GOSFORD HIGH SCHOOL

EXTENSION 1 MATHEMATICS

HSC Course Assessment Task 1

December 2010

Special Instructions

- Students are to hand in their papers in three separate bundles.
- **Students must start PART B and PART C on a new page.**

General Instructions

Time Allowed: 60 minutes plus 5 minutes reading time.

- Attempt all questions.
- Approved calculators may be used.
- Write using blue or black pen.
- Full marks may not be awarded where necessary working is not shown.

PART A: ITERATIVE METHODS.

Question 1:

(a) Show that $f(x)=2x^3+2x-1$ has a zero between $x=0$ and $x=1$. (2)

(b) By considering $f'(x)$, explain why this is the only root of $f(x) = 0$. (2)

(c) Taking $x=0$ as a first approximation to this root, use Newtons Method twice to find an approximation to two decimal places. (3)

Question 2:

(a) Show that $f(x)=x^4+4x^3+8x-4$ has a zero α between $x=0$ and $x=1$. (2)

(b) Use the method of halving the interval to determine whether α is closer to $x=0$ or $x=1$. (2)

(c) Given that $x^4+4x^3+8x-4=(x^2+2)(x^2+4x-2)$, show that $f(x) = 0$ has only two real roots and find the value of α correct to three decimal places. (3)

PART B: MATHEMATICAL INDUCTION.

(Start a new page)

Question 1:

(a) Prove by induction that:

$$1.2 + 3.4 + 5.6 + \ldots. + (2n-1)(2n) = \frac{n}{3}(n+1)(4n-1)$$

for all positive integers n. (6)

(b) Prove by induction that:

$2^n - 1$ is divisible by 3 for all even positive integers. (5)

PART C: PARAMETRIC TREATMENT OF THE PARABOLA.

(Start a new page)

Question1:

(a) Show that the points defined by the parametric equations $x=2\cos t$ and $y=\cos 2t$ lie on a parabolic arc. (3)

(b) Sketch the arc clearly showing its end points, the focus and the directrix of this parabola. (2)

Question 2:

(a) The parametric equations of a parabola are $x=6t$ and $y=3t^2$. Find the Cartesian equation of the parabola. (2)

(b) Hence show that the equation of the normal at P, where $t=p$, is

$$x+py=3p^3+6p. \qquad (4)$$

Question 3:

$P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$ whose focus is S. The line through S, perpendicular to the tangent at P, meets the tangent at L.

(a) Show that the equation of the tangent at P is $y = px - ap^2$. (3)

(b) Show that the equation of SL is $x + py = ap$. (2)

(c) Determine the locus of L. (2)

PART A

Q1/

a) $f(0) = 2(0)^3 + 2(0) - 1$

$= -1$

$f(1) = 2(1)^3 + 2(1) - 1$

$= 3$

Since $f(x)$ is continuous

& Since $f(0)$ & $f(1)$ are opposite in sign there is a zero of $f(x)$ between $x=0$ & $x=1$

b) $f(x) = 2x^3 + 2x - 1$

$f'(x) = 6x^2 + 2$

$\therefore$ $f'(x) > 0$ for all x and hence $f(x)$ is a monotonically increasing function. & continuous

$\therefore$ There is only one root of $f(x) = 0$

c) $a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$

$= 0 - -\frac{1}{2}$

$= \frac{1}{2}$.

$a_3 = a_2 - \dfrac{f(a_2)}{f'(a_2)}$

$= \frac{1}{2} - \dfrac{\frac{1}{4}}{\frac{7}{2}}$

$\doteqdot 0.43$ (2 d.p)

Q2/ Note $f(x)$ is continuous

a) $f(0) = (0)^4 + 4(0)^3 + 8(0) - 4$

$= -4$

$f(1) = (1)^4 + 4(1)^3 + 8(1) - 4$

$= 9$

Since $f(0)$ & $f(1)$ are opposite in sign, α lies between $x=0$ & $x=1$.

b) $f(0.5) = (0.5)^4 + 4(0.5)^3 + 8(0.5) - 4$

$= 0.5625$

Since $f(0)$ & $f(0.5)$ are opposite in sign α lies between $x=0$ & $x=0.5$

$\therefore$ α is closer to 0 than 1.

c) If $f(x) = 0$

$x^2 + 2 = 0$ or $x^2 + 4x - 2 = 0$

No real sol^n

$x^2 + 4x + 4 = 2 + 4$

$(x+2)^2 = 6$

$x + 2 = \pm\sqrt{6}$

$x = -2 \pm \sqrt{6}$

$\therefore$ $f(x)$ has only two real roots

Now $-2 + \sqrt{6}$ is between 0 & 1

$\therefore$ $\alpha \doteqdot 0.449$ (3 d.p).

PART B

Q1/

a) $1.2 + 3.4 + \dots + (2n-1)2n = \frac{n}{3}(n+1)(4n-1)$

Prove true for $n=1$

$LHS = 1.2$

$= 2$

$RHS = \frac{1}{3}(1+1)(4-1)$

$= 2.$

$\therefore$ true for $n=1$.

Assume true for $n=k$

ie. $1.2 + 3.4 + \dots + (2k-1)2k = \frac{k}{3}(k+1)(4k-1)$

Prove true for $n=k+1$

ie $1.2 + 3.4 + \dots + (2k-1)(2k) + (2k+1)(2k+2) = \frac{k+1}{3}(k+2)(4k+3)$

$LHS = \frac{k}{3}(k+1)(4k-1) + (2k+1)(2k+2)$

$= \frac{k}{3}(k+1)(4k-1) + 2(2k+1)(k+1)$

$= \frac{k+1}{3}\left[k(4k-1) + 6(2k+1)\right]$

$= \frac{k+1}{3}(4k^2 + 11k + 6)$

$= \frac{k+1}{3}(4k+3)(k+2)$

$= RHS$

$\therefore$ If the statement is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, by induction the statement is true for all positive integers n.

b) $2^n - 1$ is divisible by 3 for all even positive integers

Prove true for $n=2$

$2^2 - 1 = 3$

which is divisible by 3.

Assume true for $n=k$, where k is a positive even integer.

Let $2^k - 1 = 3Q$ for some integer Q

Prove true for $n=k+2$

Now $2^{k+2} - 1 = 2^2(2^k - 1) + 3$

$= 4 \times 3Q + 3$

$= 3(4Q + 1)$

which is divisible by 3.

$\therefore$ If the statement is true for $n=k$ it is true for $n=k+2$. Since it is true for $n=2$, by induction it is true for all positive even integers n.

PART C

Q1/

a) $x = 2\cos t$, $y = \cos 2t$

Since $-1 \le \cos t \le 1$

$-2 \le 2\cos t \le 2$

$\therefore -2 \le x \le 2$.

Also if $y = \cos 2t$

$y = 2\cos^2 t - 1$

$y = 2\left(\frac{x}{2}\right)^2 - 1$

$y = \frac{x^2}{2} - 1$

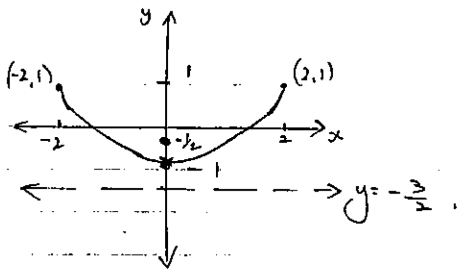
$2y = x^2 - 2$

$x^2 = 2y + 2$

$x^2 = 4 \times \frac{1}{2}(y+1)$

$\therefore$ the eqn represents a parabolic arc with vertex at $(0,-1)$ & focal length $\frac{1}{2}$ unit for $-2 \le x \le 2$.

b)



N.B. if $x=-2$, $y=1$
$x=2$, $y=1$

Q2/

a) $x = 6t$, $y = 3t^2$

If $x = 6t$

$t = \frac{x}{6}$

$\therefore y = 3\left(\frac{x}{6}\right)^2$

$y = \frac{3x^2}{36}$

$y = \frac{x^2}{12}$

or $x^2 = 12y$.

b) If $y = \frac{x^2}{12}$

$y' = \frac{2x}{12}$

$= \frac{x}{6}$

When $x = 6t$, $y' = \frac{6t}{6} = t$

$\therefore$ the gradient of the normal is $-\frac{1}{t}$

Hence the eqn of the normal is:

$y - 3t^2 = \frac{-1}{t}(x - 6t)$

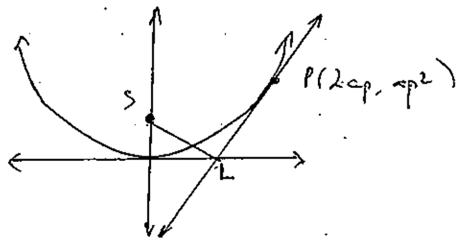
$ty - 3t^3 = -x + 6t$

ie. $x + ty = 3t^3 + 6t$.

Now when $t = p$

$x + py = 3p^3 + 6p$.

Q3



a) S is the point $(0, a)$

If $x^2 = 4ay$, $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$= \frac{x}{2a}$

When $x = 2ap$, $y' = \frac{2ap}{2a}$

$y' = p$

$\therefore$ the eqn of the tangent is:

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ — (1)

b) m of SL is $-\frac{1}{p}$

$\therefore$ the eqn of SL is:

$y = -\frac{1}{p}x + a$ — (2)

ie $py = -x + ap$

$x + py = ap$.

b) Solving (1) & (2) simultaneously

$-\frac{1}{p}x + a = px - ap^2$

$-x + ap = p^2x - ap^3$

$\therefore ap^3 + ap = p^2x + x$

Hence $x\cancel{(p^2+1)} = ap\cancel{(p^2+1)}$ since $p^2 + 1 \neq 0$

$\therefore x = ap$

$\therefore$ Substituting into (2)

$y = -\frac{1}{p} \cdot ap + a$

$y = -a + a$

$y = 0$

$\therefore$ The locus is the line $y = 0$ (or the x-axis).