GOSFORD HIGH SCHOOL

2012

HSC ASSESSMENT TASK 1

EXTENSION 1 MATHEMATICS

General Instructions:

Total marks: - 48

Attempt all Questions 1-3
- Working time: 60 minutes
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question should be started on a separate page.
- All necessary working should be shown in every question.

SECTION 1	**/ 16**
SECTION 2	**/ 16**
SECTION 3	**/ 16**
TOTAL	**/ 48**

Section I. POLYNOMIALS AND FURTHER GRAPHS (16 marks).

1. (a) Show that $x = 1$ *and* $x = 2$ are roots of the polynomial equation $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0.$ (2)

   (b) Hence sketch the graph of $y = x^4 - 2x^3 - 7x^2 + 20x - 12$ clearly showing any intercepts. (4)

2. If $x^3 - 6x^2 + kx - 12 = 0$ has one root equal to the sum of the other two roots, find the value of k. (3)

3. Consider the function $f(x) = \dfrac{x^2}{1-x^2}$.

   (a) Show that $f(x)$ is even. (1)

   (b) Find any intercepts of the graph of $y = f(x)$. (1)

   (c) State the vertical asymptote(s) of the graph of $y = f(x)$. (1)

   (d) Describe how the function behaves as $x \to \pm\infty$. (2)

   (e) Sketch the graph of $y = f(x)$. (2)

Section II. FURTHER TRIGONOMETRY (16 marks). **START A NEW PAGE.**

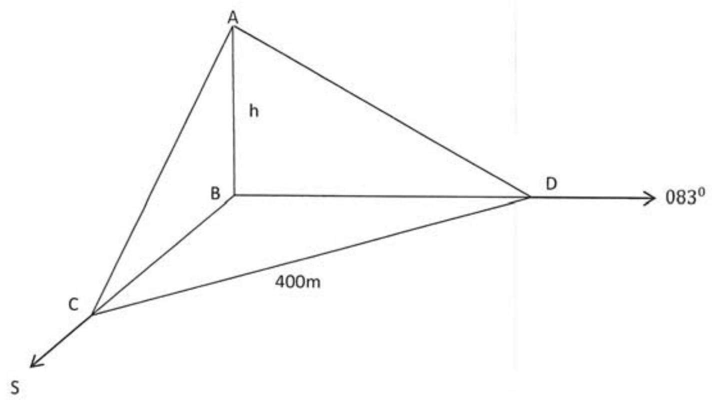
1. Find, in simplest exact form, the value of $\tan 105^0$. (3)

2. Prove that $\frac{sin2x+cosx}{cos2x+sinx} = \frac{cosx}{1-sinx}$. (3)

3. Find an expression for $secx - tanx$ in terms of t (where $t = \tan\frac{x}{2}$). (2)

4. Solve $\cos x + \sqrt{3}\sin x = \sqrt{3}$ for $0^0 \leq x \leq 360^0$. (5)

5.



From a point C, due south of the foot of a hill AB, the angle of elevation to the top of the hill is 14^0. From a point D, bearing 083^0 from the foot of the hill, the angle of elevation to the top is 10^0. Find the height h of the hill, to the nearest metre, if the distance from C to D is 400 metres. (3)

Section III. PARAMETRIC REPRESENTATION (16 marks). **START A NEW PAGE.**

1. Given the parametric equations $x = 6t, y = 3t^2$.
Eliminate t to find the Cartesian equation of the parabola. (2)

2. The points $P(2ap, ap^2)$ & $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the vertex of the parabola.

(a) Show that $pq + 4 = 0$. (3)

(b) Show that the equation of the chord PQ is $y - \frac{1}{2}(p+q)x + apq = 0$. (3)

3. (a) Show that the equation of the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ is $y = px - ap^2$. (3)

(b) The tangent at P meets the axis of the parabola at T. Find the coordinates of the point T. (1)

(c) The normal at P meets the axis of the parabola at U. Show that $TS = US$, where S is the focus of the parabola. (4)

MATHS EXT 1 2012 #1

SOLUTIONS

SECTION I

1 a) Let $P(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$

$$\begin{aligned} P(1) &= (1)^4 - 2(1)^3 - 7(1)^2 + 20(1) - 12 \\ &= 1 - 2 - 7 + 20 - 12 \\ &= 0 \end{aligned}$$

$$\begin{aligned} P(2) &= (2)^4 - 2(2)^3 - 7(2)^2 + 20(2) - 12 \\ &= 16 - 16 - 28 + 40 - 12 \\ &= 0 \end{aligned}$$

2.

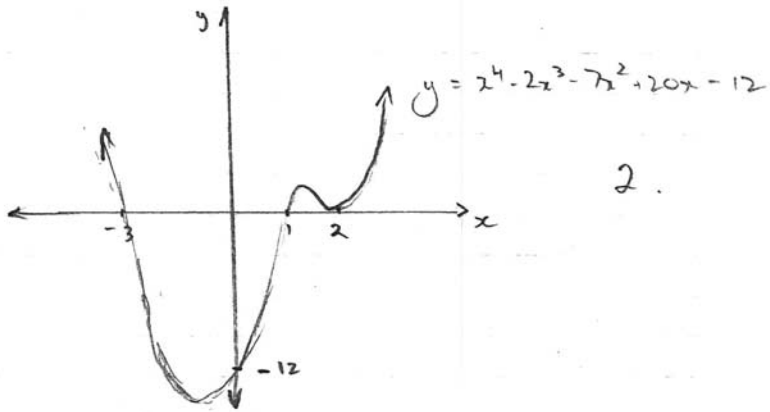
$\therefore x = 1$ & $x = 2$ are roots of the eqn.

b) $(x-1)(x-2) = x^2 - 3x + 2$

$$\begin{array}{r|l} & x^2 + x - 6 \\ x^2 - 3x + 2 & x^4 - 2x^3 - 7x^2 + 20x - 12 \\ & x^4 - 3x^3 + 2x^2 \\ & \quad x^3 - 9x^2 + 20x \\ & \quad x^3 - 3x^2 + 2x \\ & \qquad -6x^2 + 18x - 12 \\ & \qquad -6x^2 + 18x - 12 \\ & \qquad\qquad 0 \end{array}$$

2.

$$\begin{aligned} \therefore P(x) &= (x-1)(x-2)(x^2 + x - 6) \\ &= (x-1)(x-2)(x+3)(x-2) \\ &= (x-2)^2(x-1)(x+3) \end{aligned}$$



2.

2. Let the roots be $\alpha, \beta, \alpha + \beta$

$$\alpha + \beta + \alpha + \beta = \frac{-b}{a} \qquad \alpha . \beta . (\alpha + \beta) = -\frac{d}{a}$$

$$2\alpha + 2\beta = -\frac{-6}{1} \qquad \therefore \alpha\beta . 3 = -\frac{-12}{1}$$

$$\alpha + \beta = 3 \qquad 3\alpha\beta = 12$$

$$\alpha\beta = 4.$$

Also $\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta) = \frac{c}{a}$

(3)

$$\begin{aligned} 4 + \alpha . 3 + \beta . 3 &= \frac{k}{1} \\ 4 + 3(\alpha + \beta) &= k \\ 4 + 3(4) &= k \\ k &= 16. \end{aligned}$$

3. a) $f(x) = \dfrac{x^2}{1 - x^2}$

$$\begin{aligned} f(-x) &= \frac{(-x)^2}{1 - (-x)^2} \\ &= \frac{x^2}{1 - x^2} \\ &= f(x) \end{aligned}$$

(1)

$\therefore f(x)$ is even

b) If $x = 0$, $y = \dfrac{(0)^2}{1 - (0)^2}$

$= 0$ (1)

$\therefore$ Curve passes thru $(0,0)$

c) $x = \pm 1$ (1)

N.B. As $x \to 1^+$, $y \to -\infty$. As $x \to -1^+$, $y \to -\infty$

" $x \to 1^-$, $y \to \infty$ As $x \to -1^-$, $y \to \infty$

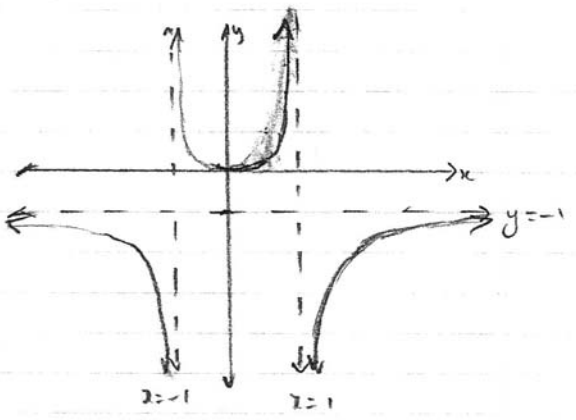
d) $\lim_{x \to \infty} \frac{x^2}{1-x^2} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}}$

$= \lim_{x \to \infty} \frac{1}{\frac{1}{x^2} - 1}$. (2)

$\therefore$ As $x \to \infty$, $f(x) \to -1$

As $x \to -\infty$, $f(x) \to -1$.

e)



(2)

SECTION II

1. $\tan 105^\circ = \tan(45^\circ + 60^\circ)$

$= \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}$

$= \frac{1 + \sqrt{3}}{1 - 1 \cdot \sqrt{3}}$ (3)

$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{4 + 2\sqrt{3}}{-2}$

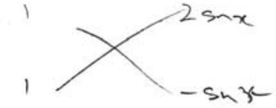
$= -2 - \sqrt{3}$.

2. Proof: LHS $= \frac{2\sin x \cos x + \cos x}{1 - 2\sin^2 x + \sin x}$

$= \frac{\cos x \,(2\sin x + 1)}{1 + \sin x - 2\sin^2 x}$ (3)

$= \frac{\cos x \,(\cancel{2\sin x + 1})}{(\cancel{1 + 2\sin x})(1 - \sin x)}$

$=$ RHS.



3. $\sec x - \tan x = \frac{1+t^2}{1-t^2} - \frac{2t}{1-t^2}$

$= \frac{1 - 2t + t^2}{1 - t^2}$ (2)

$= \frac{(1-t)^{2}}{\cancel{(1-t)}(1+t)}$

$= \frac{1-t}{1+t}$.

4. $1 \cos x + \sqrt{3} \sin x = 2\cos(x - \alpha)$ (1) where $\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = 60^\circ$ (1)

$\therefore 2\cos(x - 60^\circ) = \sqrt{3}$

$\cos(x - 60^\circ) = \frac{\sqrt{3}}{2}$

$\therefore\ x - 60^\circ = -30^\circ, 30^\circ, 330^\circ$

S	A ✓
T	C ✓

$x = 30^\circ, 90^\circ, \cancel{390^\circ}$

Sol^n is $x = 30^\circ, 90^\circ$.

(3)

5\. In $\triangle ABC$, $\frac{BC}{h} = \tan 76^\circ$

$BC = h \tan 76^\circ$

In $\triangle ABD$, $BC = h \tan 80^\circ$

In $\triangle BCD$, $400^2 = BC^2 + BD^2 - 2\,BC.BD.\cos 97^\circ$

$400^2 = h^2(\tan^2 76^\circ + \tan^2 80^\circ) - 2h^2 \tan 76^\circ \tan 80^\circ \cos 97^\circ$

$\therefore\ 400^2 = h^2\left[\tan^2 76^\circ + \tan^2 80^\circ - 2\tan 76^\circ \tan 80^\circ \cos 97^\circ\right]$

$$h^2 = \frac{400^2}{\tan^2 76^\circ + \tan^2 80^\circ - 2\tan 76^\circ \tan 80^\circ \cos 97^\circ}.$$

$h \doteqdot 55$ metres (nearest metre)

(3)

Section III

1\. $x = 6t \Rightarrow t = \frac{x}{6}$

$\therefore$ if $y = 3t^2$

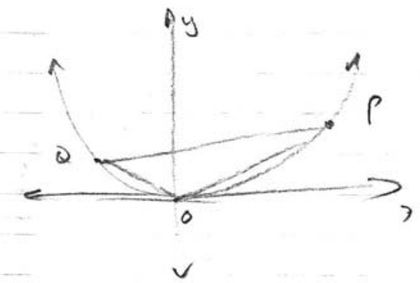
$y = 3\left(\frac{x}{6}\right)^2$

$y = \frac{3x^2}{36}$

$y = \frac{x^2}{12}$ or $x^2 = 12y$.

(2)

2



a) M of PO $= \frac{ap^2 - 0}{2ap - 0}$

$= \frac{p}{2}$

M of QO $= \frac{aq^2 - 0}{2aq - 0}$

$= q/2$

(3)

$\therefore\ \frac{p}{2} \times \frac{q}{2} = -1$

ie $\frac{pq}{4} = -1$

$pq = -4$

$\therefore\ pq + 4 = 0$

b) M of PQ $= \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{a\cancel{(p-q)}(p+q)}{2a\cancel{(p-q)}}$

$= \dfrac{p+q}{2}$

$\therefore$ Eqⁿ of PQ is

$y - ap^2 = \left(\dfrac{p+q}{2}\right)(x - 2ap)$

$y - ap^2 = \frac{1}{2}(p+q)x - \cancel{\tfrac{1}{2}}(p+q)\cancel{2}ap$

$y - \cancel{ap^2} = \frac{1}{2}(p+q)x - \cancel{ap^2} - apq$ 3

ie $y - \frac{1}{2}(p+q)x + apq = 0$

3 a) P $x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$y' = \dfrac{2x}{4a}$

$= \dfrac{x}{2a}$

At $(2ap, ap^2)$ $m = \dfrac{2ap}{2a}$

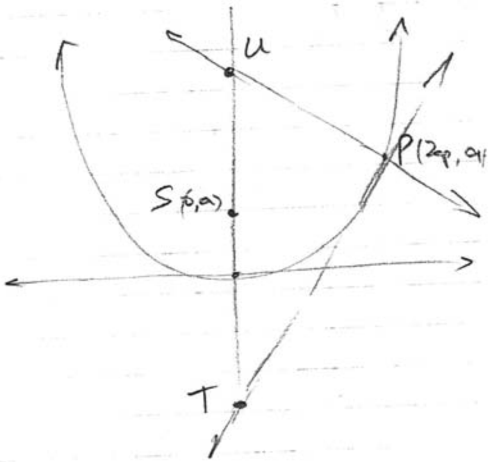
$\therefore$ the gradient of the tangent $= p$

$\therefore$ the eqⁿ of the tangent is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$



b) Axis of parabola is $x = 0$

When $x = 0$, $y = p(0) - ap^2$

$y = -ap^2$ 1

T is $(0, -ap^2)$

c) the gradient of the normal is $-\frac{1}{p}$

$\therefore$ the eqⁿ of the normal is

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

When $x = 0$, $py = ap^3 + 2ap$

$y = ap^2 + 2a$

$\therefore$ U is $(0, ap^2 + 2a)$

S is $(0, a)$ 4

$\therefore$ TS $= a + ap^2$

US $= ap^2 + 2a - a$

$= a + ap^2$

$\therefore$ TS $=$ US