times 64y^{18} - 128y^{21}$$

$$= x^{14} - 14x^{12}y^3 + 84x^{10}y^6 - 280x^8y^9 + 560x^6y^{12} - 672x^4y^{15} + 448x^2y^{16} - 128y^{21}$$ 4

(b) $\left(5x^2 - \frac{1}{x}\right)^{18}$

$$T_{r+1} = (-1)^r \, {}^{18}C_r \, (5x^2)^{18-r} \left(\frac{1}{x}\right)^r$$

$$= (-1)^r \, {}^{18}C_r \, 5^{18-r} \, x^{36-2r} \times \frac{1}{x^r}$$

$$= (-1)^r \, {}^{18}C_r \, 5^{18-r} \, x^{36-3r}$$ 3

For coefficient of x^9, $36 - 3r = 9$

$$r = 9$$

Coefficient of x^9 $= (-1)^9 \, {}^{18}C_9 \, 5^{18-9}$

$$= -{}^{18}C_9 \times 5^9$$

(c) $(1+kx)^7 = 1 + {}^7C_1(kx) + {}^7C_2(kx)^2 + {}^7C_3(kx)^3 + \cdots$

${}^7C_1 k$, ${}^7C_2 k^2$, ${}^7C_3 k^3$ form an AP

$${}^7C_2 k^2 - {}^7C_1 k = {}^7C_3 k^3 - {}^7C_2 k^2$$ 5

$$21k^2 - 7k = 35k^3 - 21k^2$$

$$35k^3 - 21k^2 - 21k^2 + 7k = 0$$

$$35k^3 - 42k^2 + 7k = 0$$

$$7k(5k^2 - 6k + 1) = 0$$

$k = 0$ or $5k^2 - 6k + 1 = 0$

$$5k^2 - 6k + 1 = 0$$

$$(k-1)(5k-1) = 0$$

$k = 1$ or $k = \frac{1}{5}$

$k = 0, 1, \frac{1}{5}$

For $k = 0$, there is no expansion.

$k = 1$, or $k = \frac{1}{5}$

Question 7 (20 marks)

(a) $a = 62$ (angles subtended at the circumference by chord BC)

$b = 180 - 62$ (opposite angles of a cyclic quadrilateral)
$= 118$

$\angle BOC = 2 \times 62$ (angles at the centre is twice the angle at the circumference)
$= 124$

6

$c = \frac{180 - 124}{2}$ (angle sum of isosceles ΔBOC)
$= 28$

(b) $\angle BAC = \angle AFB$ (angle between tangent and chord is equal to angle in the alternate segment)

$\angle BAC = \angle BDC$ (angles subtended at the circumference by chord BC)

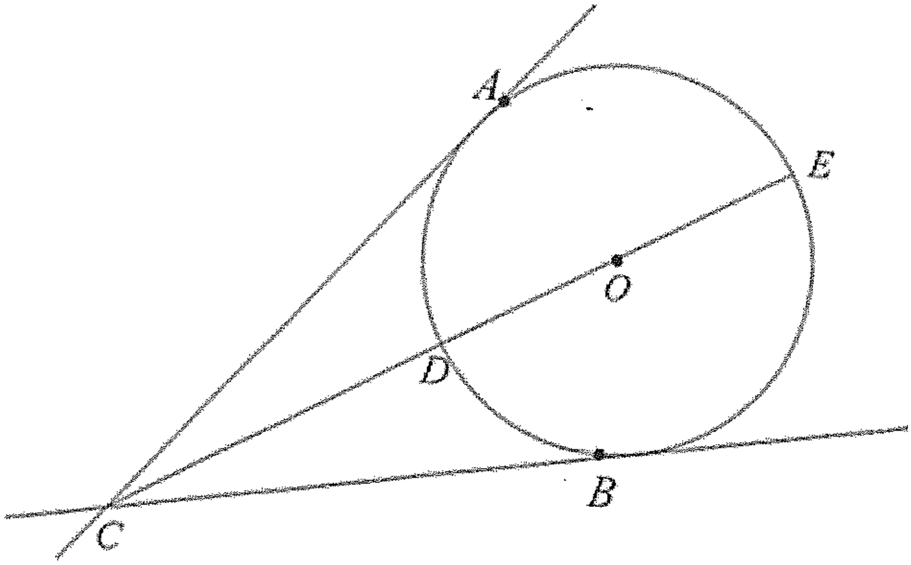
$\therefore \angle AFB = \angle BDC$

3

ie $\angle AFD = \angle FDC$

$CD \parallel FA$ (a pair of alternate angles equal)

(c)



(i) $\angle OAC = \angle OBC = 90°$ (angle between tangent and radius) 3

$\angle OAC + \angle OBC = 180°$

OACB is a cyclic quadrilateral (opposite angles of qudrilateral OACB is supplementary)

(ii) $CA^2 = CD \times CE$ (square of the tangent is equal to product of the intercepts of the secant)

$= CD \times (CD + 2DO)$

$CA = CB$ (tangents to a circle from an external point)

$CA \times CB = CD(CD + 2DO)$

4

$$CD = \frac{CA \times CB}{CD + 2DO}$$

Question 8 (18 marks)

(a) $x^2 = 8y$ $\quad a = 2 \quad T(8,6)$

(i) The equation of the chord of contact is

$$xx_1 = 2a(y+y_1)$$
$$8x = 4(y+6)$$
$$8x = 4y + 24$$
$$8x - 4y - 24 = 0 \quad (3)$$
$$2x - y - 6 = 0$$

(ii) $x^2 = 8y$ —— (1)

$2x - y - 6 = 0$ —— (2)

(2) $\Longrightarrow y = 2x - 6$

substitute in (1)

$$x^2 = 8(2x-6)$$
$$x^2 = 16x - 48 \quad (4)$$
$$x^2 - 16x + 48 = 0$$
$$(x-12)(x-4) = 0$$

$x = 12$ or $x = 4$

when $x = 12$, $y = \frac{144}{8} = 18$

when $x = 4$, $y = \frac{16}{8} = 2$

$P(12, 18) \quad Q(4, 2)$

(iii) Midpoint of PQ (2)

$$M = \left(\frac{16}{2}, \frac{20}{2}\right)$$
$$= (8, 10)$$

(iv) $T(8,6) \quad M(8,10)$

Equation of TM

$x = 8$ (1)

(v) Axis of the parabola is $x = 0$ (2)

$x = 8$ is parallel to $x = 0$ because both represent vertical lines.

(b) (i) $y = px - ap^2$ —— ①

$x = 2aq$ —— ②

substitute ② in ①

$y = p(2aq) - ap^2$

$= 2apq - ap^2$ (3)

$\therefore$ $T(2aq, 2apq - ap^2)$

(ii) Midpoint of PT

$P(2ap, ap^2)$ $T(2aq, 2apq - ap^2)$

$M = \left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2 + 2apq - ap^2}{2}\right)$

$= (a(p+q), apq)$ (2)

(iii) $pq = -1$ $\therefore$ $M(a(p+q), -a)$

locus of M is $y = -a$ (1)

Question 9 (18 marks)

$$2(1!)+5(2!)+10(3!)+\cdots+(n^2+1)(n!)=(n+1)!\times n$$

$n=1$

LHS $= 2\times 1! = 2$; RHS $=(1+1)!\times 1$
$= 2!\times 1 = 2$

LHS = RHS $\therefore$ the result is true for $n=1$

Assume the result is true for $n=k$

$$2(1!)+5(2!)+\cdots+(k^2+1)(k!)=(k+1)!\times k \quad (1)$$

Aim: To prove for $n=k+1$

$$2(1!)+5(2!)+\cdots+(k^2+1)(k!)+((k+1)^2+1)\times(k+1)! \quad (2)$$
$$=(k+2)!\times(k+1)$$

LHS of (2)

$= 2(1!)+5(2!)+\cdots+(k^2+1)(k!)+((k+1)^2+1)\times(k+1)!$

$=(k+1)!\times k+(k^2+2k+2)\times(k+1)!$

$=(k+1)!\left[k+k^2+2k+2\right]$

$=(k+1)!\,(k^2+3k+2)$

$=(k+1)!\,(k+2)(k+1)$

$=(k+2)!\times(k+1)$

= RHS of (2)

∴ if the result is true for $n=k$, then it is true for $n=k+1$. By the principle of Mathematical Induction, the result is true for all $n \geq 1$.

(b) $4 \times 2^n + 3^{3n}$ is divisible by 5 for $n \geq 0$

$n=0$

$4 \times 2^0 + 3^0 = 4 \times 1 + 1 = 5$ which is divisible by 5

Assume true for $n=k$

$4 \times 2^k + 3^{3k} = 5P$ —— ① where P is an integer

Aim: To prove for $n=k+1$

$4 \times 2^{k+1} + 3^{3(k+1)} = 5Q$ —— ② where Q is an integer

LHS of ②

$= 4 \times 2^{k+1} + 3^{3(k+1)}$

$= 4 \times 2 \times 2^k + 3^{3k+3}$

$= 2(5P - 3^{3k}) + 3^{3k} \times 3^3$ (by assumption ①)

$= 2 \times 5P - 2 \times 3^{3k} + 27 \times 3^{3k}$

$= 2 \times 5P + 25 \times 3^{3k}$

$= 5(2P + 5 \times 3^{3k})$

$= 5Q$ where $Q = 2p + 5 \times 3^{3k}$

$=$ RHS of (2)

$\therefore$ if the result is true for $n=k$, then it is true for $n=k+1$. Hence by the principle of Mathematical induction, the result is true for all integers $n \geq 0$.

(c) $5^n \geq 4n+12$ for $n > 1$

$\underline{n=2}$

LHS $= 5^2 = 25$; RHS $= 4 \times 2 + 12$
$= 20$

LHS $>$ RHS

$\therefore$ the result is true when $n = 2$

assume true for $n = k$

$5^k \geq 4k+12$ —— (1)

Aim: To prove for $n = k+1$

$5^{k+1} \geq 4(k+1)+12$ —— (2)

$5^{k+1} \geq 4k+16$ —— (2)

LHS of (2)

$= 5^{k+1} = 5 \times 5^k$

$\geq 5(4k+12)$ (by assumption (1))

$= 20k+60$

$= 4k+16k+16+64$

$> 4k+16 \quad (\text{since } 16k+44 > 0)$

$= \text{RHS of } (2)$

$\therefore$ if the result is true for $n=k$, then it is true for $n=k+1$. By the principle of Mathematical induction the result is true for $n \geq 2$.

Question 10 (18 marks)

(a) ${}^{n}C_2 + {}^{n+1}C_1 = 7$

$$\frac{n(n-1)}{1\times 2} + n+1 = 7$$

$$n^2 - n + 2n + 2 = 14$$

3

$$n^2 + n - 12 = 0$$

$$(n+4)(n-3) = 0$$

$$n = 3 \quad \text{or} \quad n = -4$$

$$\underline{\underline{n = 3 \quad (\text{since } n > 0)}}$$

(b) $(2+5x)^{20}$ $\quad a=2, \; b=5x, \; n=20$

(i) $T_{r+1} = {}^{n}C_r \, a^{n-r} \, b^r$

$$= {}^{20}C_r \, 2^{20-r} (5x)^r$$

$$= {}^{20}C_r \, 2^{20-r} (5x)^r$$

$$T_r = {}^{20}C_{r-1} \, 2^{20-r+1} (5x)^{r-1}$$

$$\frac{T_{r+1}}{T_r} = \frac{{}^{20}C_r\, 2^{20-r}(5x)^r}{{}^{20}C_{r-1}\, 2^{20-r+1}(5x)^{r-1}}$$

$$= \frac{{}^{20}C_r}{{}^{20}C_{r-1}}\; 2^{20-r-20+r-1}\,(5x)^{r-r+1}$$

$$= \frac{{}^{20}C_r}{{}^{20}C_{r-1}}\; 2^{-1}(5x) = \frac{{}^{20}C_r}{{}^{20}C_{r-1}}\;\frac{5x}{2}$$ 5

$$= \left[\frac{20!}{r!\,(20-r)!} \div \frac{20!}{(r-1)!\,(20-r+1)!}\right] \times \frac{5x}{2}$$

$$= \left[\frac{20!}{r!\,(20-r)!} \times \frac{(r-1)!\,(20-r+1)!}{20!}\right] \times \frac{5x}{2}$$

$$= \frac{20-r+1}{r} \times \frac{5x}{2}$$

$$\frac{C_{r+1}}{C_r} = \frac{21-r}{r} \times \frac{5}{2}$$

$$= \frac{105-5r}{2r}$$

$$\frac{C_{r+1}}{C_r} \geq 1$$

$$\frac{105-5r}{2r} \geq 1$$

$$105 - 5r \geq 2r$$

$$105 \geq 7r$$

$$r \leq \frac{105}{7} = 15$$ 3

Greatest coefficients C_{15} or C_{16}

$$C_{15} = {}^{20}C_{14} \times 2^6\,(5x)^{14}$$

or

$$C_{16} = {}^{20}C_{15} \times 2^5 \times 5^{15}$$

(iii) From (ii) $\frac{T_{r+1}}{T_r} = \frac{21-r}{r} \times \frac{5x}{2}$

When $x = \frac{1}{3}$, $\frac{T_{r+1}}{T_r} = \frac{21-r}{r} \times \frac{5}{2} \times \frac{1}{3}$

$$= \frac{105-5r}{6r}$$

$$\frac{T_{r+1}}{T_r} \geq 1$$

$$\frac{105-5r}{6r} \geq 1$$

$$105-5r \geq 6r$$

$$105 \geq 11r$$

$$r \leq \frac{105}{11} = 9\frac{6}{11}$$

T_{10} is the greatest term

$$T_{10} = {}^{20}C_9 \; 2^{20-9} (5x)^9$$

$$= {}^{20}C_9 \; 2^{11} 5^9 x^9$$

(3)

When $x = \frac{1}{3}$

$$T_{10} = {}^{20}C_9 \; 2^{11} 5^9 \left(\frac{1}{3}\right)^9$$

$$= {}^{20}C_9 \times 2^{11} \times \left(\frac{5}{3}\right)^9$$

(c) $(2+3x)^n = 2^n + {}^nC_1\, 2^{n-1}(3x) + {}^nC_2\, 2^{n-2}(3x)^2$

$+ {}^nC_3\, 2^{n-3}(3x)^3 + {}^nC_4\, 2^{n-4}(3x)^4 + \cdots$

$$\frac{{}^nC_4\, 2^{n-4}\, 3^4}{{}^nC_3\, 2^{n-3}\, 3^3} = \frac{15}{8}$$

$$\frac{{}^nC_4}{{}^nC_3} \times \frac{3}{2} = \frac{15}{8}$$

$$\left[\frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4} \times \frac{1\times 2\times 3}{n(n-1)(n-2)}\right] \times \frac{3}{2} = \frac{15}{8}$$

$$\frac{3(n-3)}{8} = \frac{15}{8}$$

$$\frac{3n-9}{8} = \frac{15}{8}$$

4

$$3n-9 = 15$$

$$3n = 24$$

$$n = 8$$