Name________________________ Teacher___________________

GOSFORD HIGH SCHOOL

2014

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 1

MATHEMATICS – EXTENSION 1

Time Allowed - 60 minutes plus 5 minutes reading time

- Write using a black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- Answers to the multiple choice are to be done on the answer sheet provided.
- In questions 5-7, show relevant mathematical reasoning and/or calculations.

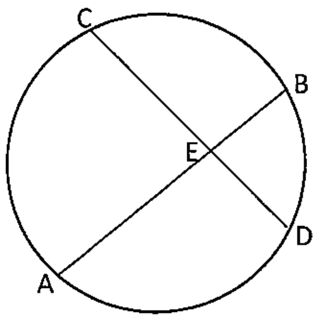
Section 1 **Multiple choice**	**4** questions worth **1 mark each.**	/4
Section 2 **Question 5**	Parametric treatment of the parabola	/12
Question 6	Polynomials	/12
Question 7	Mathematical Induction, Inequalities, Circle Geometry	/12
TOTAL		**/40**

SECTION 1: MULTIPLE CHOICE. Questions are worth 1 mark each. Answer on the multiple choice answer sheet provided.

1) Two lines have gradients of $m_1 = 2$ and $m_2 = \frac{1}{2}$ respectively. The angle between them to the nearest degree is:

A) 51° **B)** 90° **C)** 37° **D)** 72°

2) In the diagram below, $AB = \alpha, BE = \beta, CE = \gamma$ and $ED = \delta$



Which one of the following statements is true?

A) $\gamma\delta = \alpha\beta$ **B)** $\frac{\gamma}{\delta} = \frac{\alpha-\beta}{\beta}$

C) $\gamma(\gamma + \delta) = \beta(\alpha - \beta)$ **D)** $\gamma\delta = \beta(\alpha - \beta)$

3) A polynomial equation has roots α, β and γ where

$$\alpha + \beta + \delta = 3,\ \alpha\beta + \alpha\gamma + \beta\gamma = -2 \text{ and } \alpha\beta\gamma = 4$$

Which polynomial equation has the roots α, β and γ?

A) $x^3 + 3x^2 + 2x + 4$ **B)** $x^3 + 3x^2 + 2x - 4$

C) $x^3 - 3x^2 - 2x - 4$ **D)** $x^3 - 3x^2 - 2x + 4$

4) The point C divides the interval from $A(-1,2)$ to $B(3,5)$ externally in the ratio 3:1. What is the x coordinate of C?

A) 5 **B)** -3 **C)** 2 **D)** 4

SECTION 2: Questions are worth 12 marks each. Answer on your own paper. Start each question on a new sheet of paper. All necessary working must be shown.

5) **a)** $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are 2 points on the parabola $x^2 = 4ay$.
Show that:

i) the equation of the tangent at P on the parabola is
$y = px - ap^2$. **2**

ii) the point of intersection, R, of the tangents at P and Q is
$(a(p+q), apq)$ **2**

iii) the length of SP is $a(1+p^2)$ where S is the focus of the parabola
$x^2 = 4ay$. **2**

b) **i)** Find the equation of the chord of contact of the tangents to the parabola $x^2 = 8y$ from the point $P(4, -6)$. **2**

ii) Hence find the coordinates of the points of contact of these tangents with the given parabola. **2**

c) The variable point $(3t, 2t^2)$ lies on a parabola. Find the Cartesian equation for this parabola. **2**

6) **START A NEW PAGE**

a) $(x-2)$ is a factor of the polynomial $P(x) = 2x^3 + x + a$. Find the value of a. **1**

b) If α, β and $\gamma$are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$, find

i) $\alpha + \beta + \gamma$ **1**

ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

iii) $\alpha\beta\gamma$ **1**

iv) $(\alpha + 1)(\beta + 1)(\gamma + 1)$ **2**

c) **i)** If 3 and -1 are 2 roots of $P(x) = x^3 - 8x^2 + 9x + 18$, express $P(x)$ in terms of three linear factors. **1**

ii) Hence solve $P(x) < 0$ **2**

c) Consider the equation $x^3 + 6x^2 - x - 30 = 0$. One of the roots of this equation is equal to the sum of the other two roots. Find the values of the three roots. **3**

7) START A NEW PAGE

a) Prove by mathematical induction that, for $n \geq 1$,

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \ldots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$$ **3**

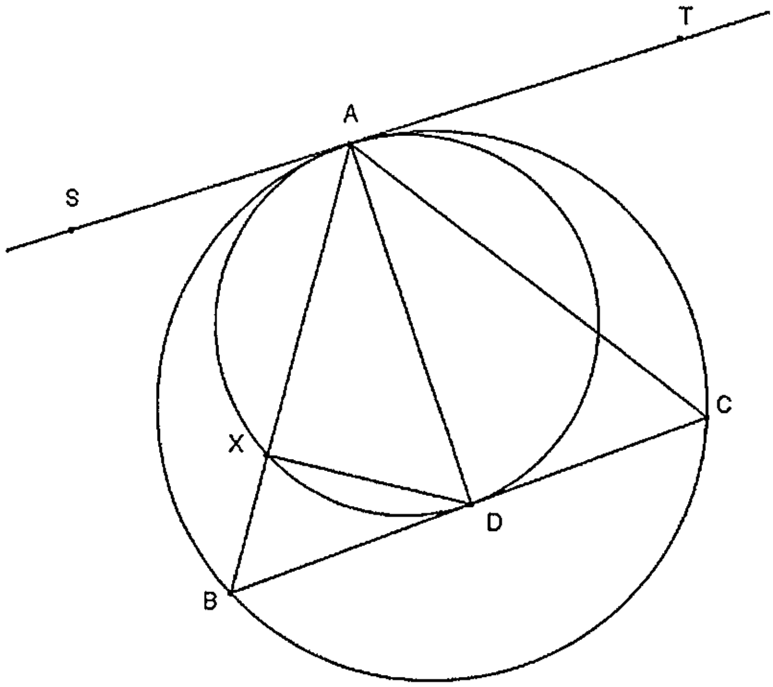
b) Prove by mathematical induction that

$$47^n + 53 \times 147^{n-1}$$

is divisible by 100 for all integers $n \geq 1$. **3**

c) Solve for x, $\frac{x^2-9}{x} > 0$ **2**

d) In the diagram, *ST* is tangent to both the circles at *A*.
The points *B* and *C* are on the larger circle and the line *BC* is tangent to the smaller circle at *D*. The line *AB* intersects the smaller circle at *X*.



Copy the diagram onto your answer sheet.

i) Explain why $\angle AXD = \angle ABD + \angle XDB$ **1**

ii) Explain why $\angle AXD = \angle TAC + \angle CAD$ **1**

iii) Hence show that *AD* bisects $\angle BAC$ **2**

SOLUTIONS

SECTION 1

$\tan\theta = \left| \frac{M_1 - M_2}{1 + M_1 M_2} \right|$

$= \left| \frac{2 - \frac{1}{2}}{1+1} \right|$

$= \frac{3/2}{2}$

$= \frac{3}{4}$

$\therefore \theta = 37^\circ$ — (C)

AE.EB = CE.ED

$(\alpha - \beta)\beta = \gamma . \delta$ — (D)

$x^3 - 3x^2 - 2x - 4 = 0$ — (C)

$(-1, 2) \quad (3, 5)$

$3 : -1$

$\frac{-1 \times -1 + 3 \times 3}{3 + -1} = \frac{1+9}{2}$

$= 5$ — (A)

SECTION 2

) a) $P(2ap, ap^2) \quad Q(2aq, aq^2) \quad x^2 = 4ay$

i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

at $x = 2ap$, $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

ie $M = p$

$\therefore y - y_1 = M(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

ii) $y = px - ap^2$ — (1)

$y = qx - aq^2$ — (2)

$px - ap^2 = qx - aq^2$

$px - qx = a(p^2 - q^2)$

$x(p-q) = a(p-q)(p+q)$

$x = a(p+q)$

Sub into (1)

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\therefore$ point of intersection is $(a(p+q), apq)$

$S(0,a) \quad P(2ap, ap^2)$

$SP = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= \sqrt{a^2(p^4 + 2p^2 + 1)}$

$= \sqrt{a^2(p^2+1)}$

$= a(p^2+1)$

1 $x^2 = 8y$ $\quad 4a = 8$, $a = 2$ $\quad x_0 = 4$, $y_0 = -6$

i) $xx_0 = 2a(y+y_0)$

$4x = 4(y-6)$

$4x = 4y - 24$

$4x - 4y + 24 = 0$

$x - y + 6 = 0$

ii) $x - y + 6 = 0$ — ①

$x^2 = 8y$ — ②

from ② $y = \frac{x^2}{8}$ — ③

Sub ③ into ①

$x - \frac{x^2}{8} + 6 = 0$

$8x - x^2 + 48 = 0$

$x^2 - 8x - 48 = 0$

$(x-12)(x+4) = 0$

$x = 12, -4$

when $x = 12$, $y = 18$

$x = -4$, $y = 2$

∴ points are $(12, 18)$ and $(-4, 2)$

) $(3t, 2t^2)$

$x = 3t$,

$t = \frac{x}{3}$, $\quad y = 2t^2$

$y = 2 \cdot \frac{x^2}{9}$

$9y = 2x^2$

$x^2 = \frac{9}{2}y$

) a) $P(2) = 0$

$16 + 2 + a = 0$

$18 + a = 0$

$a = -18$

b) $x^3 + 2x^2 - 11x - 12 = 0$

i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= -2$

ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= -11$

iii) $\alpha\beta\gamma = -\frac{d}{a}$

$= 12$

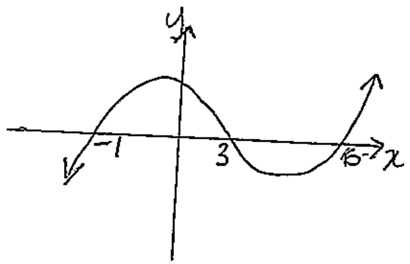
iv) $(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$

$= 12 - 11 - 2 + 1$

$= 0$

$= (x-3)(x+1)(x-6)$

ii)



$P(x) < 0$

when $x < -1$ or $3 < x < 6$

d) $x^3 + 6x^2 - x - 30 = 0$

let the roots be α, β and γ, where $\gamma = \alpha + \beta$

then $\alpha + \beta + (\alpha + \beta) = -6$

$2\alpha + 2\beta = -6$

$\alpha + \beta = -3$ — ①

ie $\gamma = -3$

now $\alpha\beta\gamma = 30$

$-3\alpha\beta = 30$

$\alpha\beta = -10$ — ②

from ① $\beta = -3 - \alpha$

sub into ②

$\alpha(-3-\alpha) = -10$

$-3\alpha - \alpha^2 = -10$

$\alpha^2 + 3\alpha - 10 = 0$

$(\alpha+5)(\alpha-2) = 0$

$\alpha = -5, 2$

$\therefore$ roots are $-5, -3, 2$

a) $1\times5 + 2\times6 + 3\times7 + \dots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$

Show true for $n=1$

LHS $= 1(1+4)$
$= 5$

RHS $= \frac{1}{6}\cdot 1(1+1)(2+13)$
$= \frac{1}{6}\cdot 30$
$= 5$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $1\times5 + 2\times6 + 3\times7 + \dots + k(k+4) = \frac{1}{6}k(k+1)(2k+13)$

Prove true for $n=k+1$, if true for $n=k$

ie Prove $1\times5 + 2\times6 + \dots + k(k+4) + (k+1)(k+5) = \frac{1}{6}(k+1)(k+2)(2k+15)$

LHS $= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5)$

$= \frac{1}{6}(k+1)[k(2k+13) + 6(k+5)]$

$= \frac{1}{6}(k+1)[2k^2 + 13k + 6k + 30]$

$= \frac{1}{6}(k+1)(2k^2 + 19k + 30)$

$= \frac{1}{6}(k+1)(2k+15)(k+2)$

$=$ RHS

$\therefore$ true for $n=k+1$ if true for $n=k$

$\therefore$ true by mathematical induction for all $n \geq 1$.

b) Prove $47^n + 53 \times 147^{n-1}$ divisible by 100 for $n \geq 1$

Show true for $n=1$

$47 + 53 \times 147^0 = 100$ $\therefore$ true for $n=1$

Assume true for $n=k$

ie $47^k + 53 \times 147^{k-1} = 100M$ where M is an integer

Prove true for $n=k+1$, if true for $n=k$

$47^{k+1} + 53 \times 147^k = 47 \cdot 47^k + 53 \times 147 \cdot 147^{k-1}$

$= 47 \cdot 47^k + 47(53 \times 147^{k-1}) + 100(53 \times 147^{k-1})$

$= 47(47^k + 53 \times 147^{k-1}) + 100(53 \times 147^{k-1})$

$= 47(100M) + 100(53 \times 147^{k-1})$ from assumption

$= 100(47M + 53 \times 147^{k-1})$

$\therefore$ divisible by 100 when $n = k+1$, if divisible by 100 when $n = k$.

$\therefore$ by the principle of mathematical induction $47^n + 53 \times 147^{n-1}$ is divisible by 100 for all integers $n \geq 1$

)
$$\frac{x^2 - 9}{x} > 0$$

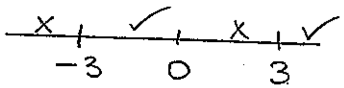
critical points

$x = 0$

solve $\frac{x^2-9}{x} = 0$

$x^2 - 9 = 0$

$x = \pm 3$



test

$x = -4$ $\quad \frac{16-9}{-4} < 0$

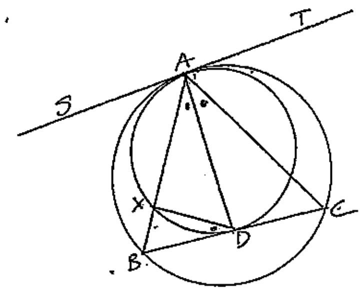
$x = -1$ $\quad \frac{1-9}{-1} > 0$

$x = 1$ $\quad \frac{1-9}{1} < 0$

$x = 4$ $\quad \frac{16-9}{4} > 0$

$\therefore -3 < x < 0$ or $x > 3$

d)



i) $\angle AXD$ is the exterior angle of $\triangle BDX$

$\therefore \angle AXD = \angle ABD + \angle XDB$ (exterior angle of $\triangle$ theorem)

ii) $\angle TAD = \angle TAC + \angle CAD$

$\angle AXD = \angle TAD$ (angle between tangent and chord equals the angle in the alternate segment)

$\therefore \angle AXD = \angle TAC + \angle CAD$

iii) $\angle CAD = \angle AXD - \angle TAC$ (from ii)

and $\angle XDB = \angle AXD - \angle ABD$ (from i)

but $\angle TAC = \angle ABC$ (angle between a chord and tangent equals the angle in the alternate segment)

$\therefore \angle CAD = \angle XDB$

but $\angle XDB = \angle XAD$ (angle between a chord and tangent equals the angle in the alternate segment)

ie $\angle BAD = \angle CAD$

$\therefore$ AD bisects $\angle BAC$