2006 Extension 1 Term 4

QUESTION ONE **MARKS**

(a) (i) Find $\int (3x+5)^8\,dx$. **1**

(ii) Use the substitution $u = 3 - x^2$ to find $\int \frac{x\,dx}{\sqrt{3-x^2}}$. **2**

(b) Express $\sqrt{12}\sin x + 2\cos x$ in the form of $A\cos(x+\alpha)$; where $A \rangle 0$ and $0 \le \alpha \le 2\pi$. **3**

(c) (i) By using one application of Simpson's Rule (i.e. 3 function values), find an approximation for the area bounded by $y = \frac{1}{x}$, $1 \le x \le 2$, and the x axis, leaving your answer as a fraction in simplest terms. **2**

(ii) Using integration, find an expression for the exact area in part (i). **1**

(iii) Using your answers to parts (i) and (ii) above , find an approximation for e correct to 4 decimal places. **1**

QUESTION TWO

(a) Differentiate $\tan(x^2)$. **1**

(b) (i) Differentiate $\left(x\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} \right)$ and simplify. **3**

(ii) Hence evaluate $\int_0^4 \sin^{-1}\frac{x}{4}\,dx$. **2**

(c) (i) Given that (0,0) is the only stationary point of $y = x\sin^{-1}\frac{x}{4}$, justify whether it is a minimum or maximum turning point. **2**

(ii) Hence , or otherwise, sketch the curve $y = x\sin^{-1}\frac{x}{4}$. **2**

QUESTION THREE **MARKS**

(a) If $x = a + b$ and $y = a - b$,

(i) Show that $\cos x + \cos y = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$. **2**

(ii) Hence find the general solutions to $\cos 4\theta + \cos 3\theta + \cos 2\theta + \cos \theta = 0$. **4**

(b)

The region in the first quadrant bounded by the curve $y = 2\tan^{-1}x$ and the y-axis between $y = 0$ and $y = \frac{\pi}{2}$ is rotated through one complete revolution about the y-axis. Find the exact volume of the solid of revolution so formed. **4**

QUESTION FOUR

(a)

Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 x\cos^2 x \, dx$. **3**

(b) A function is defined as $f(x) = 1 - \cos\frac{x}{2}$, where $0 \le x \le a$. **2**

(i) Find the largest value of a for which the inverse function $f^{-1}(x)$ exists.

(ii) Find the equation of $f^{-1}(x)$. **1**

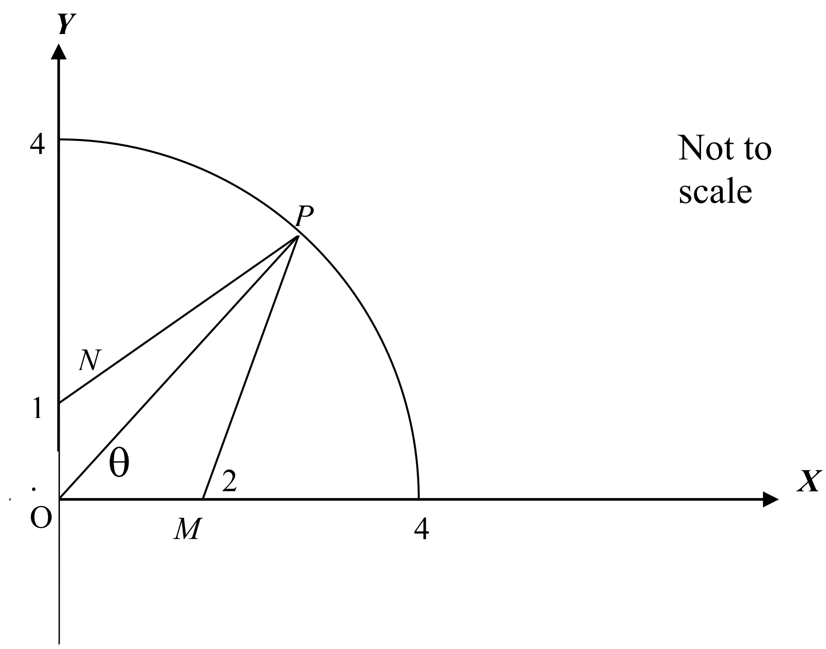
(iii) Sketch the graph of $y = f^{-1}(x)$. **2**

(iv) Find the area enclosed between the curve $y = f^{-1}(x)$, the x-axis and $x = 2$. **2**

QUESTION FIVE **MARKS**

(a) Find the value of $\lim_{x \to 0} \dfrac{x^2}{1 - \cos 2x}$. **3**

(b)



The diagram shows the part of the circle $x^2 + y^2 = 16$ that lies in the first quadrant. The point $P(x, y)$ is on the circle with centre at the origin O. M is on the x-axis at $x = 2$ and N is on the y-axis at $y = 1$.
The size of angle MOP is θ radians.

(i) Show that the area, A, of the quadrilateral $OMPN$ is given by $A = 4\sin\theta + 2\cos\theta$. **2**

(ii) Find the value of $\tan\theta$ for which A is maximum. **3**

(iii) Hence determine in surd form the coordinates of P for which A is maximum. **2**

QUESTION SIX **MARKS**

The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

(i) Find the equation of the chord PQ and hence, or otherwise, show that $pq = \frac{-b}{a}$. **3**

(ii) Prove that $p^2 + q^2 = 4m^2 + \frac{2b}{a}$. **2**

(iii) Given that the equation of the normal to the parabola at P is **2**

$$x + py = 2ap + ap^3$$

and that N, the point of intersection of the normals at P and Q, has co-ordinates $\left[-apq(p+q),\ a(2 + p^2 + pq + q^2)\right]$,
express these co-ordinates in terms of a, m, b.

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line. **3**

Verify that this line is a normal to the parabola.

END OF PAPER

Solutions to T4 3u 2006

1a) $\frac{(3x+5)^9}{27} + c$ # 1

ii) $-\frac{1}{2}\int \frac{du}{\sqrt{u}}$ $\quad u = 3-x^2$, $du = -2x\,dx$

$= -\frac{1}{2}\,\frac{u^{\frac12}}{\frac12}$ 1

$= -\sqrt{3-x^2} + c$ # 1

b) $\sqrt{12}\sin x + 2\cos x = A\cos x\cos\alpha - A\sin x\sin\alpha$

$A = \sqrt{(\sqrt{12})^2 + 2^2} = \sqrt{16} = 4$ $\quad$ $A = 4$ 1

$-4\sin\alpha = \sqrt{12}$ $\quad 2 = 4\cos\alpha$

since $\sin\alpha < 0$, $\cos\alpha > 0$, α is in 4th Quad 1

$\tan\alpha = -\frac{\frac{\sqrt{12}}{4}}{\frac12} = -\sqrt{3}$ $\quad \alpha = \frac{5\pi}{3}$ 1

$\sqrt{12}\sin x + 2\cos x = 4\cos(x + \frac{5\pi}{3})$ #

c i)

x	1	$1\frac12$	2
$y = \frac1x$	1	$\frac23$	$\frac12$

Area $\doteqdot \frac{2-1}{6}[1 + 4\times\frac23 + \frac12] = \frac16(\frac33 + \frac83 + \frac12)$ 1

$= \frac{25}{36}$ 1

ii) Exact Area $= \int_1^2 \frac1x\,dx = [\ln x]_1^2 = \ln 2 - \ln 1$

$= \ln 2$ 1

iii) $\ln 2 \doteqdot \frac{25}{36}$

$e^{\frac{25}{36}} = 2$ $\quad \therefore e = 2^{\frac{36}{25}} = 2.7132$ (4dp) 1

2a) $2x\sec(x^2)$ 1

b i) $x \cdot \frac{\frac14}{\sqrt{1 - \frac{x^2}{16}}} + \sin^{-1}\frac{x}{4} + \frac{-2x}{2\sqrt{16-x^2}}$

$= \frac{\frac14 x}{\frac14\sqrt{16-x^2}} + \sin^{-1}\frac{x}{4} - \frac{x}{\sqrt{16-x^2}}$

$= \sin^{-1}\frac{x}{4}$

ii) $\int_0^4 \sin^{-1}\frac{x}{4}\,dx = \left[x\sin^{-1}\frac{x}{4} + \sqrt{16-x^2}\right]_0^4$

$= 4\sin^{-1}1 + 0 - 0 - \sqrt{16} = 2\pi - 4$

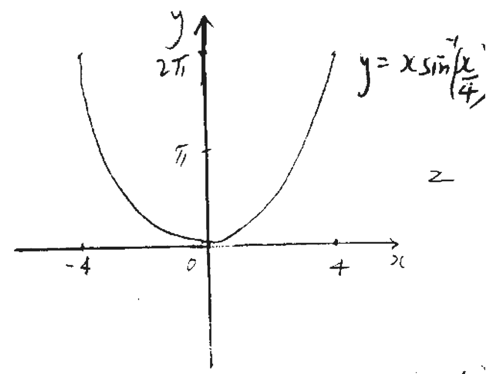
iii) $y' = \frac{x}{\sqrt{16-x^2}} + \sin^{-1}\frac{x}{4}$ (from b i)

x	-0.1	0	0.1
y'	-0.5	0	0.5
slope	⌣	—	⌣

1

Since y is continuous and differentiable for $-1 \le x \le 1$ and y' changes signs $(-0+)$ from $x = -0.1$ to $x = 0.1$

$\therefore$ min turning point at $(0,0)$ 1



2

3a) $\cos x + \cos y = \cos(a+b) + \cos(a-b)$

$= \cos a\cos b - \sin a\sin b + \cos a\cos b + \sin a\sin b$

$= 2\cos a\cos b = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$

Since $x = a+b$, $y = a-b$ $\}$ $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$

ii) $(\cos4\theta + \cos2\theta) + (\cos3\theta + \cos\theta) = 0$

$2\cos3\theta\cos\theta + 2\cos2\theta\cos\theta = 0$ (by pa

$2\cos\theta(\cos3\theta + \cos2\theta) = 0$

$2\cos\theta(2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}) = 0$

$4\cos\theta\cos\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$ 1

$\cos\theta = 0$ $\quad \theta = \pm\frac{\pi}{2} + 2n\pi$ $\quad n \in J$

$\cos\frac{5\theta}{2} = 0$ $\quad \frac{5\theta}{2} = \pm\frac{\pi}{2} + 2n\pi$ $\therefore \theta = \pm\frac{\pi}{5} + \frac{4n\pi}{5}$

$\cos\frac{\theta}{2} = 0$ $\quad \frac{\theta}{2} = \pm\frac{\pi}{2} + 2n\pi$ $\therefore \theta = \pm\pi + 4n\pi$

$\therefore$ general solns are:

$\theta = \pm\frac{\pi}{2} + 2n\pi,\ \pm\frac{\pi}{5} + \frac{4n\pi}{5}$ or $\pm\pi + 4n\pi$ $\quad n \in J$

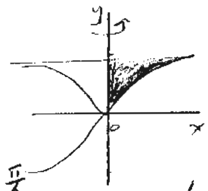
3b) $y = 2\tan^{-1}x \qquad x = \tan\frac{y}{2}$

Vol of revol $= \pi\int_0^{4} x^2\,dy$

1 $= \pi\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{y}{2}\right)dy = \pi\int_0^{\pi/2}\left(\sec^2\frac{y}{2} - 1\right)dy$ 1

1 $= \left[\pi\left(\tan\frac{y}{2}\right)2 - \pi y\right]_0^{\frac{\pi}{2}} = 2\pi\tan\frac{\pi}{4} - \frac{\pi^2}{2}$ 1

1 $= \underline{\underline{2\pi - \frac{\pi^2}{2} \text{ unit}^3}}$



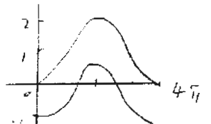
4a) $\int_0^{\frac{\pi}{6}}\left(\frac{1}{2}\sin 2x\right)^2dx = \frac{1}{4}\int_0 \frac{1-\cos 4x}{2}\,dx$ 1

1 $= \left[\frac{x}{8} - \frac{\sin 4x}{8\times 4}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{48} - \frac{\sin\frac{2\pi}{3}}{32}$

$= \frac{\pi}{48} - \frac{\sqrt{3}}{64}$ # 1

b) $y = 1 - \cos\frac{x}{2}$

Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$ 1



i) largest a is 2π 1

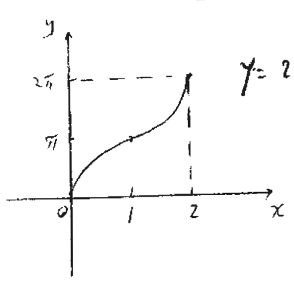
ie $f^{-1}(x)$ exists for $0 \le x \le 2\pi$

ii) For inverse: $x = 1 - \cos\frac{y}{2}$

$\cos\frac{y}{2} = 1 - x$

$\frac{y}{2} = \cos^{-1}(1-x)$

$\underline{y = 2\cos^{-1}(1-x)}$ 1



$y = 2\cos^{-1}(1-x)$

2

iv) Area $= \frac{1}{2} \times$ Area of Rectangle

$= \frac{1}{2}\times 2\times 2\pi = \underline{2\pi \text{ units}^2}$ 2

Alternatively

Area $= 4\pi - \int_0^{2\pi}\left(1 - \cos\frac{y}{2}\right)dy$ 1

$= 4\pi - \left[y - 2\sin\frac{y}{2}\right]_0^{2\pi}$

$= 4\pi - (2\pi - 0 - 0 + 0) = \underline{2\pi \text{ units}}$ 1

Q5

a) $\lim_{x\to 0} \frac{x^2}{2\sin^2 x} = \lim_{x\to 0} \frac{1}{2}\left(\frac{x}{\sin x}\right)^2 = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$ 1 1

b) i) Area $\Delta OPN = \frac{1}{2}\times 1\times 4\cos\theta = 2\cos\theta$

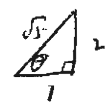
Area $\Delta OPM = \frac{1}{2}\times 2\times 4\sin\theta = 4\sin\theta$

Area $OMPN(A) = 4\sin\theta + 2\cos\theta$ 1

ii) $A' = 4\cos\theta - 2\sin\theta = 0 \qquad 0 < \theta < \frac{\pi}{2}$

$4\cos\theta = 2\sin\theta$

$\tan\theta = 2$ 1



$A'' = -4\sin\theta - 2\cos\theta = -\frac{4\times 2}{\sqrt{5}} - \frac{2}{\sqrt{5}} < 0$ (+

Since A, A', A'' are continuous and differentiable for $0 < \theta < \frac{\pi}{2}$ and only 1 TP $\therefore \tan\theta = 2$ gives max A 1

iii) $P = (4\cos\theta, 4\sin\theta) = \left(\frac{4}{\sqrt{5}}, \frac{4\times 2}{\sqrt{5}}\right)$ 1

$= \left(\frac{4}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$ 1

Q6

(i) Slope $PQ = \frac{a(q^2-p^2)}{2a(q-p)} = \frac{a(q+p)(q-p)}{2a(q-p)}$

$= \frac{q+p}{2}$ # 1

EQ of PQ $\quad y - ap^2 = \left(\frac{q+p}{2}\right)(x - 2ap)$

$y = \frac{x}{2}(p+q) - apq - ap^2 + ap^2$

$y = \frac{x}{2}(p+q) - apq$ 1

At P and Q: $y = mx + b$ and $y = \frac{x}{2}(p+q) - apq$

$$mx + b = \frac{x}{2}(p+q) - apq$$

$$m = \frac{p+q}{2} \qquad b = -apq \qquad \therefore pq = \frac{-b}{a} \quad \#$$

ii) Since $p+q = 2m$, $pq = -\frac{b}{a}$

$$p^2 + q^2 = (p+q)^2 - 2pq = (2m)^2 - 2\left(-\frac{b}{a}\right) = 4m^2 + \frac{2b}{a} \quad \#$$

iii)

$$x = -apq(p+q) = -a\left(\frac{-b}{a}\right)2m \qquad \therefore x = 2mb \quad \#$$

$$y = a(2 + p^2 + pq + q^2) = a(2 + (p+q)^2 - pq) = a\left(2 + 4m^2 - \frac{-b}{a}\right)$$

$$y = a\left(2 + 4m^2 + \frac{b}{a}\right) = 2a + 4am^2 + b \quad \#$$

iv) Since $b = \frac{x}{2m}$ $\therefore y = 2a + 4am^2 + \frac{x}{2m}$

Gradient of chord PQ is fixed $\therefore$ m is a constant but a is also a constant $\therefore$ this is an equation of a straight line in gradient-intercept form.

$$x - 2my = -4am - 8am^3$$

$$x + (-2m)y = 2a(-2m) + a(-2m)^3$$

Compare this equation with equation of normal to parabola at 1

$$x + py = 2ap + ap^3$$

p has been replaced by $(-2m)$

$\therefore$ locus of N is a straight line which is normal to the parabola at point $(2a \times -2m,\ a(-2m)^2)$

ie at $(-4am, 4am^2)$