NAME: ______________________

TEACHER: ______________________

GOSFORD HIGH SCHOOL

2012/2013
EXTENSION 1 MATHEMATICS
HSC ASSESSMENT TASK 1.

Time Allowed: 60 minutes (plus 5 min. reading time)

- Write using black or blue pen.
- Board-approved calculators may be used.
- Sections 3 and 4 should be started on a new page.
- All necessary working should be shown in Section 2 , 3 and 4

SECTION	QUESTION TYPE	MARKS	RESULT
1	MULTIPLE CHOICE	4	
2	PARABOLA	13	
3	INDUCTION	8	
4	FURTHER TRIG	18	
	TOTAL	43	

MATHEMATICS
Extension 1
Assessment Task 1
December 2012

Time: 1 hour plus 5 minutes reading time

SECTION 1

Multiple choice.

(Write your answer on your answer sheet: **NOT** on the test sheet)

1) The length of the latus rectum of the parabola $y = x^2$ is:

A) $\frac{1}{4}$ **B)** 1

C) 2 **D)** 4

2) A correct expression for $\cos 4A$ is:

A) $2\cos^2 A - 1$ **B)** $2\cos^2 2A - 2$

C) $1 - 2\sin^2 A$ **D)** $2\cos^2 2A - 1$

3) The equation of the chord of contact from the point (1,-1) to the parabola $x^2 = 8y$ is:

A) $x - 4y + 4 = 0$ **B)** $x - 2y + 2 = 0$

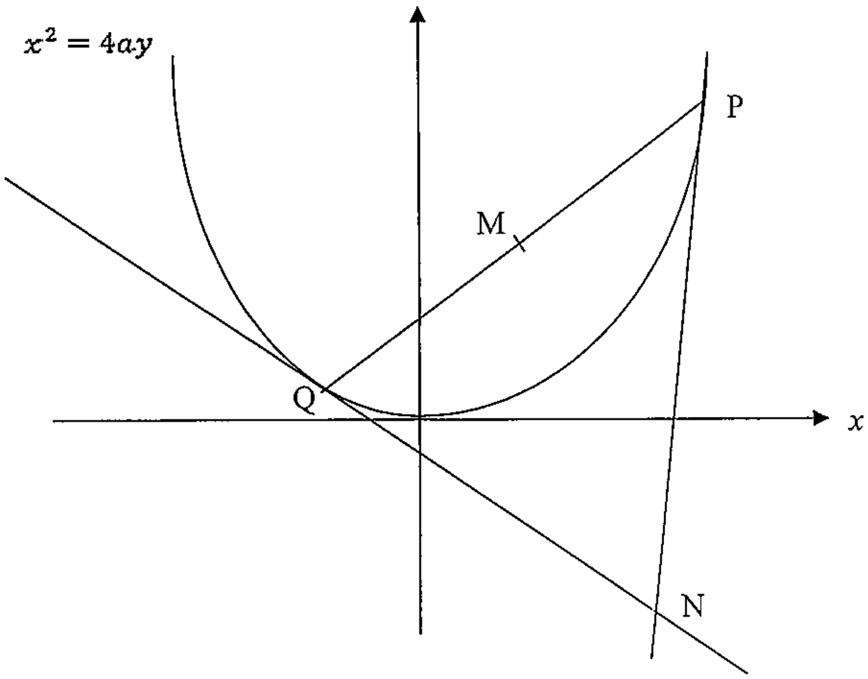
C) $x - 8y + 8 = 0$ **D)** $x + 4y - 4 = 0$

4) When written in terms of *t*, where $t = \tan\frac{\alpha}{2}$, $\frac{1+\sin\alpha+\cos\alpha}{1+\sin\alpha-\cos\alpha} \equiv$

A) t **B)** $-t$

C) $\frac{1}{t}$ **D)** $-\frac{1}{t}$

SECTION 2



1) P($2ap,ap^2$) and Q($2aq,aq^2$) are two points on the parabola $x^2 = 4ay$. M is the mid- point of the chord PQ. The tangents at P and Q meet at the point N.

(i) Show that the equation of the tangent at P is $y = px - ap^2$ **3**

(ii) Show that the co-ordinates of M are $(a(p+q), \frac{a(p^2+q^2)}{2})$ **2**

(iii) Show that the co-ordinates of N are $(a(p+q)\ , apq)$. **3**

(iv) Show that MN is parallel to the 'y' axis. **1**

(v) Find the co-ordinates of T, the mid-point of MN. **2**

(vi) Show that T lies on the parabola. **2**

SECTION 3
(start a new page)

1) By using mathematical induction show that
$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+-------+\frac{1}{n(n+1)}=\frac{n}{n+1}$ where n is a positive integer. **4**

2) Prove by Mathematical Induction that 3^n+7^n is divisible by 10 for n a positive **odd** number. **4**

SECTION 4
(start a new page)

1) Solve $\cos 2x=\cos x$ for $0^\circ \leq x \leq 360^\circ$ **3**

2) Express $\sqrt{3}\cos x+\sin x$ in the form $A\cos(x-\alpha)$ and hence find the minimum value of $\sqrt{3}\cos x+\sin x$ and a value of x that gives this minimum value. **3**

3) If $t=\tan\frac{x}{2}$ write an expression for $\sin x, \cos x$ and $\tan x$ in terms of t. Hence solve $\sin x+\cos x=1$ for $0^\circ \leq x^\circ \leq 360^\circ$ **3**

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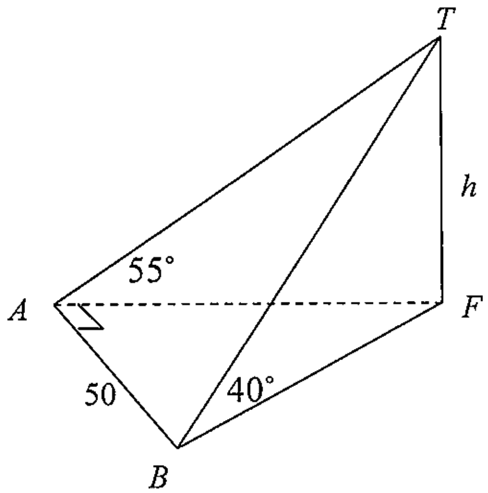
3) Solve $\sqrt{3}\sin x-\cos x=1$ for $0^\circ \leq x \leq 360^\circ$. **3**

Section 4 continued over the page.

5)



The diagram shows a tower of height h metres standing on level ground. The angle of elevation of the top T of the tower from two points A and B on the ground nearby are 55° and 40° respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF, where F is the foot of the tower.

a) Find AT and BT in terms of h. **2**

b) Using triangle BAT show that $h = \dfrac{50\sin 55^\circ \sin 40^\circ}{\sqrt{\sin^2 55^\circ - \sin^2 40^\circ}}$ **3**

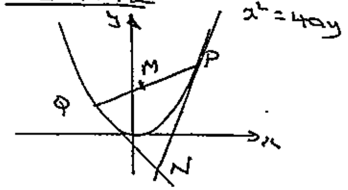
c) Hence find the height of the tower, correct to the nearest metre. **1**

Ext1 Assessment task 1 2012 Solutions

Section 1

1) B
2) D
3) A
4) C

Section 2



i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $x = 2ap$: $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

∴ eqn of the tangent

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

ii) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(\frac{2a(p+q)}{2}, \frac{a(p^2+q^2)}{2}\right)$

$= \left(a(p+q)\ \frac{a(p^2+q^2)}{2}\right)$

iii) $y = px - ap^2$ --- (1)

$y = qx - aq^2$ --- (2)

equating (1) and (2)

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p+q)(p-q)$

$x = a(p+q)$

Sub into (1)

$y = ap(p+q) - ap^2$

$y = ap^2 + apq - ap^2$

$y = apq$

∴ $N\ (a(p+q), apq)$

iv) As M and N have the same 'x' co-ordinate MN must be a vertical line. Therefore parallel to the 'y' axis.

v) $T = \left(\frac{a(p+q)+a(p+q)}{2}, \frac{apq + \frac{a(p^2+q^2)}{2}}{2}\right)$

$= \left(a(p+q), \frac{2apq + a(p^2+q^2)}{4}\right)$

vi) $x^2 = 4ay$

$a^2(p+q)^2 = 4a\left(\frac{2apq + a(p^2+q^2)}{4}\right)$

$a^2(p+q)^2 = a^2(2pq + p^2 + q^2)$

$a^2(p+q)^2 = a^2(p+q)^2$

∴ T lies on the parabola.

Section 3

1) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Prove true for $n = 1$.

$\frac{1}{1.2} = \frac{1}{1+1}$

$= \frac{1}{2} = \frac{1}{2}$ True.

Assume true for $n = k$ (where k is a positive integer)

ie. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Prove true for $n = k+1$

ie $\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Proof:

L.HS. $= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ from the assumption

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

$=$ RHS.

Therefore the statement is true for $n=k+1$ assuming true for $n=k$. Therefore it is true by Mathematical induction

2) $3^n + 7^n$ is divisible by 10 n odd.

Prove true for $n=1$.

ie $3^1 + 7^1$ is divisible by 10 — True.

Assume true for $n = k$; k odd.

ie $3^k + 7^k = 10M$, M a positive integer

Prove true for $n = k+2$.

ie $3^{k+2} + 7^{k+2}$ is divisible by 10

$$= 9.3^k + 49.7^k$$
$$= 49.3^k + 49.7^k - 40.3^k$$
$$= 49(3^k + 7^k) - 40.3^k$$
$$= 49 \times 10M - 40.3^k \quad \text{from assumption}$$
$$= 10(49M - 4.3^k) \quad \text{which is divisible by 10.}$$

Therefore the statement is true for $n = k+1$ assuming true for $n = k$. Therefore true by mathematical induction.

SECTION 4

1)
$$\cos 2x = \cos x$$
$$\cos 2x - \cos x = 0$$
$$2\cos^2 x - 1 - \cos x = 0$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$\cos x = -\tfrac{1}{2} \quad \cos x = 1$$
$$x = 120^\circ, 240^\circ, 0, 360^\circ$$

2) $\sqrt{3}\cos x + \sin x = A\cos(x - \alpha)$
$$= A\cos x\cos\alpha + A\sin x\sin\alpha$$

equating coefficients
$$A\cos\alpha = \sqrt{3} \quad \text{---- (1)}$$
$$A\sin\alpha = 1 \quad \text{--- (2)}$$
$$(2) \div (1) \quad \tan\alpha = \tfrac{1}{\sqrt{3}}$$
$$\alpha = 30^\circ$$
$$(1)^2 + (2)^2 \quad A^2(\sin^2\alpha + \cos^2\alpha) = 4$$
$$A^2 = 4$$
$$A = 2$$
$$\therefore \sqrt{3}\cos x + \sin x = 2\cos(x - 30^\circ)$$
$\therefore$ min Value $= -2$
when $x = 210^\circ$

3)
$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$
$$\tan x = \frac{2t}{1-t^2}$$
$$\sin x + \cos x = 1$$
$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$
$$2t + 1 - t^2 = 1 + t^2$$
$$2t^2 - 2t = 0$$
$$2t(t-1) = 0$$
$$t = 0, \quad t = 1$$
$$\tan\tfrac{x}{2} = 0 \qquad \tan\tfrac{x}{2} = 1$$
$$\tfrac{x}{2} = 0, 180^\circ, 360^\circ \; ; \; \tfrac{x}{2} = 45^\circ, 225^\circ$$
$x = 0, 360^\circ, 90$ for $0 \leq x \leq 360$

Check $x = 180^\circ$
$$\sin 180 + \cos 180 = 1$$
$$0 - 1 = 1$$
False $\therefore$ not a Solution

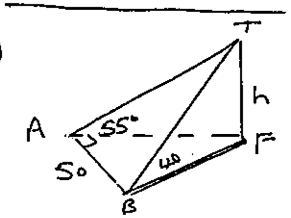
4) $\sqrt{3}\sin x - \cos x = 1$

Let $\sqrt{3}\sin x - \cos x = R\sin(x - \alpha)$
$$= R\sin x\cos\alpha - R\cos x\sin\alpha$$

equating Coefficients
$$R\cos\alpha = \sqrt{3} \quad \text{--- (1)}$$
$$R\sin\alpha = 1 \quad \text{--- (2)}$$
$$(2) \div (1) \quad \tan\alpha = \tfrac{1}{\sqrt{3}}$$
$$\alpha = 30^\circ$$
$$(1)^2 + (2)^2 : R^2(\sin^2\alpha + \cos^2\alpha) = 4$$
$$R^2 = 4$$
$$R = 2$$
$$\therefore \sqrt{3}\sin x - \cos x = 2\sin(x - 30^\circ)$$
$$\therefore \sqrt{3}\sin x - \cos x = 1$$
$$\text{i.e. } 2\sin(x - 30^\circ) = 1$$
$$\sin(x - 30^\circ) = \tfrac{1}{2}$$
$$x - 30^\circ = 30^\circ, 150^\circ$$
$$x = 60^\circ, 180^\circ$$

N.B. 't' Method could also be used.

5)



a) from ΔATF
$$\frac{h}{AT} = \sin 55^\circ$$
$$AT = \frac{h}{\sin 55^\circ}$$

from ΔBTF
$$\frac{h}{TB} = \sin 40^\circ$$
$$TB = \frac{h}{\cdots}$$

b) ΔBAT is right angled
$$\therefore TB^2 = AT^2 + AB^2 \quad \text{(Pythagoras)}$$
$$\frac{h^2}{\sin^2 40^\circ} = \frac{h^2}{\sin^2 55^\circ} + 50^2$$
$$\frac{h^2}{\sin^2 40^\circ} - \frac{h^2}{\sin^2 55^\circ} = 2500$$
$$h^2\left(\frac{1}{\sin^2 40^\circ} - \frac{1}{\sin^2 55^\circ}\right) = 50^2$$
$$h^2\left(\frac{\sin^2 55^\circ - \sin^2 40^\circ}{\sin^2 55^\circ \sin^2 40^\circ}\right) = 50^2$$
$$h^2 = \frac{50^2 \sin^2 55^\circ \sin^2 40^\circ}{\sin^2 55^\circ - \sin^2 40^\circ}$$
$$h = \frac{50\sin 55^\circ \sin 40^\circ}{\sqrt{\sin^2 55^\circ - \sin^2 40^\circ}}$$

c) 52 metres.

2012|2013 Ext 2 Assess Task 1 Solutions

Section I

$iz = i(3-i)$
$= 3i - i^2$
$= 1+3i$
$\therefore \overline{iz} = 1-3i$ Hence C

If $Re(z) = 2$
$x = 2$ ✓
If $|z| = |z-4|$
$x = 2$ ✓
If $z + \bar{z} = 4$
$x+iy+x-iy = 4$
$2x = 4$
$x = 2$ ✓
Hence C

If $w = 1+i$
$\frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i}$
$= \frac{1-i}{2}$
$= \frac{1}{2} - \frac{1}{2}i$
Hence A

4. $\frac{z_1}{z_2} = \frac{2}{4} \text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$
$= \frac{1}{2} \text{cis} \frac{4\pi}{6}$
$= \frac{1}{2} \text{cis} \frac{2\pi}{3}$
Hence A

Section II

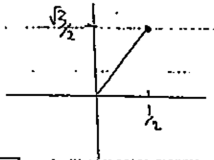
1. (i) $2z + iw$
$= 2(3-2i) + i(1-i)$
$= 6-4i + i - i^2$
$= 7-3i$ (1)

(ii) $\bar{z}w$
$= (3+2i)(1-i)$
$= 3-3i+2i-2i^2$
$= 5-i$ (1)

(iii) $\frac{4}{w}$
$= \frac{4}{1-i} \times \frac{1+i}{1+i}$
$= \frac{4(1+i)}{1-i^2}$
$= \frac{\cancel{4}^{2}(1+i)}{\cancel{2}}$
$= 2+2i$ (2)

(iv) $\left|\overline{\left(\frac{4}{w}\right)}\right|$
$= \sqrt{(2)^2+(-2)^2}$ since $\overline{\left(\frac{4}{w}\right)} = 2-2i$
$= \sqrt{8}$
$= 2\sqrt{2}$ (1)

2 (i) $\frac{1+i\sqrt{3}}{2}$
$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$



$\therefore r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$
$= 1$.

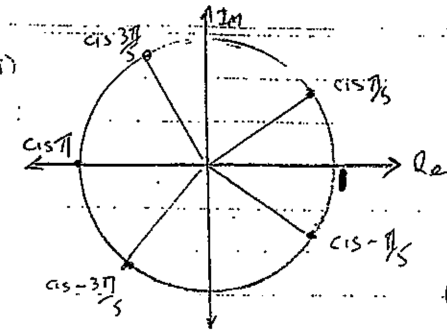
$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$
$= \tan^{-1}(\sqrt{3})$
$= \pi/3$
$\therefore \frac{1+i\sqrt{3}}{2} = \text{cis}\,\pi/3$. (2)

(ii) If $z = \text{cis}\,\pi/3$
$z^3 = \text{cis}\,3\left(\frac{\pi}{3}\right)$
$z^3 = \text{cis}\,\pi$
$= \cos\pi + i\sin\pi$
$= -1 + i.0$
$= -1$ (1)

(iii) $z^7 = z^6 . z$
$= (z^3)^2 . z$
$= (-1)^2 . z$
$= z$
$= \frac{1+i\sqrt{3}}{2}$ or $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (1)

3.(i) Let $z = \cos\theta + i\sin\theta$
If $z^5 = -1$
$\cos 5\theta + i\sin 5\theta = -1 + 0i$
$\therefore \cos 5\theta = -1$
$5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$
$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$
i.e. $\theta = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pi$ (2)
since $\frac{7\pi}{5} = \frac{-3\pi}{5}$ & $\frac{9\pi}{5} = \frac{-\pi}{5}$
The roots are $\text{cis}\frac{\pi}{5}$, $\text{cis}\frac{3\pi}{5}$, $\text{cis}\,\pi$, $\text{cis}\,-\frac{\pi}{5}$, $\text{cis}\,-\frac{3\pi}{5}$

(ii)



(1)

Section II

1(i) Let $\sqrt{-6i} = x+iy$
$0-6i = (x+iy)^2$
$0-6i = x^2-y^2+2xyi$
$x^2-y^2 = 0$, $2xy = -6$
$y = \frac{-3}{x}$
$\therefore x^2 - \left(\frac{-3}{x}\right)^2 = 0$
$x^2 - \frac{9}{x^2} = 0$
$x^4 - 9 = 0$
$(x^2-3)(x^2+3) = 0$
$x = \pm\sqrt{3}$
$y = \mp\frac{3}{\sqrt{3}}$
$y = \mp\sqrt{3}$ (2)
$\therefore \sqrt{-6i} = \sqrt{3} - i\sqrt{3}$ or $-\sqrt{3} + i\sqrt{3}$

(ii) If $z^2 + (1+i)z + 2i = 0$
$z = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 8i}}{2}$
$= \frac{-(1+i) \pm \sqrt{1+2i+i^2-8i}}{2}$

$$= \frac{-(1+i) \pm \sqrt{-6i}}{2}$$

$$= \frac{-(1+i) \pm (\sqrt{3} - i\sqrt{3})}{2}$$

$$= \frac{(\sqrt{3}-1)}{2} - \frac{(\sqrt{3}+1)i}{2}$$ (2)

$$\text{or } -\frac{(\sqrt{3}+1)}{2} + \frac{(\sqrt{3}-1)i}{2}$$

iii) $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$

$\therefore \tan\frac{5\pi}{12} = \tan\left(\pi/6 + \pi/4\right)$

$$= \frac{\tan\pi/6 + \tan\pi/4}{1 - \tan\pi/6 \tan\pi/4}$$

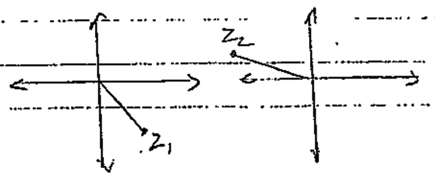
$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - 1/\sqrt{3}}$$

$$= \frac{\frac{\sqrt{3}+1}{3}}{\frac{\sqrt{3}-1}{3}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$ (2)

(iv) Let $z_1 = \frac{(\sqrt{3}-1)}{2} - \frac{(\sqrt{3}+1)i}{2}$

$$z_2 = -\frac{(\sqrt{3}+1)}{2} + \frac{(\sqrt{3}-1)i}{2}$$



$$\arg z_1 = \tan^{-1}\left[\frac{-\frac{(\sqrt{3}+1)}{2}}{\frac{(\sqrt{3}-1)}{2}}\right]$$

$$= -\tan^{-1}\left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right]$$

$$= -\frac{5\pi}{12}$$

$$\arg(z_2) = \pi - \tan^{-1}\left[\frac{\frac{(\sqrt{3}-1)}{2}}{\frac{(\sqrt{3}+1)}{2}}\right]$$

$$= \pi - \frac{\pi}{12}$$

$$= \frac{11\pi}{12}$$

$$\therefore \arg(z_1) + \arg(z_2) = -\frac{5\pi}{12} + \frac{11\pi}{12}$$

$$= \pi/2$$

2. Let $z = x+iy$

$z - A = x+iy - (1-i)$

$= x-1 + i(y+1)$

$z - B = x+iy - (2+i)$

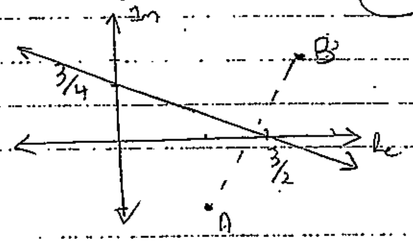
$= x-2 + i(y-1)$

If $|z-A| = |z-B|$

$$\sqrt{(x-1)^2 + (y+1)^2} = \sqrt{(x-2)^2 + (y-1)^2}$$

$\therefore x^2 - 2x + 1 + \cancel{y^2} + 2y + 1 = x^2 - 4x + 4 + \cancel{y^2} - 2y + 1$

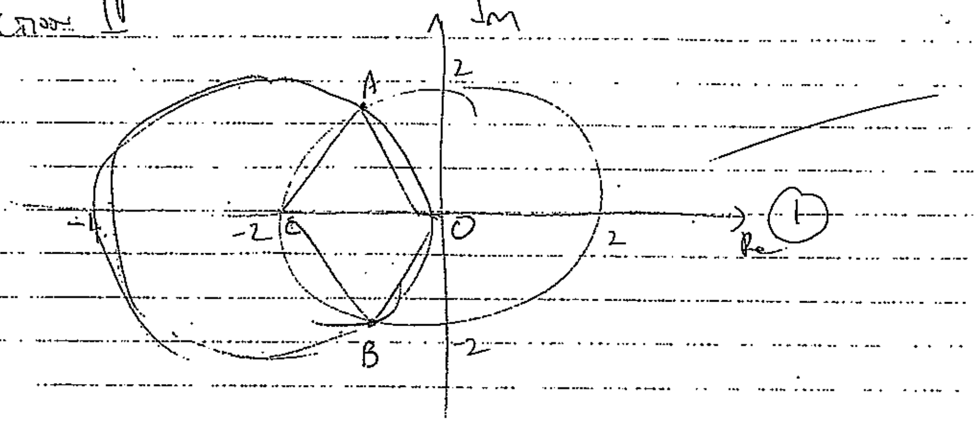
i.e. $2x + 4y - 3 = 0$ (3)



N.B. the locus is the perpendicular bisector of AB

Section II

1/ (i)



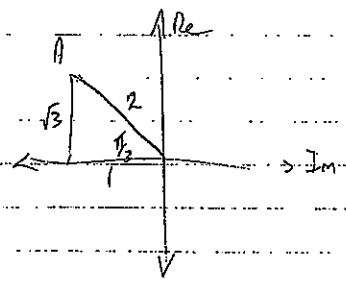
(1)

(ii) Let the points of intersection be A & B
Let the centres of the circles be O & C

$\triangle$s AOC & BOC are equilateral since OC = CA = OB = OC = BC
= 2 units

$\therefore$ A is the point cis $\frac{2\pi}{3}$

& B is the point cis $-\frac{2\pi}{3}$ (2)



$\therefore$ A is $-1 + \sqrt{3}\,i$

& by symmetry B is $-1 - \sqrt{3}\,i$

2/ (i) If $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$

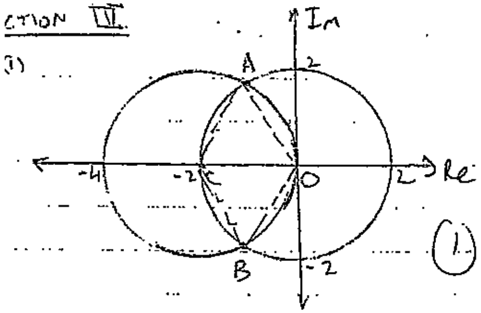
$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

$\therefore z^n + z^{-n} = 2\cos n\theta$

SECTION III.

(i)



(1)

(ii)

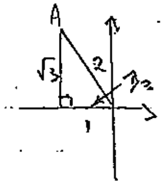
Let the points of intersection be A & B and the centres be O & C

OA = OB = OC = CA = CB = 2 units

∴ Δ AOC is equilateral

∴ A is the point cis $\frac{2\pi}{3}$

& B is the point cis $\frac{-2\pi}{3}$



A is $-1+\sqrt{3}i$

& by symmetry

B is $-1-\sqrt{3}i$

2. (i) $z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

∴ $z^n + z^{-n} = 2\cos(n\theta)$ (1)

(ii) If $z = \cos\theta + i\sin\theta$

$z^n - z^{-n} = 2i\sin(n\theta)$

∴ $z - \frac{1}{z} = 2i\sin\theta$

Now $\left(z-\frac{1}{z}\right)^3 = (2i\sin\theta)^3$

$= -8i\sin^3\theta$

∴ $-8i\sin^3\theta = z^3 - 3z^2\cdot\frac{1}{z} + 3z\cdot\frac{1}{z^2} - \frac{1}{z^3}$

$-8i\sin^3\theta = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$

$-8i\sin^3\theta = 2i\sin3\theta - 3(2i\sin\theta)$

$-8i\sin^3\theta = 2i\sin3\theta - 6i\sin\theta$

∴ $\sin^3\theta = \frac{2i\sin3\theta - 6i\sin\theta}{-8i}$

$= -\frac{\sin3\theta}{4} + \frac{3\sin\theta}{4}$

$= \frac{3\sin\theta}{4} - \frac{\sin3\theta}{4}$ (3)

3. (i) If $z^3 - 1 = 0$

$z^3 + 0z^2 + 0z - 1 = 0$

the sum of the roots is $\frac{-b}{a}$

∴ $1 + \omega + \omega^2 = 0$ (1)

(ii) $(z-\omega)(z-\omega^2)$

$= z^2 - z\omega^2 - z\omega + \omega^3$

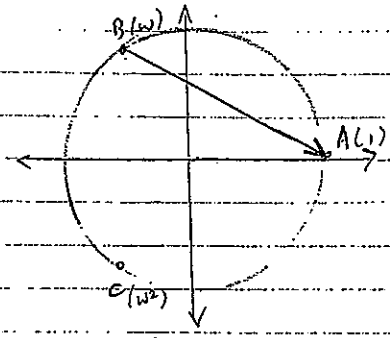
$= z^2 - z(\omega^2+\omega) + \omega^3$

$= z^2 - z(-1) + 1$

since $\omega^2 + \omega = -1$ & if ω is a root of $z^3 - 1 = 0$, $\omega^3 = 1$

∴ $(z-\omega)(z-\omega^2) = z^2 + z + 1$ (2)

iii)



(1)

iv) $\overrightarrow{BA} \cdot \overrightarrow{CA} = (1-\omega)(1-\omega^2)$

$= 1 - \omega^2 - \omega + \omega^3$

$= 1 - (\omega^2 + \omega) + \omega^3$

$= 1 - (-1) + 1$

$= 3$ (1)