NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Extension 1 Mathematics Assessment Task 1
Term 4, 2011

Name:_______________________________ **Mathematics Class:**_______

Time Allowed: **60 minutes + 2 minutes reading time**

Total Marks: **46**

Instructions:

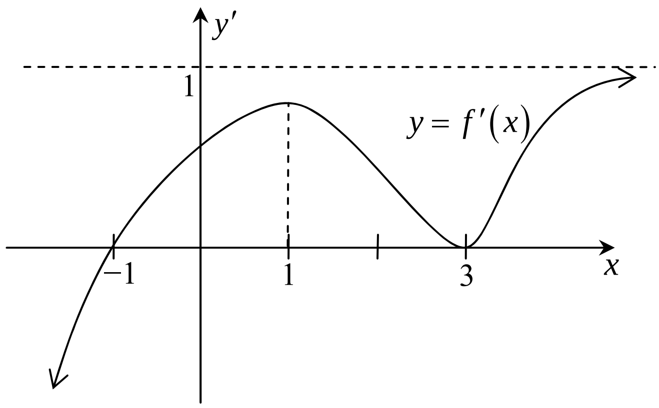
- Attempt all three questions.
- Start each question in a new booklet. Put your name on every booklet.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.

Question	A1-2	A3-8	B1	B2	C	Total
PE3		/14	/3			/17
HE7	/2			/11	/16	/29
						/46

Question 1 (16 marks)

(a) Write A, B, C or D on your answer sheet.

For parts (i) and (ii), consider the gradient function $y = f'(x)$ which is shown in the sketch below.



(i) The stationary points on the graph of $y = f(x)$ consist of: **1**

(A) One local minimum and one horizontal point of inflexion.

(B) One local maximum and one horizontal point of inflexion.

(C) One local minimum and one local maximum.

(D) Two horizontal points of inflexion.

(ii) The graph of $y = f(x)$ has one asymptote. **1**
The equation of this asymptote could be:

(A) $x = 1$ (B) $y = 1$

(C) $y = x + 2$ (D) None of the above.

(iii) The polynomial $x^3 - 2x^2 - 5x + 6$ factorises to: **1**

(A) $(x+1)(x-2)(x+3)$ (B) $(x-1)(x+2)(x+3)$

(C) $(x+1)(x-2)(x-3)$ (D) $(x-1)(x+2)(x-3)$

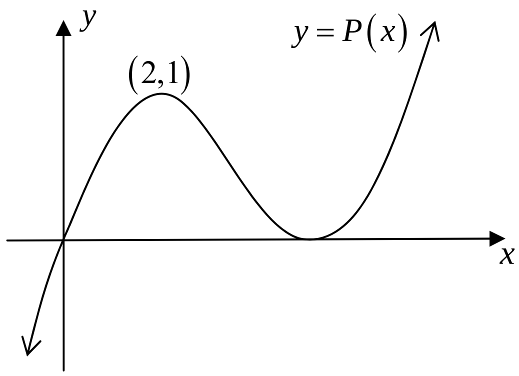
(iv) If $(x-2)^3(x+1)^2$, $(x+3)^4(x+1)$ and $(x-2)^5(x+3)^3$ are all factors of a polynomial $P(x)$, then the smallest possible degree of $P(x)$ is **1**

(A) 5 (B) 8

(C) 11 (D) 18

Question 1 continued

(v) **1**



The diagram shows the graph of the polynomial $y = P(x)$.
You are given that $P'(0) = 2$.
For what values of k will the equation $P(x) = kx$ have exactly three distinct roots?

(A) $k > 0$ (B) $0 < k < 2$

(C) $0 < k < 1$ (D) all real k

For the remainder of this section, show full working.

(b) (i) Show that $x - 2$ is a factor of $x^3 - 4x^2 + 3x + 2$. **1**

(ii) Sketch the graph of $P(x) = x^3 - 4x^2 + 3x + 2$. **3**

(iii) Hence solve the inequation $x^3 - 4x^2 + 3x + 2 < 0$. **1**

(c) The polynomial $P(x)$ has a remainder of $ax - 2$ when divided by $x^2 - x - 6$, **3**
and a remainder of 13 when divided by $x - 3$.

By first finding the value of a, find the remainder when $P(x)$ is divided by $x + 2$.

(d) If $5x^3 + 3x^2 - 7x - 3 = ax(x-1)(x+2) + bx(x-1) + cx - 3$ for all values of x, find **3**
the values of a, b and c.

Question 2 (14 marks) Start a new booklet

(a) The polynomial equation $x^3 + bx^2 + cx + d = 0$ has roots α, β and γ. **3**

Find a simplified expression for $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\alpha\gamma} + \dfrac{\gamma}{\alpha\beta}$ in terms of b, c and d.

(b) Consider the function $f(x) = \dfrac{x^2}{x+2}$.

(i) Write down the equation of the vertical asymptote. **1**

(ii) Show that $f(x) = x - 2 + \dfrac{4}{x+2}$. **1**

(iii) Hence write down the equation of the non-vertical asymptote. **1**

(iv) Show that $f'(x) = \dfrac{x(x+4)}{(x+2)^2}$. **3**

(v) Find the coordinates of each stationary point and determine their nature in each case. **3**

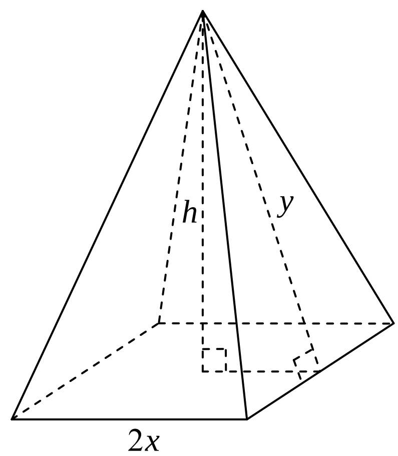
(vi) Draw a sketch of $y = f(x)$ showing all this information. **2**

Question 3 (16 marks) Start a new booklet

(a) (i) Show that $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}=-\dfrac{h(2x+h)}{x^2(x+h)^2}$. **2**

(ii) Hence differentiate the function $f(x)=\dfrac{1}{x^2}$ by first principles. **2**

(b) A closed right pyramid has a square base of side length 2*x* cm, and its triangular faces each have perpendicular height *y* cm, as shown in the following diagram.



(i) This pyramid is to be constructed using $400\ \text{cm}^2$ of cardboard. **2**
Show that $y=\dfrac{100}{x}-x$.

(ii) Show that the perpendicular height h of the pyramid is given by **2**
$$h^2=\frac{10000}{x^2}-200.$$

(iii) Show that the volume V of the pyramid is given by. **2**
$$V=\frac{40}{3}x\sqrt{100-2x^2}.$$

(iv) Show that $\dfrac{dV}{dx}=\dfrac{160\left(25-x^2\right)}{3\sqrt{100-2x^2}}$. **3**

(v) Find the base length and perpendicular height of such a pyramid with maximum volume. Justify your answer. **3**

End of paper

Extension 1 Assessment 1 Solutions

Question 1

(a) (i) A (ii) C (iii) D (iv) C (v) B

(b) (i) Let $P(x) = x^3 - 4x^2 + 3x + 2$.

$P(2) = 2^3 - 4(2)^2 + 3(2) + 2$

$= 0$

$\therefore\ x - 2$ is a factor of $P(x)$

(ii) Let the zeros of $P(x)$ be $\alpha,\ \beta$ and 2.

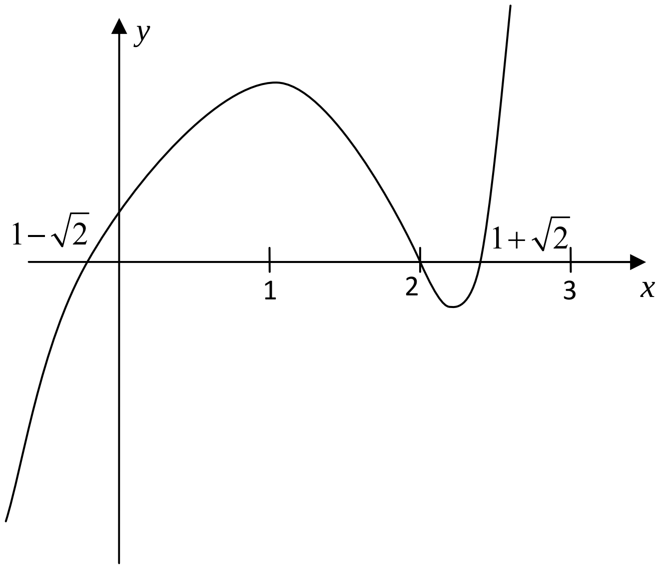
So $\alpha + \beta + 2 = 4$ $\qquad 2\alpha\beta = -2$

$\alpha + \beta = 2$ $\qquad \alpha\beta = -1$

So the quadratic with roots α and β is $x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = 1 \pm \sqrt{2}$$



(iii) $x < 1 - \sqrt{2},\ \ 2 < x < 1 + \sqrt{2}$

(c) $P(x) = (x-3)(x+2)Q(x) + ax - 2$

$P(3) = 13$

$13 = 3a - 2$

$a = 5$

remainder $= P(-2)$

$= -12$

(d) $5x^3 + 3x^2 - 7x - 3 = ax(x-1)(x+2) + bx(x-1) + cx - 3$

$(x = 1)$: $\quad 5 + 3 - 7 - 3 = c - 3 \quad \Rightarrow \quad c = 1$

$(x = -2)$: $\quad -40 + 12 + 14 - 3 = 6b - 2 - 3 \quad \Rightarrow \quad b = -2$

$(x = 2)$: $\quad 40 + 12 - 14 - 3 = 8a - 4 + 2 - 3 \quad \Rightarrow \quad a = 5$

Question 2

(a) $$\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)}{\alpha\beta\gamma}$$

$$=\frac{(-b)^2-2(c)}{-d}$$

$$=\frac{2c-b^2}{d}$$

(b) (i) $x=-2$

(ii) $$RHS=x-2+\frac{4}{x+2}$$

$$=\frac{(x-2)(x+2)+4}{x+2}$$

$$=\frac{x^2-4+4}{x+2}$$

$$=\frac{x^2}{x+2}$$

(iii) $y=x-2$

(iv)

$$f'(x)=\frac{(x+2)\cdot 2x-x^2\cdot 1}{(x+2)^2}$$

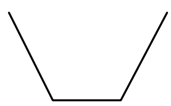
$$=\frac{2x^2+4x-x^2}{(x+2)^2}$$

$$=\frac{x^2+4x}{(x+2)^2}$$

$$=\frac{x(x+4)}{(x+2)^2}$$

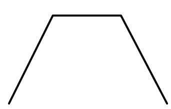
(v) Stationary points at $x=0,-4$.

x	-1	0	1
$\frac{dy}{dx}$	-3	0	$\frac{5}{9}$



so, local minimum at $(0,0)$

x	-5	-4	-3
$\frac{dy}{dx}$	$\frac{5}{9}$	0	-3



so, local maximum at $(-4,-8)$

(v) [Alternative solution]

$$f(x) = x - 2 + \frac{4}{x+2}$$

$$= x - 2 + 4(x+2)^{-1}$$

$$f'(x) = 1 - 4(x+2)^{-2}$$

$$= 1 - \frac{4}{(x+2)^2}$$

$$f''(x) = 8(x+2)^{-3}$$

$$= \frac{8}{(x+2)^3}$$

Stat Pts: $1 - \frac{4}{(x+2)^2} = 0$

$$(x+2)^2 = 4$$

$$x = 0, -4$$

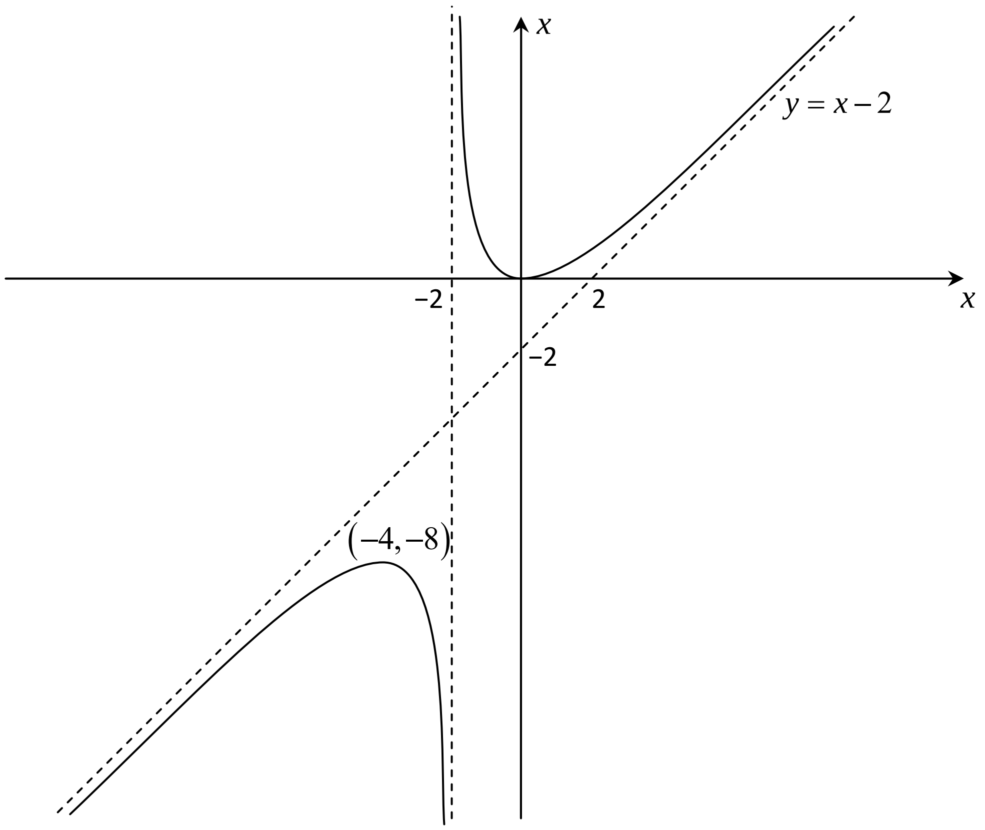
$f(0) = 0,\ f''(0) = 1 > 0$

so, local min. at $(0,0)$

$f(-4) = -8,\ f''(-4) = -1 < 0$

so, local max. at $(-4,-8)$

(vi)



Question 3

(a) (i)

$$\frac{1}{(x+h)^2}-\frac{1}{x^2}=\frac{x^2-(x+h)^2}{x^2(x+h)^2}$$
$$=\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}$$
$$=\frac{-2xh-h^2}{x^2(x+h)^2}$$
$$=-\frac{h(2x+h)}{x^2(x+h)^2}$$

(ii)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$
$$=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$$
$$=\frac{-2x}{x^2\cdot x^2}$$
$$=-\frac{2}{x^3}$$

(b) (i) $4x^2+4\left(\frac{1}{2}\cdot 2x\cdot y\right)=400$

$$x^2+xy=100$$
$$xy=100-x^2$$
$$y=\frac{100}{x}-x$$

(ii) $h^2=y^2-x^2$ by Pythagoras' Theorem

$$=\left(\frac{100}{x}-x\right)^2-x^2$$
$$=\frac{10000}{x^2}-200+x^2-x^2$$
$$=\frac{10000}{x^2}-200$$

(iii)

$$V = \frac{1}{3}(2x)^2 h$$

$$= \frac{4}{3}x^2\sqrt{\frac{10000}{x^2} - 200}$$

$$= \frac{4}{3}x^2 \cdot \frac{10}{x}\sqrt{100 - 2x^2}$$

$$= \frac{40}{3}x\sqrt{100 - 2x^2}$$

(iv)

$$\frac{dV}{dx} = \frac{40}{3}\left[\left(100 - 2x^2\right)^{\frac{1}{2}} \cdot 1 \quad + \quad x \cdot \frac{1}{2}\left(100 - 2x^2\right)^{-\frac{1}{2}} \cdot (-4x)\right]$$

$$= \frac{40}{3}\left[\left(100 - 2x^2\right)^{\frac{1}{2}} \quad - \quad 2x^2\left(100 - 2x^2\right)^{-\frac{1}{2}}\right]$$

$$= \frac{40}{3}\left(100 - 2x^2\right)^{-\frac{1}{2}}\left[\left(100 - 2x^2\right) - 2x^2\right]$$

$$= \frac{160}{3}\left(100 - 2x^2\right)^{-\frac{1}{2}}\left(25 - x^2\right)$$

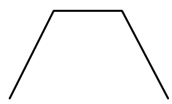
$$= \frac{160\left(25 - x^2\right)}{3\sqrt{100 - 2x^2}}$$

(v)

$$\frac{dV}{dx} = 0$$

$$x = 5 \quad (x > 0)$$

x	4	5	6
$\frac{dV}{dx}$	$\frac{240}{\sqrt{17}}$	0	$\frac{-880}{3\sqrt{7}}$



i.e. maximum Volume when $x = 5$

Dimensions:

Base: 10 cm

Perp Ht: $10\sqrt{2}$ cm