NORTH SYDNEY BOYS HIGH SCHOOL

2013 YEAR 12 ASSESSMENT TASK 1

Mathematics Extension 1

General Instructions

- **Working time – 60 minutes**
- **Reading time – 5 minutes**

- Answer all questions on the lined paper in the booklet provided.
- Write using blue or black pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Each new question is to be started on a **new page** with the exception of the multiple choice which can be answered on the same page.
- Attempt all questions.

Class Teacher: Please colour

- O Mr Berry
- O Mr Fletcher
- O Mr Lam
- O Mr Lin
- O Mr Lucas
- O Ms Ziaziaris

Student No./Name:___________________________

(To be used by the exam markers only.)

Question No	1-4	5	6	7	8	9	10	Total	%
Mark	/4	/8	/12	/3	/8	/7	/8	/50	/100

ANSWER ALL QUESTIONS IN THE ANSWER BOOKLET PROVIDED

QUESTION 1

A parabola has equation $(x+3)^2 = -16y$.

The coordinates of the focus will be

A. (-3, 4)
B. (-3, -4)
C. (3, 4)
D. (3, -4)

QUESTION 2

Find the general solution of $2\cos x = \sqrt{3}$.

A. $2\pi n \pm \frac{\pi}{6}$

B. $\pi n \pm \frac{\pi}{6}$

C. $2\pi n + \frac{\pi}{6}$

D. $\pi n + \frac{\pi}{6}$

QUESTION 3

How many points of intersection does the line $y = mx - 2m^2$ have with the parabola $x^2 = 8y$.

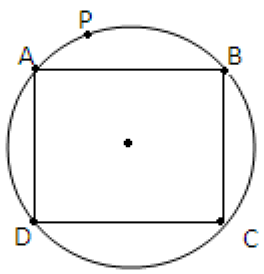
A. Unable to distinguish
B. 0
C. 1
D. 2

exam continues overleaf.....................

QUESTION 4

ABCD is a square inscribed in a circle. P is a point on the minor arc AB. Find the size of

$\angle APB.$



A. 90 B. 135 C. 45 D. 120

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QUESTION 5

(a) Find the Cartesian equation of the curve whose parametric equations are 2

$x = 2\cos\theta$

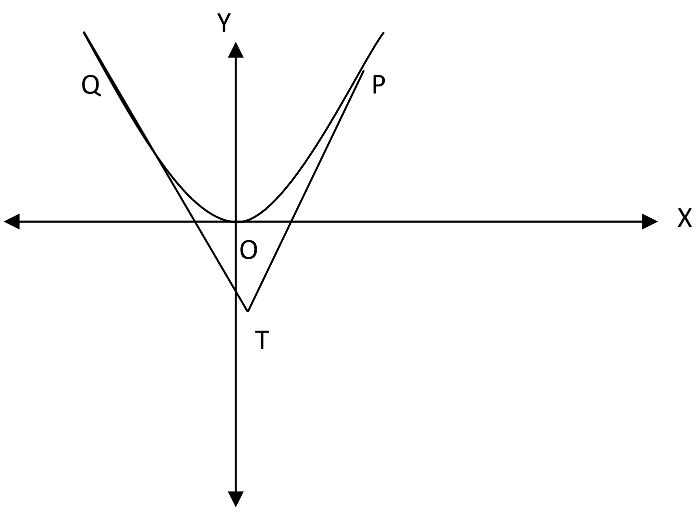
$y = \sqrt{3}\sin\theta$

(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.
The tangents at P and Q meet at T.

(i) Derive the equation of the tangent at P. 2

(ii) Show that the coordinates of T are $\left[a(p+q), apq\right]$. 3

(iii) If T lies on the line $y = -2a$ find a relationship between p and q. 1



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QUESTION 6

(a) (i) Prove that $\dfrac{\sin 3x}{\cos 2x \cos x} = \tan 2x + \tan x$ 2

(ii) Hence, solve $\tan 2x + \tan x = 0$ for $0 \le x \le 2\pi$ 2

(b) Given that $\tan\alpha = -\dfrac{2}{3}$ and $\sin\beta = \dfrac{1}{4}$ and $\dfrac{\pi}{2} < \alpha < \beta < \pi$.

Find the exact value for

(i) $\sin 2\alpha$ 2

(ii) $\cos(\beta - \alpha)$ 3

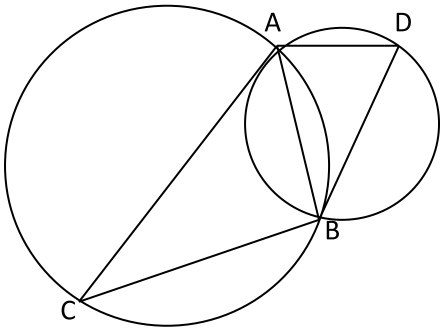
(c) The lines $y = mx$ and $y = 2mx$, where $m > 0$ are inclined to each other at an angle θ such that $\tan\theta = \dfrac{1}{3}$. Find the possible values of m . 3

QUESTION 7

AC is a tangent to circle ABD and BD is a tangent to circle ABC. AB is a common chord, meeting the tangents at A and B.

(i) Copy or trace diagram into your booklet

(ii) Prove AD ll CB 3



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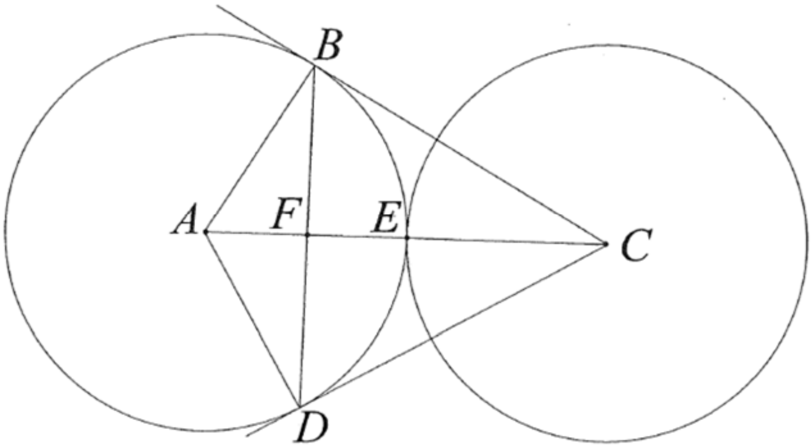
Question 8



Two circles with equal radii and centres A and C touch externally at E as shown in the diagram.

The lines BC and DC are tangents from C to the circle with centre A.

(i)	Copy or trace the diagram into your booklet.	
(ii)	Explain why ABCD is a cyclic quadrilateral.	2
(iii)	Show that E is the centre of the circle that passes through A, B, C and D.	2
(iv)	Show that $\angle BCA = \angle DCA = 30^\circ$.	2
(v)	Deduce that ΔBCD is equilateral.	2

exam continues overleaf......................

QUESTION 9

(a) Consider the quadratic function $x^2-(k+2)x+4=0$. For what values of k is the quadratic function positive definite. 2

(b) The quadratic equation $ax^2+bx+c=0$ has roots $x=\tan\alpha$ and $x=\tan\beta$.

(i) Show that $\tan(\alpha+\beta)=\dfrac{b}{c-a}$ 2

(ii) Show that $\tan^2(\alpha-\beta)=\dfrac{b^2-4ac}{(a+c)^2}$ 3

Question 10

The straight line $y=mx+b$ meets the parabola $x^2=4ay$ at the points $P(2ap,ap^2)$ and $Q(2aq,aq^2)$.

(a) Derive the equation of the chord PQ and hence or otherwise, show that $pq=-\dfrac{b}{a}$. 2

(b) Prove that $p^2+q^2=4m^2+\dfrac{2b}{a}$ 2

(c) Given that the equation of the normal to the parabola at P is $x+py=ap^3+2ap$, and that, N, the point of intersection of the normals at P and Q has the coordinates $[\ -apq(p+q),a(2+p^2+pq+q^2)\ ]$, express these coordinates in the terms of a, m and b. 2

(d) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N, and show that it is a normal to the parabola. 2

END OF EXAMINATION

Suggested Solutions

Section I

(Lin)

1. (B) **2.** (A) **3.** (C) **4.** (B)

Section II

Question 5 (Berry)

(a) (2 marks)

$$\begin{cases} x = 2\cos\theta \\ y = \sqrt{3}\sin\theta \end{cases}$$

Make $\cos\theta$ and $\sin\theta$ the subject,

$$\begin{cases} \frac{x}{2} = \cos\theta \\ \frac{y}{\sqrt{3}} = \sin\theta \end{cases}$$

Squaring both equations and adding,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = \cos^2\theta + \sin^2\theta$$
$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

(b) i. (2 marks)

$$x^2 = 4ay$$
$$y = \frac{1}{4a}x^2$$
$$\frac{dy}{dx} = \frac{2}{4a}x = \frac{1}{2a}x\bigg|_{x=2ap}$$
$$= \frac{2ap}{2a} = p$$

$\therefore$ gradient of tangent at $P(2ap, ap^2)$ is p. Using point-gradient formula,

$$\frac{y - ap^2}{x - 2ap} = p$$
$$y - \underset{+ap^2}{ap^2} = px - \underset{+ap^2}{2ap^2}$$
$$y = px - ap^2$$

ii. (3 marks) Similarly, equation of tangent at Q is $y = qx - aq^2$. Point of intersection:

$$\begin{cases} y = px - ap^2 \\ y = qx - aq^2 \end{cases}$$
$$px - ap^2 = qx - aq^2$$
$$px - qx = ap^2 - aq^2$$
$$x(p-q) = a(p-q)(p+q)$$
$$x = a(p+q)$$

Finding y coordinate,

$$y = p(a)(p+q) - ap^2$$
$$= ap^2 + apq - ap^2 = apq$$
$$\therefore T\left(a(p+q), apq\right)$$

iii. (1 mark)
As T lies on $y = -2a$,

$$apq = -2a$$
$$\therefore pq = -2$$

Question 6 (Lam)

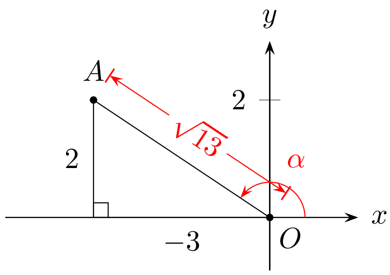
(a) i. (2 marks)

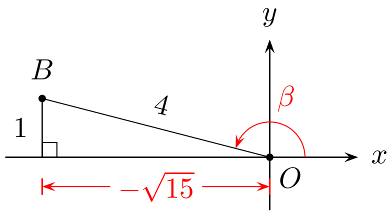
$$\frac{\sin 3x}{\cos 2x\cos x} = \frac{\sin(2x+x)}{\cos 2x\cos x}$$
$$= \frac{\sin 2x\cos x + \cos 2x\sin x}{\cos 2x\cos x}$$
$$= \frac{\sin 2x\cancel{\cos x}}{\cos 2x\cancel{\cos x}} + \frac{\cancel{\cos 2x}\sin x}{\cancel{\cos 2x}\cos x}$$
$$= \tan 2x + \tan x$$

ii. (2 marks)

$$\tan 2x + \tan x = \frac{\sin 3x}{\cos 2x\cos x} = 0$$
$$\therefore \sin 3x = 0 \quad (0 \le 3x \le 6\pi)$$
$$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$$
$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$$

(b) $\tan\alpha = -\frac{2}{3}$, $\sin\beta = \frac{1}{4}$ implies second quadrant:





i. (2 marks)

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= 2\times\frac{2}{\sqrt{13}}\times\left(-\frac{3}{\sqrt{13}}\right) \\ &= -\frac{12}{13}\end{aligned}$$

ii. (3 marks)

$$\begin{aligned}&\cos(\beta-\alpha) \\ &= \cos\beta\cos\alpha+\sin\beta\sin\alpha \\ &= \left(-\frac{\sqrt{15}}{4}\right)\times\left(-\frac{3}{\sqrt{13}}\right) \\ &\quad+\left(\frac{1}{4}\right)\times\left(\frac{2}{\sqrt{13}}\right) \\ &= \frac{3\sqrt{15}+2}{4\sqrt{13}}\end{aligned}$$

(c) (3 marks)

$$\begin{cases} y = mx \\ y = 2mx \\ \tan\theta = \frac{1}{3}\end{cases}$$

The angle between two lines:

$$\begin{aligned}\tan\theta &= \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{2m-m}{1+2m^2}\right| \\ &= \left|\frac{m}{1+2m^2}\right| = \frac{1}{3}\end{aligned}$$

Case 1:

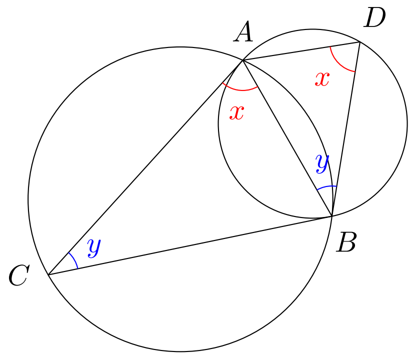
$$\begin{aligned}\frac{m}{1+2m^2} &= \frac{1}{3} \\ 3m &= 1+2m^2 \\ 2m^2-3m+1 &= 0 \\ (2m-1)(m-1) &= 0 \\ \therefore m &= \frac{1}{2}, 1\end{aligned}$$

Case 2:

$$\begin{aligned}\frac{m}{1+2m^2} &= -\frac{1}{3} \\ -3m &= 1+2m^2 \\ 2m^2+3m+1 &= 0 \\ (2m+1)(m+1) &= 0 \\ \therefore m &= -\frac{1}{2}, -1\end{aligned}$$

As $m > 0$, $m = \frac{1}{2}$ or 1 only.

Question 7 (Lin) (3 marks)



- Let $\angle CAB = x$. Then $\angle ADB = x$ ($\angle$ in the alternate segment)
- Let $\angle ABD = y$. Then $\angle ACB = y$ ($\angle$ in the alternate segment)
- In $\triangle ABC$

$$\angle ABC = 180^\circ - (x+y)$$

($\angle$ sum of $\triangle$)

- Similarly in $\triangle ABD$,

$$\angle BAD = 180^\circ - (x+y)$$

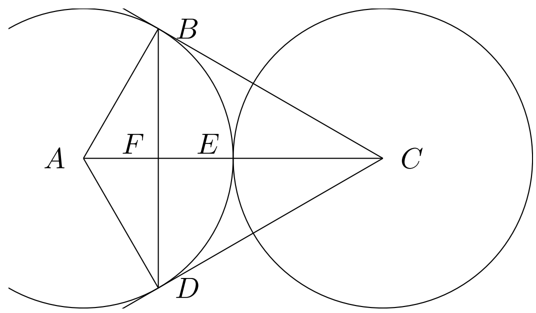
- $\therefore AD \parallel BC$:

$$\angle BAD = \angle ABC = 180^\circ - (x+y)$$

and $\angle BAD$ and $\angle ABC$ are alternate angles.

Question 8 (Fletcher)

(a) (2 marks)



- $\angle ADC = 90^\circ$ (radius is $\perp$ to tangent at point of contact)
- Similarly, $\angle ADC = 90^\circ$.
- Hence $ADCB$ is a cyclic quadrilateral, as $\angle ABC + \angle ADC = 180^\circ$ – one pair of opposite $\angle$ supplementary.

(b) (2 marks)

- $AE = CE$ (given, as both circle have same radii).
- As $\angle ABC = \angle ADC = 90^\circ$, then AC must be the diameter ($\angle$ in a semicircle)

(c) (2 marks)

- Let $AD = x$. Then $AC = 2x$.
- In $\triangle DCA$, $\sin \angle DCA = \frac{x}{2x} = \frac{1}{2}$
- $\therefore \angle DCA = 30^\circ$. Similarly for $\angle BCA$.

(d) (2 marks)
In $\triangle CBF$ and $\triangle CDF$,

- FC (common)
- $\angle BCF = \angle DCF = 30^\circ$ (previously proven)
- $DC = BC$ (tangents drawn from external point are equal)

$\therefore \triangle CBF \equiv \triangle CDF$ (SAS).

Hence $\angle CBD = \angle CDB$, and

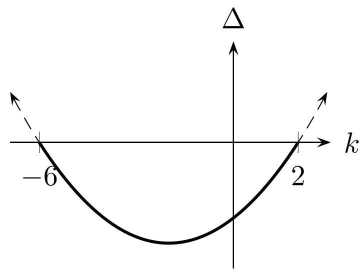
$$\begin{aligned} 2\angle CBD + 60^\circ &= 180^\circ \\ 2\angle CBD &= 120^\circ \\ \therefore \angle CBD &= 60^\circ = \angle CDB \end{aligned}$$

Hence $\triangle CBD$ is equilateral.

Question 9 (Lucas)

(a) (2 marks) $a > 0$, $\Delta < 0$ for positive definite.

$$\begin{aligned} (k+2)^2 - 4(1)(4) &< 0 \\ (k+2)^2 - 16 &< 0 \\ (k+2-4)(k+2+4) &< 0 \\ (k-2)(k+6) &< 0 \end{aligned}$$



$$\therefore -6 < k < 2$$

(b) i. (2 marks) Let $ax^2 + bx + c = 0$ have roots $x_1 = \tan\alpha$ and $x_2 = \tan\beta$

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{x_1 + x_2}{1 - x_1 x_2} \end{aligned}$$

Sum of roots:

$$x_1 + x_2 = -\frac{b}{a}$$

Product of roots:

$$x_1 x_2 = \frac{c}{a}$$

$$\begin{aligned} \therefore \tan(\alpha+\beta) &= \frac{-\frac{b}{a}}{1 - \frac{c}{a}} = \frac{-\frac{b}{a}}{\frac{a-c}{a}} \\ &= -\frac{b}{a-c} = \frac{b}{c-a} \end{aligned}$$

ii. (3 marks)

$$\begin{aligned}
&\tan^2(\alpha-\beta)\\
&=\left(\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right)^2\\
&=\frac{\tan^2\alpha-2\tan\alpha\tan\beta+\tan^2\beta}{1+2\tan\alpha\tan\beta+\tan^2\alpha\tan^2\beta}\\
&=\frac{{x_1}^2+{x_2}^2-2x_1x_2}{1+2x_1x_2+(x_1x_2)^2}\\
&=\frac{(x_1+x_2)^2-2x_1x_2-2x_1x_2}{1+2x_1x_2+(x_1x_2)^2}\\
&=\frac{(x_1+x_2)^2-4x_1x_2}{1+2x_1x_2+(x_1x_2)^2}\\
&=\frac{\left(-\frac{b}{a}\right)^2-\frac{4c}{a}}{1+\frac{2c}{a}+\frac{c^2}{a^2}}\times\frac{a^2}{a^2}\\
&=\frac{b^2-4ac}{a^2+2ac+c^2}\\
&=\frac{b^2-4ac}{(a+c)^2}
\end{aligned}$$

Question 10 (Ziaziaris)

(a) (2 marks) Equation of chord PQ:

$$\begin{aligned}
\frac{y-ap^2}{x-2ap}&=\frac{ap^2-aq^2}{2ap-2aq}\\
\frac{y-ap^2}{x-2ap}&=\frac{\not{a}\cancel{(p-q)}(p+q)}{2\not{a}\cancel{(p-q)}}\\
y-ap^2&=\frac{p+q}{2}x-2ap\times\frac{p+q}{2}\\
y-ap^2&=\frac{p+q}{2}x-ap^2-apq\\
y&=\frac{p+q}{2}x-apq
\end{aligned}$$

As the straight line $y=mx+b$ is also the chord PQ, equate y intercepts:

$$\begin{aligned}
\therefore -apq&=b\\
\therefore pq&=-\frac{b}{a} \qquad (10.1)
\end{aligned}$$

(b) (2 marks)
Previously, the y intercepts were equated. Equate gradients:

$$\begin{aligned}
m&=\frac{p+q}{2}\\
2m&=p+q \qquad (10.2)\\
\therefore p^2+q^2&=(p+q)^2-2pq\\
&=(2m)^2-2\left(-\frac{b}{a}\right)\\
&=4m^2+\frac{2b}{a}\\
\therefore p^2+q^2&=4m^2+\frac{2b}{a} \qquad (10.3)
\end{aligned}$$

(c) (2 marks)

$$\begin{cases} x=-apq(p+q) & (1)\\ y=a\left(2+p^2+pq+q^2\right) & (2)\end{cases}$$

(1): substitute from (10.2) and (10.3)

$$x=-a\left(-\frac{b}{a}\right)\times 2m=2bm$$

(2): substitute from (10.1) and (10.3)

$$\begin{aligned}
y&=a\left(2+4m^2+\frac{2b}{a}+-\frac{b}{a}\right)\\
&=a\left(2+4m^2+\frac{b}{a}\right)\\
&=2a+4am^2+b\\
\therefore\ & N\left(2bm,4am^2+2a+b\right)
\end{aligned}$$

(d) (2 marks) – m fixed (constant).

$$\begin{cases} x=2bm & (3)\\ y=4am^2+2a+b & (4)\end{cases}$$

Change subject of (3) to b: $b=\frac{x}{2m}$, and substitute to (4):

$$\begin{aligned}
y=4am^2+2a+\frac{x}{2m}2my&=8am^3+4am+x\\
x+(-2m)y&=-8am^3-4am\\
x+(-2m)y&=a(-2m)^3+2a(-2m)
\end{aligned}$$

Which resembles the form

$$x+py=ap^3+2ap$$

Hence the locus is a normal to the parabola where $p=-2m$.