NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 1)

2014 HSC Course Assessment Task 1

December 5, 2013

General instructions

- Working time – 50 minutes.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 5)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

- ◯ 12M3A – Mr Zuber
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Lowe
- ◯ 12M4A – Ms Ziaziaris
- ◯ 12M4B – Mr Lam
- ◯ 12M4C – Mr Ireland

Marker's use only.

QUESTION	1-5	6	7	8	9	10	Total	%
MARKS	/5	/8	/6	/7	/5	/6	/37	

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. What is the equation of the parabola with focus $S(4, 0)$ and directrix $x = -2$? **1**

(A) $y^2 = 24(x - 4)$ (C) $y^2 = -12(x - 1)$

(B) $y^2 = -24(x - 4)$ (D) $y^2 = 12(x - 1)$

2. Which of the following would represent solutions to $\cos 2x + \sin x = 0$? **1**

(A) $\sin x = 0$, $\cos x = -\frac{1}{2}$ (C) $\sin x = \frac{1}{2}$, $\sin x = -1$

(B) $\sin x = -\frac{1}{2}$, $\sin x = 1$ (D) $\sin x = 0$, $\cos x = -1$

3. Which of the following calculations would return the acute angle between the lines $x - 2y = 6$ and $y = 3x - 1$? **1**

(A) $\tan\theta = \left|\dfrac{3-2}{1+6}\right|$ (C) $\tan\theta = \left|\dfrac{3+2}{1-6}\right|$

(B) $\tan\theta = \left|\dfrac{3+\frac{1}{2}}{1-\frac{3}{2}}\right|$ (D) $\tan\theta = \left|\dfrac{3-\frac{1}{2}}{1+\frac{3}{2}}\right|$

4. When the polynomial $P(x) = kx^3 + x^2 - (2k - 1)x + 2$ is divided by $(x + 1)$, the remainder is 4. **1**

What is the value of k?

(A) -2 (B) 0 (C) 2 (D) 4

5. Which of the following represent all values of θ, for which $\tan\theta = \cot\theta$? **1**

(A) $\theta = n\pi \pm \frac{\pi}{4}$ (C) $\theta = n\pi + \frac{\pi}{4}$

(B) $\theta = 2n\pi \pm \frac{\pi}{4}$ (D) $\theta = n\pi - \frac{\pi}{4}$

End of Section I.
Examination continues overleaf.

Question 6 (8 Marks) Commence a NEW page. **Marks**

(a) i. Show that $\dfrac{1+\cos 2A}{\sin 2A} = \cot A$. **3**

ii. Hence find the exact value of $\cot 15^\circ$ in simplest form. **2**

(b) If $\cos\theta = 0.8$, $-\frac{\pi}{2} \le \theta \le 0$, find the exact value of $\cos\dfrac{\theta}{2}$. **3**

Question 7 (8 Marks) Commence a NEW page. **Marks**

(a) Derive the equation of the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$. **2**

(b) Given Q is the point $(2aq, aq^2)$, O is the origin, show that if OQ is parallel to the tangent, then $q = 2p$. **1**

(c) If M is the midpoint of PQ, find the equation of the locus of M as P and Q vary along the parabola such that OQ remains parallel to the tangent at P. **3**

Question 8 (7 Marks) Commence a NEW page. **Marks**

(a) The equation $x^3 - mx + 2 = 0$ has two equal roots.

i. Write down the expressions for the sum of the roots and for the product of the roots. **2**

ii. Hence find the value of m. **2**

(b) If α, β and γ are roots of the equation $x^3 - x^2 + 4x - 1 = 0$, find the value of $(\alpha+1)(\beta+1)(\gamma+1)$. **3**

Question 9 (8 Marks) Commence a NEW page. **Marks**

(a) From what external point are the tangents to the parabola $x^2 = 4y$ to be drawn such that the equation of the chord of contact is $2y - 3x + 2 = 0$? **2**

(b) Given $A(4, 2)$ and $B(-2, -8)$, show that the locus of the point $P(x, y)$ moving such that $\angle APB$ is a right angle, is given by **3**

$$x^2 - 2x + y^2 + 6y = 24$$

Question 10 (8 Marks) Commence a NEW page. **Marks**

Two of the roots of the equation $x^3 + ax^2 + b = 0$, (a, $b \in \mathbb{R}$ and non zero) are reciprocals of each other.

(a) Show that the third root is $-b$. **1**

(b) Show that $a = b - \dfrac{1}{b}$. **3**

(c) Show that the two roots, which are reciprocals, will be real if $-\dfrac{1}{2} \le b \le \dfrac{1}{2}$. **2**

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

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1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (D) **2.** (B) **3.** (D) **4.** (C) **5.** (A)

Section II

Question 5 (Berry)

(a)

Question 6 (Berry)

(a)

Question 7 (Ziaziaris/Lam)

(a) (2 marks)

$$x^2 = 4ay \quad \Rightarrow \quad y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

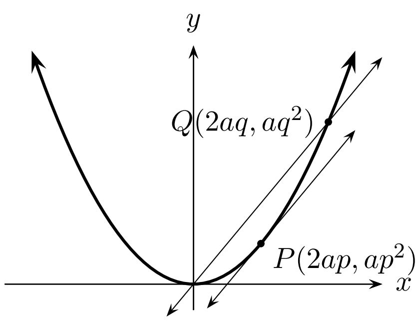
When $x = 2ap$,

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

Applying point gradient formula,

$$\begin{aligned} y - ap^2 &= p(x - 2ap) \\ y &= px - 2ap^2 + ap^2 \\ &= px - ap^2 \end{aligned}$$

(b) (1 mark)



$$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$$

Gradient of tangent at P is p. As $OQ \parallel$ tangent,

$$\frac{q}{2} = p$$
$$\therefore q = 2p$$

(c) (3 marks)

✓ [1] for midpoint.

✓ [1] for p in terms of x.

✓ [1] for final answer.

$$\begin{aligned} MP_{PQ} &= \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right) \\ &= \left(a(p+q), \frac{a}{2}\left(p^2 + q^2\right)\right) \end{aligned}$$

From the midpoint,

$$\begin{cases} x = a(p+q) \\ y = \frac{a}{2}(p^2 + q^2) \end{cases}$$

Since $q = 2p$,

$$\begin{cases} x = a(p + 2p) = 3ap \\ y = \frac{a}{2}(p^2 + 4p^2) = \frac{a}{2} \times 5p^2 \end{cases}$$

Rearrange the x equation and substitute into y:

$$\begin{aligned} p &= \frac{x}{3a} \\ \therefore y &= \frac{a}{2} \times 5\left(\frac{x}{3a}\right)^2 \\ &= \frac{a}{2} \times 5 \times \frac{x^2}{9a^2} \\ &= \frac{5x^2}{18a} \\ \therefore x^2 &= \frac{18}{5}ay \end{aligned}$$

Question 8 (Berry)

(a)

Question 9 (Berry)

(a)

1. D 2. B 3. D 4. C 5. A

2014 YR 12 ASSESSMENT TASK SOLUTIONS
EXT 1 TASK # 1

6 a) (i) LHS $= \frac{1+\cos 2A}{\sin 2A}$

$= \frac{1 + 2\cos^2 A - 1}{2\sin A\cos A}$

$= \frac{2\cos^2 A}{2\sin A\cos A}$

$= \frac{\cos A}{\sin A}$

$= \cot A$

$=$ RHS.

(ii) $\cot 15° = \frac{1+\cos 30°}{\sin 30°}$

$= \left(1 + \frac{\sqrt{3}}{2}\right) \div \frac{1}{2}$

$= \frac{2+\sqrt{3}}{2} \times \frac{2}{1}$

$= 2+\sqrt{3}$

b) $\cos\theta = 0{\cdot}8$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$\therefore \frac{\cos\theta + 1}{2} = \cos^2\frac{\theta}{2}$

$\frac{0{\cdot}8 + 1}{2} = \cos^2\frac{\theta}{2}$

$0{\cdot}9 = \cos^2\frac{\theta}{2}$

$\cos\frac{\theta}{2} = \pm\sqrt{0{\cdot}9}$

But in 4th quad

$\therefore \cos\frac{\theta}{2} = \sqrt{0{\cdot}9} = \frac{3}{\sqrt{10}}$

7. a) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $x = 2ap$

$\frac{dy}{dx} = p$

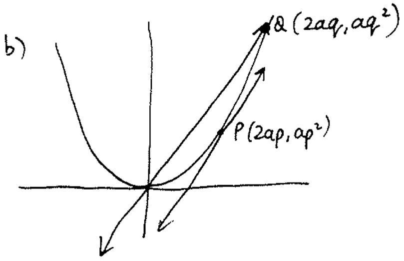
Eq'n of tangent:

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

b)



$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$

m of tang at P $= p$.

Since OQ ∥ tangent at P then

$p = \frac{q}{2}$

$\therefore 2p = q$

OR. Equ'n of QO: $y - aq^2 = p(x - 2aq)$

$y - aq^2 = px - 2apq$

But (0,0) satisfies equ'n

$0 - aq^2 = 0 - 2apq$

$-q = -2p$

$\therefore 2p = q$

c) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), a\frac{(p^2+q^2)}{2}\right)$

ie. $x = a(p+q)$

But $q = 2p$

$\therefore x = a3p$

$\therefore p = \frac{x}{3a}$

Sub p into $y = a\frac{(p^2+q^2)}{2}$

$y = \frac{a}{2}\left(\frac{x^2}{9a^2} + (2p)^2\right)$

$y = \frac{a}{2}\left(\frac{x^2}{9a^2} + 4\left(\frac{x^2}{9a^2}\right)\right)$

$y = \frac{a}{2}\left(\frac{5x^2}{9a^2}\right)$

$18ay = 5x^2$

8. a) $x^3 - mx + 2 = 0$

(i) Let the roots be α, α, β.

$\therefore 2\alpha + \beta = 0$ — ①

$\alpha^2\beta = -2$. — ②

(ii) $\beta = -2\alpha$ — ③

Sub ③ into ②

$\alpha^2(-2\alpha) = -2$

$\alpha^3 = 1$

$\therefore \alpha = 1$

$\therefore \beta = -2$.

$\therefore$ Roots 1, 1, −2

$1 + -2 - 2 = -m$

$-3 = -m$

$\therefore m = 3$.

b) $x^3 - x^2 + 4x - 1 = 0$.

$(a+1)(b+1)(c+1) = (ab + a + b + 1)(c+1)$

$= abc + ab + ac + a + bc + b + c + 1$

$= abc + (ab + ac + bc) + (a + b + c) + 1$

$= 1 + 4 + 1 + 1$

$= 7$

⑨. a) Chord of contact

$xx_0 = 2a(y + y_0)$

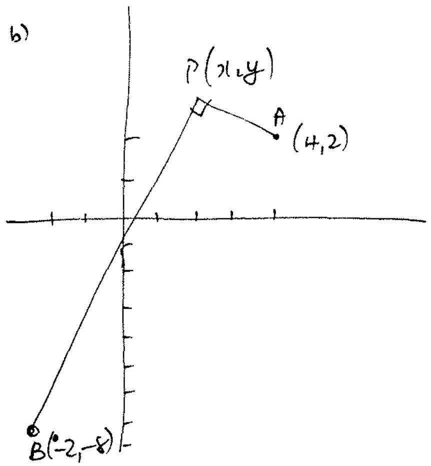
ie. $xx_0 - 2ay - 2ay_0 = 0$

$+3x - 2y - 2 = 0$.

$\therefore x_0 = 3, \ y_0 = 1$

External point is $(3,1)$

b)



Let P be the point (x,y)

$m_{AP} = \frac{y-2}{x-4}$

$m_{PB} = \frac{y+8}{x+2}$

Since AP ⊥ PB then

$\frac{y-2}{x-4} \times \frac{y+8}{x+2} = -1$

$y^2 + 8y - 2y - 16 = -(x^2 + 2x - 4x - 8)$

$y^2 + 8y - 2y - 16 = -x^2 - 2x + 4x + 8$

(10) a) <u>Equ'n of PQ :</u> $m = m$ $(0, -2)$

$y + 2 = m(x - 0)$

$y + 2 = mx$

$y = mx - 2$.

b) Sub $2t, t^2$

$t^2 = m(2t) - 2$

$t^2 - 2mt + 2 = 0$.

c) $p + q = 2m$

$pq = 2$

$R\left(-pq(p+q),\ (p+q)^2 - pq + 2\right)$

ie. $R\left(-2(2m),\ (2m)^2 - 2 + 2\right)$

$R(-4m,\ 4m^2)$

$x^2 = 4y$ is original parabola

LHS $= x^2$
$= (-4m)^2$
$= 16m^2$

RHS $= 4y$
$= 4(4m^2)$
$= 16m^2$

(11) a) $x^3 + ax^2 + b = 0$

Let the roots be $\alpha, \frac{1}{\alpha}, \beta$

Product of roots : $\alpha \cdot \frac{1}{\alpha} \cdot \beta = -b$

$\therefore \boxed{\beta = -b}$

b) Sum of roots : $\alpha + \frac{1}{\alpha} + \beta = -a$

$\alpha + \frac{1}{\alpha} - b = -a$

$\therefore \alpha + \frac{1}{\alpha} = b - a$

OR (Sub in $P(-b) = 0$
$-b^3 + ab^2 + b = 0$
$ab^2 = b^3 - b$
$a = b - \frac{1}{b}$)

Sum of roots 2 at a time : $\alpha \cdot \frac{1}{\alpha} + \frac{1}{\alpha} \cdot \beta + \beta\alpha = 0$

$1 + \frac{\beta}{\alpha} + \alpha\beta = 0$

$1 - \frac{b}{\alpha} - b\alpha = 0$ As $\beta = -b$

$1 - b\left(\frac{1}{\alpha} + \alpha\right) = 0$

$\frac{1}{\alpha} + \alpha = \frac{1}{b}$

But $\alpha + \frac{1}{\alpha} = b - a$

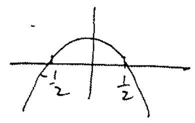
$\therefore \frac{1}{b} = b - a$

$\therefore \boxed{a = b - \frac{1}{b}}$

c) $\alpha, \frac{1}{\alpha}$ are roots of equ'n

$x^2 - \left(\alpha + \frac{1}{\alpha}\right)x + \alpha \cdot \frac{1}{\alpha} = 0$

$x^2 - \frac{1}{b}x + 1 = 0$ (1)



Real when $\Delta \geq 0$.

$\frac{1}{b^2} - 4 \geq 0$

$1 - 4b^2 \geq 0$

$(1 - 2b)(1 + 2b) \geq 0$

i.e. $\boxed{-\frac{1}{2} \leq b \leq \frac{1}{2}}$

HIGH QUALITY MATHEMATICAL COMMUNICATION IS REQUIRED!

"Show" questions require clearly communicated mathematical reasoning. They are the mathematics analogue of a well crafted English essay. **You are telling a story** – it must have a clear beginning, clearly marked "paragraphs", with an argument presented in a logical sequence.

Reason $A + C = B + C$, $\therefore A = B$ (1)

Reason $2A + Z = Q + Z$, $2A = Q$ (2)

(1) + (2) $3A = B + Q$, $A = (B+Q)/3$

In particular:

1. Each new equation introduced into the flow must have an explanation where it comes from. eg: "Sum of roots two at a time", "Substitute (1) into (2)"
2. Add labels (1), (2) or (A), (B) to the end of each block of reasoning so that you can refer to it when you start a new block of reasoning linking equations together.
3. Work in ONE column, going down the page.
4. Each line must have an equals sign – you are making a statement of truth and carrying this truth throughout the argument.

Running out of time, or 'working out' while exploring a solution is not an excuse to make a mess. If you expect it to be marked, then you owe it to the marker to clearly explain what you are doing.

It takes only a few extra seconds to label equations, to write a two word justification for a new equation. You are thinking it in your head anyway – so write it down.

MARKS CAN AND SHOULD BE DEDUCTED FOR POOR SETTING OUT.

STUDENT SAMPLE OF QUALITY WORK

Question 11

a) let roots be $\alpha, \frac{1}{\alpha}, \beta$

then product of roots are

$$\alpha \cdot \frac{1}{\alpha} \cdot \beta = \frac{-b}{1}$$

$$\beta = -b$$

$\therefore$ other root is $-b$

b) sum of roots 2 at a time

$$\alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{\beta}{\alpha} = 0$$

$$1 + \alpha\beta + \frac{\beta}{\alpha} = 0$$

$$\alpha + \alpha^2\beta + \beta = 0 \quad (1)$$

Sum of roots

$$\alpha + \frac{1}{\alpha} + \beta = -a$$

$$\alpha + \frac{1}{\alpha} - b = -a$$

$$\alpha + \frac{1}{\alpha} = -a + b \quad (2)$$

From (1): $1 + \alpha\beta + \frac{\beta}{\alpha} = 0$

$$1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = 0$$

(2) $\rightarrow$ (1)

$$1 + \beta(-a + b) = 0$$

$$1 - b(-a + b) = 0$$

$$-b(-a+b) = -1$$

$$-a + b = \frac{1}{b}$$

$$\therefore a = b - \frac{1}{b}$$

continued...